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ON THE CORRESPONDENCE BETWEEN CLASSICAL AND QUANTUM MEASUREMENTS ON A BOSONIC FIELD

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We study the correspondence between classical and quantum measurements on a harmonic oscillator that describes a one-mode bosonic field with annihilation and creation operators a and $a^\dagger$ with commutation $[a, a^\dagger] = 1$. We connect the quantum measurement of an observable $\hat{O} = \hat{O}(a, a^\dagger)$ of the field with the possibility of amplifying the observable $\hat{O}$ ideally through a quantum amplifier which achieves the Heisenberg-picture evolution $\hat{O} \to g\hat{O}$, where g is the gain of the amplifier. The "classical" measurement of $\hat{O}$ corresponds to the joint measurement of the position $\hat{q} = 1/2(a^\dagger + a)$ and momentum $\hat{p} = i/2(a^\dagger - a)$ of the harmonic oscillator, with following evaluation of a function $f(\alpha, \bar{\alpha})$ of the outcome $\alpha = q + ip$. For the electromagnetic field the joint measurement is achieved by a heterodyne detector. The quantum measurement of $\hat{O}$ is obtained by preamplifying the heterodyne detector through an ideal amplifier of $\hat{O}$ and rescaling the outcome by the gain g. We give a general criterion which states when this preamplified heterodyne detection scheme approaches the ideal quantum measurement of $\hat{O}$ in the limit of infinite gain. We show that this criterion is satisfied and the ideal measurement is achieved for the case of the photon number operator $a^\dagger a$ and for the quadrature $\hat{X}_\phi = (a^\dagger e^{i\phi} + ae^{-i\phi})/2$, where one measures the functions $f(\alpha, \bar{\alpha}) = |\alpha|^2$ and $f(\alpha, \bar{\alpha}) = \mathrm{Re}(\alpha e^{-i\phi})$ of the field, respectively. For the photon number operator $a^\dagger a$ the amplification scheme also achieves the transition from the continuous spectrum $|\alpha|^2 \in \mathbb{R}$ to the discrete one $n \in \mathbb{N}$ of the operator $a^\dagger a$. Moreover, for both operators $a^\dagger a$ and $\hat{X}_\phi$ the method is robust to nonunit quantum efficiency of the heterodyne detector. On the other hand, we show that the preamplified heterodyne detection scheme does not work for arbitrary observable of the field. As a counterexample, we prove that the simple quadratic function of the field $\hat{K} = i(a^{\dagger 2} - a^2)/2$ has no corresponding polynomial function $f(\alpha, \bar{\alpha})$ — including the obvious choice $f = \mathrm{Im}(\alpha^2)$ — that allows the measurement of $\hat{K}$ through the preamplified heterodyne measurement scheme.

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1. Introduction

In the standard formulation of Quantum Mechanics an abstract concept of physical observable is formulated in terms of real eigenvalues and sharp probability distributions, which leads to the well known correspondence between observables and self-adjoint operators on the Hilbert space.[1] A natural extension of this formulation is based on the general concept of Positive Operator–Valued Measure (POVM),[2,3] which allows the description of joint measurements of non-commuting observables, with generally complex eigenvalues and probability distributions that are not sharp for any quantum state. From an operational point of view, however, we have no prescription on how to achieve the ideal quantum measurement (i.e. with minimum noise) of a generic operator and the problem of finding a *universal detector* is still an open one. Quantum homodyne tomography — the only known method for measuring the state itself of the field — can also be regarded as a kind of universal detection,[4] however it is far from being ideal, due to the occurrence of statistical measurement errors that are intrinsic of the method.

In this paper we study the possibility of achieving the ideal measurement of an observable $\hat{O} = \hat{O}(a, a^\dagger)$ of one mode of the electromagnetic field by means of a fixed detection scheme — the heterodyne detector — after ideal preamplification $\hat{O} \to g\hat{O}$ of the observable $\hat{O}$, g denoting the amplifier gain, seeking a connection between the problem of measuring $\hat{O}$ and that of amplifying $\hat{O}$ ideally. As heterodyne detection corresponds to the ideal joint measurement of the canonical pair $\hat{q} = 1/2(a^\dagger + a)$ and $\hat{p} = i/2(a^\dagger - a)$ of a harmonic oscillator in the phase space, in this way we also try to set a link between classical and quantum measurements. We will give a necessary and sufficient condition that establishes when the preamplified heterodyne detection scheme approaches the ideal quantum measurement of $\hat{O}$ in the limit of infinite gain. We show that such condition is satisfied for the photon number operator $a^\dagger a$ — corresponding to the function $f(\alpha, \bar{\alpha}) = |\alpha|^2$ of the heterodyne outcome $\alpha \in \mathbb{C}$ — and for the quadrature operator $\hat{X}_\phi = (a^\dagger e^{i\phi} + a e^{-i\phi})/2$ — corresponding to the function $f(\alpha, \bar{\alpha}) = \mathrm{Re}(\alpha e^{-i\phi})$. For the photon number operator $a^\dagger a$ the amplification scheme also achieves the transition from the continuous spectrum $|\alpha|^2 \in \mathbb{R}$ to the discrete spectrum $\mathbb{S}_{a^\dagger a} \equiv \mathbb{N}$ of $a^\dagger a$. Moreover, for both operators $a^\dagger a$ and $\hat{X}_\phi$ the methods is also robust to nonunit quantum efficiency of the heterodyne detector. On the other hand, we will see that the preamplified heterodyne scheme does not work for arbitrary observable of the field. As a counterexample, we show that, unexpectedly, the simple quadratic function of the field $\hat{K} = i(a^{\dagger 2} - a^2)/2$ has no corresponding polynomial function $f(\alpha, \bar{\alpha})$ — including the obvious choice $f = \mathrm{Im}\, \alpha^2$) — which allows the measurement of $\hat{K}$ through the preamplified heterodyne measurement scheme.

The paper is organized as follows. In Sec. 2 we derive the POVM of the heterodyne measurement of a function f of the field, for generally nonunit quantum efficiency. In Sec. 3 we analyze the ideal amplification of an observable $\hat{O}$ and prove that it can be always achieved by a unitary transformation. In Sec. 4 we give a necessary and sufficient condition for the preamplified heterodyne detection scheme to approach the ideal measurement of $\hat{O}$. Section 5 is devoted to the two examples $\hat{O} = a^\dagger a$ and $\hat{O} = \hat{X}_\phi$ which satisfy the requirements of the general criterion of Sec. 3. There we also prove explicitly that the ideal measurement of $\hat{O}$ is achieved by the preamplified heterodyne detection scheme in the limit of infinite gain of the amplifier, also for nonunit quantum efficiency of the heterodyne detector. Section 6 is devoted to the counterexample $\hat{K} = i(a^{\dagger 2} - a^2)/2$, where, in order to prove that the preamplified heterodyne scheme does not work, we also derive the explicit analytical form of the ideal amplification map for $\hat{K}$. Section 7 concludes the paper by summarizing the main results.

