LATEST DEVELOPMENTS IN QUANTUM TOMOGRAPHY

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Latest developments in quantum tomography are presented. The method for measuring the state of radiation is derived in a simple group-theoretical framework that allows generalization to arbitrary quantum systems. Some recently developed topics are synthetically reviewed, including tomography of many radiation modes using only one local oscillator, generalization to N-level systems, and new "adaptive" techniques for noise reduction. A set of newly proposed experiments is presented, based on a *conditioned* tomographic technique.

INTRODUCTION

Optical homodyne tomography is now a well assessed method to measure the quantum state of radiation (for a review see Ref. 1). The density operator ρ can be measured in a given representation by averaging a set of special functions over homodyne data. The method has been extended to estimate the expectation value $\langle O \rangle$ of an arbitrary operator O, making homodyne tomography the first universal detector for radiation.[2]

In this paper I will present some recent progress in quantum tomography, with the method extended to any quantum system, as any number of radiation modes or N-level systems. The method is based on the possibility of measuring a set of observables—socalled *quorum*—which are irreducible for the unitary representation of the dynamical group of the system. For a set of radiation modes the quorum is given by all linear combinations of the creation and annihilation operators of the modes. For a set of N-level systems it is the set of all linear combinations of the angular momentum operators of each system. The symmetries of the quorum can be exploited to reduce the statistical errors of the method, and this is the basis of recently discovered "adaptive" techniques.

In the first section, devoted to homodyne tomography, I will present a synthetic derivation of the method in a simple group-theoretical framework that allows generalization to arbitrary quantum system. In the same section I will also illustrate: *i*) the emergence of bounds for quantum efficiency (on the basis of a simple example); *ii*) the

Quantum Communication, Computing, and Measurement 2
Edited by Kumar et al., Kluwer Academic / Plenum Publishers, New York, 2000.

equivalence classes of estimators, and the generating function of all s-ordered monomials in the field operator; *iii*) the new adaptive method for reducing statistical errors; *iv*) the generalization to many modes of radiation using only one local oscillator. In the subsequent section I will outline the generalization to other dynamical systems, treating spin tomography as an example. Finally, in the last section I will shortly outline a set of new experiments which are now feasible with the homodyne tomographic method, including the possibility of detecting small Schrődinger cats of radiation, checking Bell inequalities, observing the Greenberger-Horne-Zeilinger state, and finally checking the quantum state-reduction rule.

HOMODYNE TOMOGRAPHY

The method allows to estimate the ensemble average $\langle O\rangle$ of a given (generally complex and unbounded) operator O of the radiation field in a unknown state ρ. In this section I focus attention on a single mode described by boson operators a and $a^\dagger$, with $[a,a^\dagger]=1$; the generalization to many modes will be given in the following section.

In homodyne tomography the problem is to estimate the ensemble average $\langle O\rangle$ from homodyne measurement of quadrature operators $X_\phi=\frac{1}{2}(a^\dagger e^{i\phi}+ae^{-i\phi})$ at different phases ϕ with respect to the local oscillator (LO). I assume that the homodyne detector is properly used in the strong LO limit: this is the only assumption of the method, which works also for nonunit quantum efficiency η of the homodyne detector (the overall quantum efficiency η including also the effect of any source of Gaussian noise[3]). The problem is to estimate the expectation $\langle O\rangle$ of a given operator O for an unknown state ρ by averaging a suitable function over homodyne data. The method provides a rule that to every operator O assigns an unbiased estimator $\mathcal{E}_\eta[O](x;\phi)$, such that for arbitrary state one has

$$\langle O\rangle=\int_0^\pi\frac{\mathrm{d}\phi}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,p_\eta(x;\phi)\mathcal{E}_\eta[O](x;\phi)\,, \tag{1}$$

$p_\eta(x;\phi)$ denoting the probability distribution of the outcomes x for the quadrature X_ϕ detected with quantum efficiency η. Notice that, due to the symmetry $X_{\phi+\pi}=-X_\phi$, the phase averaging can be restricted to the window $\phi\in[0,\pi]$. According to the central-limit theorem, the mathematical expectation (1) can be estimated by averaging the estimator $\mathcal{E}_\eta[O](x;\phi)$ over experimental data only if $\mathcal{E}_\eta[O](x;\phi)$ has moments bounded up to the third order. As in the strong LO approximation the probability $p_\eta(x;\phi)$ must decay as a Gaussian for large x, it follows that the integral in Eq. (1) can be experimentally sampled for any *a priori* unknown probability distribution $p_\eta(x;\phi)$ only if $\mathcal{E}_\eta[O](x;\phi)$ increases slower than $\exp(kx^2)$ for large x and is bounded for $|x|<+\infty$. In this case one is guaranteed that the integral in Eq. (1) can be statistically sampled over a sufficiently large set of data, and the mean values for different experiments will be Gaussian distributed around the mathematical expectation (1), allowing estimation of confidence intervals. On the contrary, if the kernel $\mathcal{E}_\eta[O](x;\phi)$ is unbounded at some $(x;\phi)$, then the ensemble average $\langle O\rangle$ cannot be measured using homodyne tomography.

The analytic form for the estimator $\mathcal{E}_\eta[O](x;\phi)$ is given by[1]

$$\mathcal{E}_\eta[O](x;\phi)=\frac{1}{4}\int_0^\infty\mathrm{d}t\,e^{\frac{1-\eta}{8\eta}t}\,\mathrm{Tr}\left[O\cos\sqrt{t}(X_\phi-x)\right]\,. \tag{2}$$

Eq. (2) can be easily derived from the identity

$$O=\int\frac{\mathrm{d}^2\alpha}{\pi}\mathrm{Tr}[OD(\alpha)]D^\dagger(\alpha)\,, \tag{3}$$

where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ denotes the displacement operator. Changing to polar variables $\alpha = \frac{i}{2}ke^{i\phi}$, and using the relation between moment generating functions

$$\langle \exp(-ikX_\phi) \rangle = e^{\frac{1-\eta}{8\eta}k^2} \int_{-\infty}^{+\infty} \mathrm{d}x\, p_\eta(x;\phi) e^{-ikx} \ , \tag{4}$$

one obtains

$$\langle O \rangle = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\pi} \int_0^{+\infty} \frac{\mathrm{d}k\, k}{4}\, e^{\frac{1-\eta}{8\eta}k^2}\, \mathrm{Tr}[O e^{ikX_\phi}] \int_{-\infty}^{+\infty} \mathrm{d}x\, p_\eta(x;\phi) e^{-ikx} \ . \tag{5}$$

Eq. (5) can be rewritten as the expectation of an estimator if the integrals over k and x can be exchanged, namely

$$\langle O \rangle = \int_0^{2\pi} \frac{\mathrm{d}\phi}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}x\, p_\eta(x;\phi) \int_0^{+\infty} \frac{\mathrm{d}k\, k}{4}\, e^{\frac{1-\eta}{8\eta}k^2}\, \mathrm{Tr}[O e^{ik(X_\phi - x)}] \ , \tag{6}$$

and using the symmetry $p(x;\phi+\pi) = p(-x;\phi)$ one obtains Eqs. (1) and (2).

The route for generalization to arbitrary quantum systems

In this subsection I briefly analyze the derivation of the starting identity (3), which is the core of the method, and which will be exploited later for generalization to arbitrary quantum system other than the bosonic field. The basic definition of the quantum tomographic method is to measure a *quorum* of observables from which the ensemble average of any operator can be obtained for any state of the system. In the case of homodyne tomography the quantum system is a harmonic oscillator, and the quorum is given by the set of all linear combinations of creation and annihilation operators $a^\dagger$ and a; such set, apart from a trivial multiplication constant, corresponds to the set of all quadratures X_ϕ for $\phi \in [0, \pi]$. In order to obtain a trace identity of the form (3) one can exploit a unitary irreducible representation of a Lie group generated by the quorum.[4] In the present case the group is given by the central extension of the abelian group of displacements over the complex plane, with unitary irreducible representation given by the displacement operators $D(\alpha)$, and with composition law $D(\alpha)D(\beta) = D(\alpha+\beta)\exp[2\mathrm{Im}(\alpha\beta^*)]$. Since the representation is irreducible (there is no Hilbert subspace which is left invariant by the group) according to the Shur's lemma only the identity commutes with the whole group representation. Now it is easy to show that the integral $\int \mathrm{d}^2\alpha\, D(\alpha) O D^\dagger(\alpha)$ commutes with all displacement operators (by multiplying the integral by a fixed $D(z)$, using the composition law and shifting the integration variable), which means that the integral is a multiple of the identity. The value of the integral can be obtained by taking its vacuum expectation, and using completeness of coherent states $|\alpha\rangle = D(\alpha)|0\rangle$. In this way one is lead to the identity

$$\mathrm{Tr}(O) = \int \frac{\mathrm{d}^2\alpha}{\pi} D(\alpha) O D^\dagger(\alpha) \ . \tag{7}$$