2. Heterodyne Detection

Heterodyne detection corresponds to measuring the complex field $\hat{Z} = a + b^\dagger$, a and b denoting the signal and the image-band modes of the detector, respectively. The measurement is an exact joint measurement of the commuting observables $\mathrm{Re}(\hat{Z})$ and $\mathrm{Im}(\hat{Z})$, but can also be regarded as the joint measurement of the noncommuting operators $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$, by considering the image-band mode in the vacuum state. In this way the vacuum fluctuations of b introduce an additional 3 dB noise, which can be proved to be the minimum added noise in an ideal joint measurement of a conjugated pair of noncommuting observables.[5]

The probability density in the complex plane $p(\alpha, \bar{\alpha})$ for heterodyne detection is given by the Fourier transform of the generating function of the moments of $\hat{Z}$, namely

$$p(\alpha, \bar{\alpha}) = \int \frac{d^2\lambda}{\pi^2} \langle e^{\lambda \hat{Z}^\dagger - \bar{\lambda}\hat{Z}} \rangle e^{\bar{\lambda}\alpha - \lambda\bar{\alpha}} \doteq \langle \delta^{(2)}(\alpha - \hat{Z}) \rangle \,, \tag{1}$$

where the overbar denotes the complex conjugate, $d^2\lambda = d\,\mathrm{Re}\,\lambda \; d\,\mathrm{Im}\,\lambda$, $\langle \cdots \rangle$ represents the ensemble quantum average on both signal and image-band modes and $\delta^{(2)}(\alpha)$ is the Dirac delta-function in the complex plane. The partial trace over the image-band mode in Eq. (1) can be evaluated as follows

$$\begin{aligned} \langle e^{\lambda \hat{Z}^\dagger - \bar{\lambda}\hat{Z}} \rangle &= \mathrm{Tr}_a[\hat{\varrho}\hat{D}_a(\lambda)]_b\langle 0|\hat{D}_b(-\bar{\lambda})|0\rangle_b = \mathrm{Tr}_a[\hat{\varrho}\hat{D}_a(\lambda)]e^{-1/2|\lambda|^2} \\ &\doteq \mathrm{Tr}_a[\hat{\varrho} :\hat{D}_a(\lambda):_A] \,, \end{aligned} \tag{2}$$

where $\hat{D}(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a)$ denotes the displacement operator ($\hat{D}_a$ for mode a and $\hat{D}_b$ for mode b), $|0\rangle_b$ represents the vacuum for mode b only, $\hat{\varrho}$ is the density matrix for the signal mode and $::_A$ denotes anti-normal ordering. The probability density versus the outcome α is given by

$$d^2\alpha \; p(\alpha, \bar{\alpha}) = \mathrm{Tr}[\hat{\varrho} d\hat{\mu}(\alpha, \bar{\alpha})] \,, \tag{3}$$

where the probability operator-valued measure (POVM) $d\hat{\mu}(\alpha, \bar{\alpha})$ can be written as follows

$$
\begin{aligned}
d\hat{\mu}(\alpha,\bar{\alpha}) &= d^2\alpha \int \frac{d^2\lambda}{\pi^2} e^{\bar{\lambda}\alpha-\lambda\bar{\alpha}} :\hat{D}(\lambda):_A \\
&= d^2\alpha \int \frac{d^2\beta}{\pi} \int \frac{d^2\lambda}{\pi^2} e^{\bar{\lambda}(\alpha-\beta)-\lambda(\bar{\alpha}-\bar{\beta})} |\beta\rangle\langle\beta| \\
&= \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \doteq d^2\alpha :\delta^{(2)}(\alpha-a):_A \,,
\end{aligned} \tag{4}
$$

using the resolution of the identity in terms of coherent states $\hat{1} = \int(d^2\beta/\pi)|\beta\rangle\langle\beta|$.

In a "classical" measurement of the function $w = f(\alpha,\bar{\alpha})$ on the phase space, one evaluates the function f of the outcome α of the complex photocurrent $\hat{Z}$. Correspondingly, the probability distribution of w is given by the marginal probability density

$$
p(w) = \int d^2\alpha\; p(\alpha,\bar{\alpha})\; \delta(w - f(\alpha,\bar{\alpha}))\,. \tag{5}
$$

The POVM $d\hat{H}_f(w)$ that provides such probability density is the marginal POVM of $d\hat{\mu}(\alpha,\bar{\alpha})$ and can be written as follows

$$
d\hat{H}_f(w) = dw \int d\hat{\mu}(\alpha,\bar{\alpha})\delta(w - f(\alpha,\bar{\alpha})) = dw :\delta(w - f(a,a^\dagger)):_A\,. \tag{6}
$$

In this way one has a correspondence rule between POVM's $d\hat{H}_f(w)$ and classical observables $w = f(\alpha,\bar{\alpha})$ on the phase space $\alpha \in \mathbb{C}$.

The quantum efficiency η of the heterodyne detector can be taken into account by introducing auxiliary vacuum field modes for both the signal and the idler and by rescaling the output photocurrent by an additional factor $\eta^{1/2}$. The overall effect resorts to a Gaussian convolution of the ideal POVM with variance $\Delta^2_\eta = (1-\eta)/\eta$. Then, the POVM in Eq. (6) rewrites

$$
d\hat{H}_f(w) = dw\Gamma_{(1-\eta)/\eta}[:\delta(w - f(a,a^\dagger)):_A]\,, \tag{7}
$$

where Γ_{σ^2} denotes the completely positive (CP) map that describes the effect of additional Gaussian noise of variance σ^2, namely

$$
\Gamma_{\sigma^2}[\hat{A}] = \int \frac{d^2\beta}{\pi\sigma^2} e^{-|\beta|^2/\sigma^2} \hat{D}(\beta)\hat{A}\hat{D}^\dagger(\beta)\,, \tag{8}
$$

for any operator $\hat{A}$. We do not know *a priori* if the measurement described by the POVM in Eq. (6) or (7) corresponds to an approximate quantum measurement of some observable of the field. We can argue that, for example, for $f(\alpha,\bar{\alpha}) = |\alpha|^2$ the measurement would approximate the ideal detection of the number of photons $a^\dagger a$. In the following we give a necessary and sufficient condition to establish when the heterodyne POVM $d\hat{H}_f(w)$ approaches the ideal quantum measurement of an observable $\hat{O}$ by preamplifying the heterodyne through an ideal amplifier of $\hat{O}$ in the limit of infinite amplifier gain. In the following section we introduce the general concept of ideal amplification of an observable, and prove that it can be always achieved by a unitary transformation.