Using identity (7) to evaluate the trace under the integral in Eq. (3) one obtains

$$\begin{aligned} \int \frac{\mathrm{d}^2\alpha}{\pi} \mathrm{Tr}[O D(\alpha)] D^\dagger(\alpha) &= \int \frac{\mathrm{d}^2\alpha}{\pi} \int \frac{\mathrm{d}^2\beta}{\pi} D(\beta) D(\alpha) O D^\dagger(\beta) D^\dagger(\alpha) \\ &= \int \mathrm{d}^2\beta D(\beta) O D^\dagger(\beta) \int \frac{\mathrm{d}^2\alpha}{\pi^2} e^{\beta\alpha^* - \beta^*\alpha} = O \ , \end{aligned} \tag{8}$$

which proves identity (3).

An example of unbiased estimator, and the emergence of bounds for η

Eq. (2) can be used for estimating the matrix element $\rho_{n+l,n}$ of the density operator of the unknown state. Here the operator O is given by $O = |n\rangle\langle n+l|$. By taking the normal ordering in Eq. (2) one obtains

$$\mathcal{E}_\eta[|n\rangle\langle n+l|](x;\phi) = \frac{1}{4}\int_0^\infty \mathrm{d}t\, e^{-\frac{2\eta-1}{8\eta}t}\langle n+l|\!:\cos(\sqrt{t}(X_\phi - x))\!:\!|n\rangle =$$

$$= \frac{1}{4}e^{il\phi}\sqrt{\frac{n!}{(n+l)!}}\mathrm{Re}\left[\left(\frac{i}{2}\right)^l \int_0^\infty \mathrm{d}t\, e^{-\frac{2\eta-1}{8\eta}t - i\sqrt{t}x}\, t^{l/2} L_n^l(t/4)\right], \quad (9)$$

$L_n^l(x)$ denoting the customary generalized Laguerre polynomials. From Eq. (9) we see that $\mathcal{E}_\eta[|n\rangle\langle m|](x;\phi)$ is bounded only if $\eta > \frac{1}{2}$, which means that the matrix element can be experimentally estimated only for quantum efficiency η above the bound $\eta_b = \frac{1}{2}$. This example shows a typical feature of the tomographic method when applied to infinite dimensional Hilbert space, namely that for every operator O generally there is a bound $\eta_b[O] > 0$ below which the ensemble average $\langle O\rangle$ cannot be estimated. This is not an artifact of the method, but is due to the perfect unbiasedness of the estimator, which works without any *a priori* knowledge of the state. For η approaching the bound $\eta_b[O]$ the statistical error in the estimation of $\langle O\rangle$ becomes unbounded (in Ref. 5 an asymptotic evaluation of errors for the diagonal matrix elements $\rho_{n,n}$ is derived for large n). Finally, it is worth noticing that for estimating the matrix elements $\rho_{n+l,n}$ an algorithm numerically more efficient than Eq. (9) is used, based on a factorization formula for the estimator that holds for $\eta = 1$, and exploiting the inversion of the Bernoulli convolution for $\frac{1}{2} < \eta < 1$.[1]

O	$\mathcal{E}_\eta[O](x;\phi)$	η_b
$:a^{\dagger n}a^m:_s$	Eq. (13)	0
$:D(\alpha):_s$	Eq. (12)	0
$W_s \doteq \frac{2}{\pi(1-s)}\left(\frac{s+1}{s-1}\right)^{a^\dagger a}$	$-\frac{2}{\pi s_\eta}\Phi\left(1,\frac{1}{2};\frac{2x^2}{s_\eta}\right)$	$(1-s)^{-1}$
$\|n\rangle\langle n+l\|$	Eq. (9)	$\frac{1}{2}$
$\|\alpha\rangle\langle\beta\|$	$\kappa\langle\alpha\|\beta\rangle\Phi\left(1,\frac{1}{2};-\kappa(x-w_\phi)^2\right)$	$\frac{1}{2}$
$\|x\rangle\langle x'\|$	Ref. 6	1
$\|\alpha;\zeta\rangle\langle\beta;\zeta\|$	Ref. 6	$(1+e^{-\|\zeta\|^2})^{-1}$

Table 1. Estimators for various field operators and relative bounds for quantum efficiency $\eta > \eta_b$. Legenda: $\Phi(\alpha;\beta;z)$ confluent hypergeometric function, $s_\eta = s - 1 + \eta^{-1}$, $\kappa = 2\eta/(2\eta-1)$, $w_\phi = \frac{1}{2}(\beta e^{i\phi} + \overline{\alpha}e^{-i\phi})$.