3. Ideal Amplification of Quantum Observables

For a given self-adjoint operator $\hat{W}$, the *ideal amplifier* of $\hat{W}$ is a device that achieves the transformation

$$\mathcal{A}_g^{(\hat{W})}(\hat{W}) = g\hat{W}\,, \tag{9}$$

where $g > 1$ denotes the gain of the amplifier. The transformation (9) is to be regarded as the Heisenberg-picture evolution of the field throughout the device when the transformation is applied to $\hat{W}$. If the spectrum $\mathbb{S}_{\hat{W}}$ of $\hat{W}$ is $\mathbb{S}_{\hat{W}} = \mathbb{R}$ or $\mathbb{S}_{\hat{W}} = \mathbb{R}^+$, the evolution $\mathcal{A}_g^{(\hat{W})}$ can be written as follows

$$\mathcal{A}_g^{(\hat{W})}(|w\rangle\langle w|) = g^{-1}|g^{-1}w\rangle\langle g^{-1}w|\,, \tag{10}$$

where $|w\rangle$ denotes the eigenvector of $\hat{W}$ pertaining to the eigenvalue $w \in \mathbb{S}_{\hat{W}}$. The corresponding Schrödinger-picture of the evolution (10) is given by the dual map

$$\mathcal{A}_g^{\vee(\hat{W})}(|w\rangle\langle w|) = g|gw\rangle\langle gw|\,, \tag{11}$$

where $|w\rangle$ now has to be regarded as a (Dirac-sense) normalized state vector. For integer spectrum $\mathbb{S}_{\hat{W}} = \mathbb{N}$ or $\mathbb{S}_{\hat{W}} = \mathbb{Z}$, Eq. (10) rewrites as follows

$$\mathcal{A}_g^{W}(|n\rangle\langle n|) = |g^{-1}n\rangle\langle g^{-1}n|\ \chi_{\mathbb{Z}}(g^{-1}n)\,, \tag{12}$$

where $\chi_{\mathbb{Z}}(x)$ is the characteristic function on integers, namely $\chi_{\mathbb{Z}}(x) = 1$ for $x \in \mathbb{Z}$, $\chi_{\mathbb{Z}}(x) = 0$ otherwise. It is easy to check that both Eqs. (10) and (12) imply Eq. (9). In the following we will consider only the cases of spectra $\mathbb{S}_{\hat{W}} = \mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{Z}$, as these are the only ones that are left invariant under amplification, i.e. $g\mathbb{S}_{\hat{W}} \subset \mathbb{S}_{\hat{W}}$ (this will exclude, for example, the case of phase amplification[3]). Moreover, for the sake of notation, if not explicitly written, we will assume $\mathbb{S}_{\hat{W}} = \mathbb{R}$.

Among all possible extensions of the amplification map (11) to all state vectors, the following ones are physically meaningful

$$\mathcal{A}_g^{\vee(\hat{W})}(|w\rangle\langle w'|) = g|gw\rangle\langle gw'|\,, \tag{13}$$

$$\mathcal{A}_g^{\vee(\hat{W})}(|w\rangle\langle w'|) = g|gw\rangle\langle gw'|\delta(w - w')\,. \tag{14}$$

In fact, both maps in Eqs. (13) and (14) are linear normal *completely positive* (CP) maps, and hence they can be realized through a unitary transformations on an extended Hilbert space.[6] The proof runs as follows. The map A is completely positive normal if and only if one has

$$\sum_{i,j=1}^{n} \langle\xi_i|A^\vee(|\eta_i\rangle\langle\eta_j|)|\xi_j\rangle \geq 0 \tag{15}$$

for all finite sequence of vectors $\{|\eta_i\rangle\}$ and $\{|\xi_i\rangle\}$. Upon expanding $|\eta_i\rangle$ and $|\xi_i\rangle$ on the orthonormal basis $\{|w\rangle\}$, for the map (13) one has

$$\sum_{i,j=1}^{n} \langle \xi_i | A_g^{\vee(\hat{W})}(|\eta_i\rangle\langle\eta_j|)|\xi_j\rangle = g\int dw_1 dw_2 dw_3 dw_4 \langle w_1|gw_2\rangle\langle gw_3|w_4\rangle \times \sum_{i,j=1}^{n} \bar{\xi}_i(w_1)\eta_i(w_2)\bar{\eta}_j(w_3)\xi_j(w_4) = g\left|\sum_{i=1}^{n}\int dw dw' \langle w|gw'\rangle \bar{\xi}_i(w)\eta_i(w')\right|^2 \geq 0\,, \quad (16)$$

whereas for the map (14) one has

$$\sum_{i,j=1}^{n} \langle \xi_i | A_g^{\vee(\hat{W})}(|\eta_i\rangle\langle\eta_j|)|\xi_j\rangle = g\int_{-\infty}^{+\infty}\frac{d\lambda}{2\pi}\left|\sum_{i=1}^{n}\int dw dw' \langle w|gw'\rangle \bar{\xi}_i(w)\eta_i(w')e^{i\lambda w'}\right|^2 \geq 0\,. \quad (17)$$

In the Schrödinger picture the two maps (13) and (14) are achieved by the following unitary transformations in an extended Hilbert space

$$\hat{U}_g|w\rangle \otimes |\psi\rangle = g^{1/2}|gw\rangle \otimes |\psi'\rangle\,, \quad (18)$$

$$\hat{U}_g|w\rangle \otimes |\psi\rangle = g^{1/2}|gw\rangle \otimes |\psi'(w)\rangle\,. \quad (19)$$

with $\langle\psi'(w_1)|\psi'(w_2)\rangle = \delta(w_1 - w_2)$. Equations (13) and (14) are obtained by Eqs. (18) and (19) when the evolution is viewed as restricted to the signal mode only, namely

$$\mathcal{A}_g^{\vee(\hat{W})}(\hat{\varrho}) = \langle\psi|\hat{U}_g\hat{\varrho}\otimes 1\hat{U}_g^\dagger|\psi\rangle\,. \quad (20)$$

We name the device corresponding to Eq. (18) an *ideal coherence-preserving quantum amplifier* of $\hat{W}$, because it achieves the ideal amplification of $\hat{W}$ without measuring $\hat{W}$ (ψ' does not depend on w; for $\psi' = \psi$ the device is "passive"). On the other hand, the transformation (19) achieves the ideal amplification of $\hat{W}$ by measuring $\hat{W}$, then performing the processing $w \to gw$ and finally preparing the state $|gw\rangle$. The measurement stage is the one which is responsible for the vanishing of all off-diagonal elements in Eq. (14). (Equation (20) together with Eqs. (18) and (19) imply Eqs. (13) and (14) also for a nonorthogonal set $\{|w\rangle\}$, however, generally not when $\langle\psi'_A(w_2)|\psi'_A(w_1)\rangle \neq 0$ for $w_1 \neq w_2$). Since we want to exploit the ideal amplification of $\hat{W}$ in order to achieve its ideal quantum measurement, we will consider only the coherence-preserving quantum amplification in Eq. (13) or (18), since the other kind of amplifier needs by itself the ideal measurement of $\hat{W}$.

4. Approaching Ideal Quantum Measurements by Preamplified Heterodyning

Let $d\hat{H}_f(u)$ be the POVM pertaining to the heterodyne measurement of the function $f(\alpha, \bar{\alpha})$ of the field and let consider a preamplified heterodyne detection scheme corresponding to the following procedure:

(1) the signal mode of the field is amplified by an ideal amplifier for $\hat{W}$ with gain g;
(2) the field is heterodyne detected and the function f is evaluated;
(3) the final result is rescaled by a factor g.

The above procedure corresponds to the following transformation

$$d\hat{H}_f(u) \longrightarrow \mathcal{A}_g^{(\hat{W})}[d\hat{H}_f(gu)]\,. \tag{21}$$

We say that the preamplified heterodyne detection of the function f of the field approaches the ideal quantum measurement of the observable $\hat{W}$ in the limit of infinite gain g if

$$\lim_{g\to\infty} \mathcal{A}_g^{(\hat{W})}[d\hat{H}_f(gu)] = du\delta(u - \hat{W})\,, \tag{22}$$

where the limit is to be regarded in the weak sense (i.e. for matrix elements) and the operator Dirac delta explicitly writes as follows

$$\delta(u - \hat{W}) = \int_{\mathbb{S}_{\hat{W}}} dw|w\rangle\langle w|\delta(u - w)\,, \tag{23}$$

and the integral is to be understood as a sum for discrete spectrum $\mathbb{S}_{\hat{W}}$. A necessary and sufficient condition for validity of Eq. (22) is the following

$$\lim_{g\to\infty} \int \mathcal{A}_g^{(\hat{W})}[d\hat{H}_f(gu)]u^l = \hat{W}^l\,, \qquad l = 0, 1, 2, \ldots\,, \tag{24}$$

where again the limit holds for expectations on any state. One can prove that condition (24) is necessary — i.e. Eq. (22) implies Eq. (24) — by simply substituting Eq. (22) into Eq. (24) and exchanging the integral with the limit. On the other hand Eq. (24) implies

$$\lim_{g\to\infty} \int \mathcal{A}_g^{(\hat{W})}[d\hat{H}_f(gu)] \exp(iku) = \exp(ik\hat{W})\,, \tag{25}$$

and taking the Fourier transform of both sides of the last identity one finds Eq. (22), proving that Eq. (24) is also a sufficient condition. Another sufficient condition in a form more convenient than Eq. (24) is the following

$$\int d\hat{H}_f(u)u^l = \hat{W}^l + o(\hat{W}^l)\,, \tag{26}$$

where $o(g(x))$ is an asymptotic notation equivalent to the vanishing of the limit $\lim_{x\to\infty} o(g(x))/g(x) = 0$,[7] whereas, for an operator $\hat{V}$, by $o(\hat{V})$ we mean

$\lim_{\kappa\to\infty}\kappa^{-1}o(\kappa\hat{V}) = 0$ in the weak sense. In fact, by amplifying both sides of Eq. (26) and rescaling the variable u by the gain g one obtains

$$\int \mathcal{A}_g^{(\hat{W})}[d\hat{H}_f(gu)]u^l = \hat{W}^l + g^{-l}o(g^l\hat{W}^l)\,, \tag{27}$$

which implies Eq. (24).

5. Two Examples

In this section we show that condition (24) holds for both the photon number $\hat{W} = a^\dagger a$ and the quadrature $W = \mathrm{Re}(ae^{-i\phi})$, corresponding to the functions of the field $f(\alpha,\bar{\alpha}) = |\alpha|^2$ and $f(\alpha,\bar{\alpha}) = \mathrm{Re}(\alpha e^{-i\phi})$ respectively. This means that both the quadrature and the photon number operators can be ideally measured through the preamplified heterodyne detection scheme in the limit of infinite gain. We also show that in both cases the detection scheme is robust to non-unit quantum efficiency of the heterodyne detector.

5.1. *Measurement of the quadrature*

The POVM $d\hat{H}(x)$ that corresponds to the function $f(\alpha,\bar{\alpha}) = \mathrm{Re}(\alpha e^{-i\phi})$ of the field is given by

$$d\hat{H}_f(x) = dx{:}\delta\left(x - \frac{1}{2}(a^\dagger e^{i\phi} + ae^{-i\phi})\right){:}_A = dx\int\frac{du}{2\pi}e^{iu(x-\hat{X}_\phi)}e^{-(1/8)u^2}$$
$$= dx\sqrt{\frac{2}{\pi}}e^{-2(\hat{X}_\phi - x)^2}\,. \tag{28}$$

Non-unit quantum efficiency introduces additive Gaussian noise and replaces the POVM (28) with the following one

$$d\hat{H}_f(x) = dx\sqrt{\frac{2\eta}{\pi(2-\eta)}}e^{-(2\eta/2-\eta)(\hat{X}_\phi - x)^2}\,. \tag{29}$$

We can see that the POVM in Eq. (29) satisfies the sufficient condition (26) for approaching the ideal quantum measurement of $\hat{X}_\phi$. In fact, the moments of the POVM (29) are given by

$$\int d\hat{H}_f(x)x^l = \int_{-\infty}^{+\infty} dx\sqrt{\frac{2\eta}{\pi(2-\eta)}}e^{-(2\eta/2-\eta)x^2}(\hat{X}_\phi + x)^l = \hat{X}_\phi^l + O(\hat{X}_\phi^{l-2})\,, \tag{30}$$

where $O(g(x))$ is the customary asymptotic notation equivalent to the condition $\lim_{x\to\infty} O(g(x))/g(x) < \infty$,[7] implying that $O(\hat{X}^{l-2}) \equiv o(\hat{X}^l)$. On the other hand, one can directly verify the limit in Eq. (22) as follows

$$\mathcal{A}_g^{\hat{X}_\phi}[d\hat{H}_f(gx)] = dx\sqrt{\frac{2g^2\eta}{\pi(2-\eta)}}e^{-(2g^2\eta/2-\eta)(\hat{X}_\phi - x)^2} \xrightarrow{g\to\infty} dx\delta(\hat{X}_\phi - x)\,. \tag{31}$$