Equivalence classes of unbiased estimators

A remarkable consequence of the inversion symmetry for the quadrature operator $p_\eta(x;\phi+\pi) = p_\eta(-x;\phi)$ is that there are "null estimators", which have zero expectation for arbitrary probability $p_\eta(x;\phi)$, i. e. for arbitrary state ρ. Null estimators are obtained as linear combinations of the following functions[7]

$$\mathcal{N}_{k,n}(x;\phi) = x^k e^{\pm i(k+2+2n)\phi} \qquad k,n \geq 0\,. \quad (10)$$

The functions $\mathcal{N}_{k,n}(x;\phi)$ have zero expectation for arbitrary probability as a consequence of the Wilcox formula

$$\langle X_\phi^k\rangle = \frac{k!}{2^k}\sum_{p=0}^{[[k/2]]}\sum_{s=0}^{k-2p}\frac{\langle a^{\dagger s}a^{k-2p-s}\rangle}{2^p p!s!(k-2p-s)!}e^{i(2p+2s-k)\phi} \tag{11}$$

along with the identity $\int_0^\pi \mathrm{d}\phi\, e^{iq\phi}=0$ for $q\neq 0$ even, $[[x]]$ denoting the integer part of x. Hence, for every operator O one actually has an equivalence class of infinitely many unbiased estimators, which differ by a linear combination of functions $\mathcal{N}_{k,n}$. Here I denote the equivalence relation by the symbol $\simeq$, i. e. $\mathcal{N}_{k,n}\simeq 0$ for $k,n\geq 0$. Non trivial examples of equivalence are:[8] *i)* $e^{\pm 2(n+1)i\phi}f(xe^{\pm i\phi})\simeq 0$, with $f(z)$ analytic function of z; the relations involving the Dirac comb over $[0,\pi]$ *ii)* $\delta_\pi(\phi)\simeq\frac{1}{\pi}$, and *iii)* $f(x^2)\delta_\pi(\phi)\simeq\frac{2}{\pi}\mathrm{Re}\left[f\left(x^2e^{2i\phi}\right)/\left(1-e^{-2i\phi}\right)-\frac{1}{2}f(x^2)\right]$ with $f(z)$ analytic.

Using the above equivalences it is straightforward to evaluate the generating function of all s-ordered monomials. One has[8]

$$\mathcal{E}_\eta[:D(\alpha):_s](x;\phi)=\exp\left(\frac{1}{2}s_\eta|\alpha|^2\right)\frac{\alpha e^{-i\phi}e^{2x\alpha e^{-i\phi}}+\alpha^* e^{i\phi}e^{-2x\alpha^* e^{i\phi}}}{\alpha e^{-i\phi}+\alpha^* e^{i\phi}}\,, \tag{12}$$

and

$$\mathcal{E}_\eta[:a^{\dagger n}a^m:_s](x;\phi)=\frac{e^{i(m-n)\phi}H_{m+n}^{\min(n,m)}\left(\sqrt{\frac{2}{s_\eta}}x\right)}{\binom{n+m}{n}\left(\sqrt{\frac{2}{s_\eta}}x\right)^{n+m}}\,, \tag{13}$$

where $s_\eta\doteq s-1+\eta^{-1}$ and $H_n^h(x)=\sum_{l=0}^h\frac{(-)^l n!}{l!(n-2l)!}(2x)^{n-2l}$ denotes the "truncated" Hermite polynomial. Notice the equivalence $e^{i(m-n)\phi}H_{m+n}^{\min(n,m)}(\kappa x)\simeq e^{i(m-n)\phi}H_{m+n}(\kappa x)$, between truncated and customary Hermite polynomials, hence the agreement with the previous result by Richter[9] for the normal ordered case $s=1$.

A list of estimators with their quantum efficiency bounds η_b is given in Table 1. For polynomials in the field operators there is no bound. For the Wigner function $W_s(\alpha,\alpha^*)=\langle D^\dagger(\alpha)W_sD(\alpha)\rangle$ (the operator W_s is defined in table 1) the bound depends on the ordering parameter s. For coherent-state and number matrix elements one has $\eta_b=\frac{1}{2}$; for squeezed representations $\eta_b>\frac{1}{2}$; finally, for the quadrature representation $\eta_b=1$. Essentially, the bound is related to the "fuzziness" of the state representation[1], and at present no analytical state representation is known having bound $\eta_b<\frac{1}{2}$ (see also Ref. 5).