The ideal amplification of the quadrature operator $\hat{X}_\phi$ is achieved by means of a phase-sensitive amplifier[8,10] which rescales the couple of conjugated quadratures as follows

$$\hat{X}_\phi \to \frac{1}{g}\hat{X}_\phi\,, \qquad \hat{X}_{\phi+(\pi/2)} \to g\hat{X}_{\phi+(\pi/2)}\,, \tag{32}$$

g being the gain at the amplifier. The Heisenberg transformations in Eq. (32) are achieved by the unitary operator

$$\hat{U}_g = \exp[-i\log g(\hat{X}_\phi \hat{X}_{\phi+(\pi/2)} - \hat{X}_{\phi+(\pi/2)}\hat{X}_\phi)]\,. \tag{33}$$

5.2. *Measurement of the photon number*

The case of the ideal measurement of the photon number $a^\dagger a$ through preamplified heterodyning is more interesting than the case of the quadrature $\hat{X}_\phi$, because here the amplification not only removes the excess noise due to the quantum measurement, but also changes the spectrum, from continuous to discrete. We consider the POVM that corresponds to heterodyning the function $f(\alpha,\bar{\alpha}) = |\alpha|^2$ of the field. This can be written as follows

$$\begin{aligned} d\hat{H}_f(h) &= dh{:}\delta(h - a^\dagger a)){:}_A = dh\int\frac{du}{2\pi}e^{-iuh}\sum_{n=0}^{\infty}(iu)^n a^n a^{\dagger n} \\ &= dh\int\frac{du}{2\pi}e^{-iuh}\sum_{n=0}^{\infty}(iu)^n\binom{a^\dagger a + n}{n} = dh\int\frac{du}{2\pi}e^{-iuh}(1-iu)^{-a^\dagger a - 1} \\ &= dhe^{-h}\frac{h^{a^\dagger a}}{(a^\dagger a)!}\,. \end{aligned} \tag{34}$$

The POVM in Eq. (34) satisfies the sufficient condition (26). In fact, one has

$$\begin{aligned} \int dhe^{-h}\frac{h^{a^\dagger a + l}}{(a^\dagger a)!} &= \frac{(a^\dagger a + l)!}{(a^\dagger a)!} \\ &= (-)^l\sum_{k=0}^{l} s_{l+1}^{(k+1)}(-a^\dagger a)^k = (a^\dagger a)^l + O[(a^\dagger a)^{l-1}]\,, \end{aligned} \tag{35}$$

where $s_l^{(k)}$ denotes a Stirling number of the first kind. Hence, if the field is amplified through an ideal photon number amplifier[9–11] and then heterodyne detected, in the limit of infinite gain the scheme achieves ideal photon number detection. Indeed, using the ideal photon number amplification map[12,13]

$$a^\dagger a \longrightarrow \hat{V}^\dagger a^\dagger a\hat{V} = g a^\dagger a\,, \tag{36}$$

with the isometry $\hat{V}$ given by

$$\hat{V} = \sum_{n=0}^{\infty}|gn\rangle\langle n|\,, \tag{37}$$

one obtains the preamplified POVM

$$\mathcal{A}_g^{a^\dagger a}[d\hat{H}_f(gh)] = \hat{V}^\dagger d\hat{H}_f(gh)\hat{V} = dhge^{-gh}\sum_{n=0}^{\infty}\frac{(gh)^{gn}}{(gn)!}|n\rangle\langle n|\,. \tag{38}$$

In the limit of infinite gain $g \to \infty$ the POVM in Eq. (38) achieves the ideal POVM for the photon-number operator measurement. This can be shown as follows. Upon writing the POVM (38) in the form

$$\mathcal{A}_g^{a^\dagger a}[d\hat{H}_f(gh)] = dh\sum_{n=0}^{\infty}p_n^{(g)}(h)|n\rangle\langle n|\,, \tag{39}$$

we need to show that the function

$$p_n^{(g)}(h) = ge^{-gh}\frac{(gh)^{gn}}{(gn)!}\,, \tag{40}$$

approaches a Dirac delta-comb over integer values $h \in \mathbb{N}$. Using the Stirling's inequality

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n-1}\right)\,, \tag{41}$$

one obtains

$$\gamma_n^{(g)}(h)\left(1+\frac{1}{12gn-1}\right)^{-1} < p_n^{(g)}(h) < \gamma_n^{(g)}(h)\,, \tag{42}$$

where

$$\gamma_n^{(g)}(h) = \frac{1}{\sqrt{2\pi g^{-1}n}}\exp\left[gn\left(1-\frac{h}{n}+\log\frac{h}{n}\right)\right]\,. \tag{43}$$

From the inequality $\log x \le x - 1$ (with equality iff $x = 1$) it follows that

$$\lim_{g\to\infty}\gamma_n^{(g)}(h) = \begin{cases} 0 & h \neq n \\ +\infty & h = n \end{cases}\,, \tag{44}$$

and hence, from Eq. (42), one has

$$\lim_{g\to\infty}p_n^{(g)}(h) = \begin{cases} 0 & h \neq n \\ +\infty & h = n \end{cases}\,. \tag{45}$$

Moreover, from the expansion for h near to n

$$1-\frac{h}{n}+\log\frac{h}{n} = -\frac{1}{2}\left(1-\frac{h}{n}\right)^2 + O\left(\left(1-\frac{h}{n}\right)^3\right)\,, \tag{46}$$

one has the Gaussian asymptotic approximation for $g \to \infty$

$$p_n^{(g)}(h) \simeq \frac{1}{\sqrt{2\pi g^{-1}n}}\exp\left[-\frac{(h-n)^2}{2g^{-1}n}\right] \stackrel{g\to\infty}{\longrightarrow} \delta(h-n)\,, \tag{47}$$

which proves the statement.

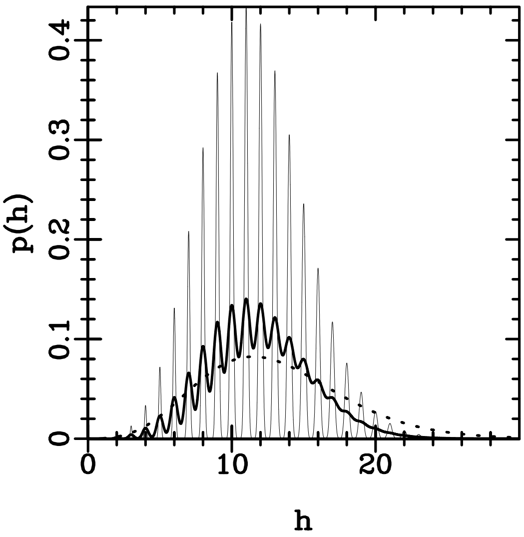

Fig. 1. Probability density $p(h)$ for a coherent state with mean photon number $\langle a^\dagger a\rangle = 12$ obtained through heterodyne detection of $f(\alpha, \bar{\alpha}) = |\alpha|^2$, preamplified by an ideal photon number amplifier. Different line-style denote different value of the gain g at the amplifier: the dashed line corresponds to $g = 1$ (no amplification); the thick line corresponds to $g = 10^2$; the thin line to $g = 10^3$.

In Fig. 1 we show the probability distribution of the outcome $h = |\alpha|^2$ form preamplified heterodyne detection of a coherent state, for different values of the amplifier gain g. Notice the emergence of a discrete spectrum from a continuous one for increasingly large gains, in agreement with Eq. (47).

It is easy to show that the preamplified heterodyne detection scheme is robust to non-unit quantum efficiency also in the present case of measuring $a^\dagger a$. In fact, the sufficient condition (26) is still satisfied for nonunit quantum efficiency, as one can check through Eqs. (8) and (35) as follows

$$\begin{aligned} &\int \frac{d^2\beta}{\pi}\frac{\eta}{1-\eta}e^{-\eta/1-\eta|\beta|^2}\hat{D}(\beta)\{(a^\dagger a)^l + O[(a^\dagger a)^{l-1}]\}\hat{D}^\dagger(\beta) \\ &\quad = \int \frac{d^2\beta}{\pi}\frac{\eta}{1-\eta}e^{-\eta/1-\eta|\beta|^2}\{[(a^\dagger - \bar{\beta})(a-\beta)]^l + O[((a^\dagger - \bar{\beta})(a-\beta))^{l-1}]\} \\ &\quad = (a^\dagger a)^l + O[(a^\dagger a)^{l-1}]\,. \end{aligned} \tag{48}$$

6. A Counterexample

The necessary and sufficient condition (24) establishes when a self-adjoint operator $\hat{W}$ is approximated by the classical observable f using a preamplified heterodyne scheme. One could now address the inverse problem, namely: Given a self-adjoint operator $\hat{W}$ is it possible to find a function of the field such that the preamplified heterodyne measurement approximates the measurement of $\hat{W}$? As we have shown in the previous section, this is certainly true for $\hat{X}_\phi$ and $a^\dagger a$. For a generic observable $\hat{W}$, the problem becomes very difficult. However, on the basis of a counterexample, we will prove that the inverse problem has no solution for some operator $\hat{W}$, namely there are observables which cannot be measured through the preamplified heterodyne detection scheme.

Consider the operator

$$\hat{K} \equiv -\frac{i}{2}(a^2 - a^{\dagger 2}) = \hat{X}\hat{Y} + \hat{Y}\hat{X}\,, \tag{49}$$

where $\hat{X}$ and $\hat{Y}$ are the conjugated quadratures $\hat{X} \equiv \hat{X}_0$ and $\hat{Y} = \hat{X}_{\pi/2}$. We show that there is no polynomial function of the field that satisfies either the necessary condition (24).

In order to construct the CP amplification map for $\hat{K}$, one has to find the eigenstates of $\hat{K}$. These are given in Ref. 14 and here we report them. One has

$$\hat{K}|\psi^\mu_\pm\rangle = \mu|\psi^\mu_\pm\rangle\,, \tag{50}$$

with

$$\psi^\mu_\pm(x) \doteq \langle x|\psi^\mu_\pm\rangle = \frac{1}{\sqrt{2\pi}}|x|^{i\mu-1/2}\theta(\pm x)\,, \tag{51}$$

where $|x\rangle$ denotes the eigenvector of the quadrature $\hat{X}$, and $\theta(x)$ is the customary step-function ($\theta(x) = 1$ for $x > 0$, $\theta(x) = 1/2$ for $x = 0$, $\theta(x) = 0$ for for $x < 0$). The vectors $|\psi^\mu_s\rangle$ form a complete orthonormal set

$$\langle\psi^\mu_r|\psi^\nu_s\rangle = \delta_{rs}\delta(\mu-\nu)\,. \tag{52}$$

The amplification of $\hat{K}$ is achieved by the unitary operator $\hat{U}_g$ satisfying the relations

$$\hat{U}_g^\dagger\hat{K}\hat{U}_g = g\hat{K}\,, \qquad \hat{U}_g|\psi^\mu_s\rangle = g^{1/2}|\psi^{g\mu}_s\rangle\,. \tag{53}$$

In terms of the eigenvectors of $\hat{K}$ the unitary operator $\hat{U}_g$ has the form

$$\hat{U}_g = \sum_{s=\pm}\int_{-\infty}^{+\infty} d\mu g^{1/2}|\psi^{g\mu}_s\rangle\langle\psi^\mu_s| = g^{1/2}\int_{-\infty}^{+\infty} dx|x|^{1/2(g-1)}|x\rangle\langle x^{*g}|\,, \tag{54}$$

where in the last identity in Eq. (54) we have written $\hat{U}_g$ in terms of the eigenstates $|x\rangle$ of the quadrature $\hat{X}$, upon introducing the notation

$$x^{*g} \equiv x|x|^{g-1} = \mathrm{sgn}(x)|x|^g\,, \tag{55}$$

where $\mathrm{sgn}(x)$ denotes the customary sign function. The analytic form (54) of $\hat{U}_g$ is derived as follows

$$\begin{aligned}\hat{U}_g &= g^{1/2}\sum_{s=\pm}\int_{-\infty}^{+\infty} d\mu|\psi^{g\mu}_s\rangle\langle\psi^\mu_s| \\ &= g^{1/2}\int_{-\infty}^{+\infty} dx\int_{-\infty}^{+\infty} dx'|x'\rangle\langle x|\sum_{s=\pm}\int_{-\infty}^{+\infty} d\mu\psi^{g\mu}_s(x')\bar{\psi}^\mu_s(x)\end{aligned}$$

$$= g^{1/2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx'|x'\rangle\langle x|\,|x'|^{1/2(g-1)} \sum_{s=\pm}\int_{-\infty}^{+\infty} d\mu\psi_s^{\mu}(x'^{*g})\bar{\psi}_s^{\mu}(x)$$

$$= g^{1/2}\int_{-\infty}^{+\infty} dx|x|^{1/2(g-1)}|x\rangle\langle x^{*g}|\,. \tag{56}$$

The Heisenberg evolution of the conjugated quadratures $\hat{X}$ and $\hat{Y}$ by the amplification $\hat{U}_g$ can be evaluated through the following steps

$$\hat{U}_g^{\dagger}\hat{X}\hat{U}_g = g\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' x|xx'|^{1/2(g-1)}|x^{*g}\rangle\langle x|x'\rangle\langle x'^{*g}|$$

$$= g\int_{-\infty}^{+\infty} dx x^{*g}|x^{*g}\rangle\langle x^{*g}| = \hat{X}^{*1/g}\,; \tag{57}$$

$$\hat{U}_g^{\dagger}\hat{Y}\hat{U}_g = \hat{U}_g^{\dagger}\int_{-\infty}^{+\infty} dx|x\rangle\left(-\frac{i}{2}\partial_x\right)\langle x|\hat{U}_g$$

$$= g\int_{-\infty}^{+\infty} dx|x|^{1/2(g-1)}|x^{*g}\rangle\left(-\frac{i}{2}\partial_x\right)\langle x^{*g}||x|^{1/2(g-1)}$$

$$= -\frac{i}{4}(g-1)\hat{X}^{*(-1/g)} + \int_{-\infty}^{+\infty} du|u\rangle\left(-\frac{i}{2}|u|^{1-(1/g)}\partial_u\right)\langle u|$$

$$= -\frac{i}{4}(g-1)\hat{X}^{*(-1/g)} + g\hat{X}^{*(-1/g)}\hat{X}\hat{Y}$$

$$= \hat{X}^{*(-1/g)}\left(\frac{1}{2}g\hat{K} + \frac{i}{4}\right) = \left(\frac{1}{2}g\hat{K} - \frac{i}{4}\right)\hat{X}^{*(-1/g)}\,. \tag{58}$$

For what follows we also need to evaluate the Heisenberg evolution of the operator $\hat{X}^2 + \hat{Y}^2 = a^{\dagger}a + 1/2$. From Eqs. (57)–(58) one has

$$\hat{U}_g^{\dagger}\left(a^{\dagger}a + \frac{1}{2}\right)\hat{U}_g = |\hat{X}|^{2/g} + \frac{1}{4}\left(g\hat{K} - \frac{i}{2}\right)|\hat{X}|^{(-2/g)}\left(g\hat{K} + \frac{i}{2}\right)$$

$$= |\hat{X}|^{2/g} + \frac{1}{4}\hat{X}^{*(-1/g)}\left(g^2\hat{K}^2 + \frac{1}{4}\right)\hat{X}^{*(-1/g)}\,. \tag{59}$$

Now, let us consider a quadratic function of the field $f(\alpha,\bar{\alpha}) = -i(\alpha^2 - \bar{\alpha}^2 + ic|\alpha|^2)/2$, c an arbitrary constant, and let us evaluate the corresponding POVM $d\hat{H}_f(u)$ pertaining to heterodyne detection of the function f of the field. From Eq. (6) one has

$$
\begin{aligned}
d\hat{H}_f(u) &= du{:}\delta(u - f(a, a^\dagger)){:}_A \\
&= du \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{-i\lambda u} e^{\lambda(a^2/2)} {:}e^{i\lambda(c/2)a^\dagger a}{:}_A e^{-\lambda(a^{\dagger 2}/2)} \\
&= du \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{-i\lambda u} e^{\lambda(a^2/2)} \left(1 - i\lambda\frac{c}{2}\right)^{-(a^\dagger a + 1/2)} e^{-\lambda(a^{\dagger 2}/2)} \left(1 - i\lambda\frac{c}{2}\right)^{-1/2},
\end{aligned}
\tag{60}
$$

where we used the relation

$$
{:}e^{z a^\dagger a}{:}_A = \sum_{n=0}^{\infty} \frac{z^n}{n!} a^n a^{\dagger n} = \sum_{n=0}^{\infty} z^n \binom{a^\dagger a + n}{n} = (1 - z)^{-a^\dagger a - 1}. \tag{61}
$$

The product of operators in the last equality of Eq. (60) can be recast in the form of a single exponential function using the Baker–Campbell–Hausdorff (BCH) formula for the $su(1, 1)$ algebra [see Appendix]. According to the prescription in Eq. (21), we need to evaluate the preamplified POVM

$$
\mathcal{A}_g^{(\hat{K})}[d\hat{H}_f(gu)] \equiv g\hat{U}_g^\dagger(d\hat{H}_f(gu))\hat{U}_g, \tag{62}
$$

in the limit of infinite gain $g \to \infty$. As shown in the Appendix, for the leading term in g one has

$$
\begin{aligned}
\mathcal{A}_g^{(\hat{K})}[d\hat{H}_f(gu)] = du \int \frac{d\lambda}{2\pi} &\exp\left(-i\lambda u + i\lambda\hat{K} + \frac{1}{8} i\lambda c g \hat{K}^2\right) \\
&\times \exp\left[-\frac{1}{8}\lambda^2\left(1 + \frac{c^2}{4}\right)\hat{K}^2\right], \qquad g \gg 1.
\end{aligned}
\tag{63}
$$

The preamplified POVM in the limit of infinite gain writes as follows

$$
\begin{aligned}
\mathcal{A}_g^{(\hat{K})}[d\hat{H}_f(gu)] &\xrightarrow{g\to\infty} du \int \frac{d\lambda}{2\pi} \exp\left(-i\lambda u + i\lambda\hat{K} + \frac{i}{8}\lambda g c \hat{K}^2\right) \\
&\quad \times \exp\left(-\frac{1}{8}\lambda^2\hat{K}^2\right) \\
&= du\sqrt{\frac{2}{\pi\hat{K}^2}} \exp\left(-\frac{2(\hat{K} + \frac{1}{8} g c \hat{K}^2 - u)^2}{\hat{K}^2}\right).
\end{aligned}
\tag{64}
$$

The POVM in Eq. (64) satisfies the necessary condition (24) for $l = 0, 1$ upon choosing $c = 0$. However, the same condition for $l = 2$ is not satisfied, because one has

$$
\int du u^2 \sqrt{\frac{2}{\pi\hat{K}^2}} \exp\left(-\frac{2(\hat{K} - u)^2}{\hat{K}^2}\right) = \frac{5}{4}\hat{K}^2. \tag{65}
$$

Therefore, there is no quadratic function $f(\alpha, \bar{\alpha})$ of the field that allows to approximate the ideal quantum measurement of the operator $\hat{K} = -i(a^2 - a^{\dagger 2})/2$. It is clear that also higher-degree polynomial functions of the field cannot satisfy condition (24), since in such case higher powers in $a^\dagger$ and a will appear in Eq. (60) and the BCH formula will have no longer closed form. In conclusion of this section we notice that Eq. (64) for $c = 0$ can also be easily obtained by the following formal asymptotic analysis

$$\begin{aligned} d\hat{H}_f(gu) &= du \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{-i\lambda u} e^{g^{-1}\lambda(a^2/2)} e^{g^{-1}-\lambda(a^{\dagger 2}/2)} \\ &= du \int_{-\infty}^{+\infty} e^{-i\lambda u} \exp\left\{ ig^{-1}\lambda \frac{1}{2}(a^2 - a^{\dagger 2}) - \frac{1}{8} g^{-2}\lambda^2 [a^2, a^{\dagger 2}] + O(g^{-3}) \right\} \\ &= du \int_{-\infty}^{+\infty} e^{-i\lambda u} \exp\left\{ ig^{-1}\lambda \hat{K} - \frac{1}{2} g^{-2}\lambda^2 \left(a^\dagger a + \frac{1}{2} \right) + O(g^{-3}) \right\} . \quad (66) \end{aligned}$$

By amplifying the first and last members of Eq. (66) and using Eq. (59) one has

$$\begin{aligned} &\mathcal{A}_g^{(\hat{K})}[d\hat{H}_f(gu)] \\ &\quad = du \int_{-\infty}^{+\infty} e^{-i\lambda u} \exp\left\{ i\lambda\hat{K} - \frac{1}{8}\lambda^2 \hat{X}^{*(-1/g)} \left(\hat{K}^2 + \frac{1}{4g^2} \right) \hat{X}^{*(-1/g)} + O(g^{-3}) \right\} \\ &\quad = du \int_{-\infty}^{+\infty} e^{-i\lambda u} \exp\left\{ i\lambda\hat{K} - \frac{1}{8}\lambda^2 \hat{K}^2 + O(g^{-1}) \right\} , \qquad (67) \end{aligned}$$

namely Eq. (64).

7. Conclusions

One may think that the heterodyne detector could be regarded as a universal detector, as it achieves the ideal measurement of the field operator a and hence, in principle, it should achieve the measurement of any operator $\hat{O} = \hat{O}(a, a^\dagger)$ of the field. However, due to the fact that the measurement of a corresponds to a joint measurement of two noncommuting conjugate observables, an intrinsic unavoidable 3 dB noise is added to the measurement, even in the ideal case. We have considered the possibility of reducing such noise by means of a suitable ideal preamplification of $\hat{O}$, which we have shown to be feasible through a unitary transformation. We have shown that in the limit of infinite gain such preamplified heterodyne detection scheme can achieve the ideal measurement of $a^\dagger a$ and $\hat{X}_\phi$, even for non-unit quantum efficiency, also realizing the transition from continuous to discrete spectrum in the case of the operator $a^\dagger a$. However, the scheme does not work for arbitrary operator and, as a counterexample, we proved that the ideal measurement cannot be achieved even for the simple quadratic form $\hat{K} = i(a^{\dagger 2} - a^2)/2$, apparently with no simple physical explanation other than the algebraic nature of the operator $\hat{K}$ itself and its ideal amplification map. In the present study we have seen some of the problems that would appear in building a universal detection machine, and we hope that this work will shed new light on the route for achieving such a challenging task.

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Appendix. The BCH Formula

Upon defining $k_+ = (1/2)a^{\dagger 2}$, $k_- = (1/2)a^2$ and $k_3 = 1/2(a^\dagger a + 1/2)$, one recognizes the following commutation rules for the $su(1,1)$ algebra

$$[\hat{k}_+, \hat{k}_-] = -2\hat{k}_3\,, \qquad [\hat{k}_3, \hat{k}_\pm] = \pm\hat{k}_\pm\,. \tag{68}$$

One needs the analytic form of the coefficients $B_\pm, B_3$ and $A_\pm$, A_3 in the following identity

$$\exp(A_-\hat{k}_-)\exp(2A_3\hat{k}_3)\exp(A_+\hat{k}_+) = \exp(2B_3\hat{k}_3 + B_+\hat{k}_+ + B_-\hat{k}_-)\,. \tag{69}$$

By using the faithful representation of the $su(1,1)$ algebra in terms of the Pauli matrices with $i\hat{\sigma}^\pm \equiv \hat{k}_\pm$, $\hat{\sigma}^3 \equiv 2\hat{k}_3$, Eq. (69) can be rewritten as follows

$$\begin{pmatrix} 1 & 0 \\ iA_- & 1 \end{pmatrix}\begin{pmatrix} e^{A_3} & 0 \\ 0 & e^{-A_3} \end{pmatrix}\begin{pmatrix} 1 & iA_+ \\ 0 & 1 \end{pmatrix} = \cosh\Gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh\Gamma}{\Gamma}\begin{pmatrix} B_3 & iB_+ \\ iB_- & -B_3 \end{pmatrix}, \tag{70}$$

where $\Gamma = (B_3^2 - B_+B_-)^{1/2}$. From Eq. (70) one obtains the relation

$$B_3 = \frac{1}{2}\frac{\Gamma}{\sinh\Gamma}\left[(1 + A_+A_-)\,e^{A_3} - e^{-A_3}\right]\,, \tag{71}$$

$$\sinh\Gamma = \left\{\left[\frac{(1 + A_+A_-)\,e^{A_3} + e^{-A_3}}{2}\right]^2 - 1\right\}^{1/2}\,, \tag{72}$$

$$B_\pm = \frac{2A_\pm e^{\pm A_3}}{(1 - A_+A_-)e^{A_3} - e^{-A_3}}B_3\,. \tag{73}$$

For the purpose of the paper, we are just interested in the asymptotic expression of the POVM $\mathcal{A}_g^{(\hat{K})}[d\hat{H}_f(gu)]$ in Eq. (62) for $g \to \infty$. By comparing Eqs. (69) and (60) one has $A_\pm = \mp g^{-1}\lambda$ and $A_3 = -\ln[1 - ig^{-1}\lambda(c/2)]$. From Eqs. (71)–(73) one obtains the asymptotic values of $B_\pm$ and B_3 for $g \to \infty$, namely

$$B_\pm \simeq \mp g^{-1}\lambda\,, \qquad B_3 \simeq \frac{1}{2}ig^{-1}\lambda c - \frac{1}{2}g^{-2}\lambda^2\left(1 + \frac{c^2}{4}\right)\,. \tag{74}$$

Hence, from Eq. (60) it follows

$$g d\hat{H}_f(gu) \overset{g\gg 1}{\longrightarrow} du \int \frac{d\lambda}{2\pi} \left(1 + i\lambda g^{-1} \frac{c}{4}\right) e^{-i\lambda u}$$
$$\times \exp\left\{ ig^{-1}\lambda\hat{K} + \frac{1}{2}\left[i\lambda g^{-1}c - g^{-2}\lambda^2 \left(1 + \frac{c^2}{4}\right)\right] \left(a^\dagger a + \frac{1}{2}\right)\right\} . \quad (75)$$

By applying the amplification map to the POVM $d\hat{H}_f(gu)$ through Eqs. (53) and (59), one obtains $\mathcal{A}_g^{(\hat{K})}[d\hat{H}_f(gu)]$ in Eq. (63).

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