Adaptive method

The estimator can be chosen within the equivalence class in order to minimize the statistical error. This is the basis of the recently discovered "adaptive tomography" of Refs. 7, 10. For arbitrary operator O a particular representative $\mathcal{E}_\eta[O](x;\phi)$ of the equivalence class of estimators is given in Eq. (2). However, one can add a linear combination of null estimators $\mathcal{N}_{k,n}(x;\phi)$ given in Eq. (10), leading to the new estimator

$$\mathcal{J}_\eta[O](x;\phi)=\mathcal{E}_\eta[O](x;\phi)+\mu\cdot\mathcal{F}(x;\phi)+\nu\cdot\mathcal{F}^*(x;\phi)\simeq\mathcal{E}_\eta[O](x;\phi)\,, \tag{14}$$

where $\mathcal{F}(x;\phi)$ and $\mathcal{F}^*(x;\phi)$ denote the vectors of null estimators $\mathcal{F}(x;\phi)\doteq\{\mathcal{N}_{k,n}(x;\phi)\}$, and μ and ν are vectors of coefficients to be determined. Minimization of the variance $\overline{\Delta\mathcal{J}_\eta^2}$ leads to the linear set of equations for μ and ν

$$\mathbf{A}\mu=\mathbf{b}\,,\qquad \mathbf{A}\nu=\mathbf{c}\,, \tag{15}$$

where

$$\mathbf{A} = \overline{\mathcal{F}\mathcal{F}^*}\,, \qquad \mathbf{b} = -\overline{\mathcal{E}_\eta[O]\mathcal{F}^*}\,, \qquad \mathbf{c} = -\overline{\mathcal{E}_\eta[O]\mathcal{F}}\,, \tag{16}$$

the overbar denoting the experimental average $\overline{\mathcal{F}} \doteq \lim_{N\to\infty} N^{-1}\sum_{n=1}^N \mathcal{F}(x_n;\phi_n)$ over homodyne outcomes x_n at (random) phases ϕ_n. Solving the linear system (15) one can show that the variance is reduced by the amount

$$\overline{\Delta\mathcal{J}_\eta^2} - \overline{\Delta\mathcal{E}_\eta^2} = -\mathbf{b}\cdot\mathbf{A}^{-1}\mathbf{b}^* - \mathbf{c}\cdot\mathbf{A}^{-1}\mathbf{c}^*\,. \tag{17}$$

The method works as follows. One first obtains the matrix $\mathbf{A}$ and vectors $\mathbf{b}$ and $\mathbf{c}$ by averaging over homodyne outcomes according to Eqs. (16). Then the linear system (15) is solved, and the optimized estimator in Eq. (14) is obtained. Finally, the ensemble average $\langle O\rangle$ is recovered by averaging the optimized estimator. In this way, the estimator is "adapted" to data and the method becomes nonlinear. For simple operators the optimized estimator can be derived analytically for some classes of states. For example, for the number operator $O \equiv a^\dagger a$ one can prove[7] that for coherent states, squeezed vacuum, and Schrödinger-cat states only the null function $\mathcal{F} = \exp(2i\phi)$ contributes to the optimization, and the optimized estimator is given by[7, 10]

$$\mathcal{J}_\eta[a^\dagger a](x;\phi) = 2x^2 - \frac{1}{2} + 2\mathrm{Re}[\mu\exp(2i\phi)]\,, \qquad \mu = -\overline{2x^2\exp(2i\phi)} \equiv -\frac{1}{2}\langle a^{\dagger 2}\rangle\,, \tag{18}$$

with variance

$$\overline{\Delta\mathcal{J}_\eta^2[a^\dagger a]} \equiv \langle\Delta(a^\dagger a)^2\rangle + \frac{1}{2}[\langle a^{\dagger 2}a^2\rangle - \langle a^{\dagger 2}\rangle\langle a^2\rangle + 2\langle a^\dagger a\rangle + 1]\,. \tag{19}$$

Notice that the leading noise term $\langle a^{\dagger 2}a^2\rangle$ is canceled by the nonlinear term $\langle a^{\dagger 2}\rangle\langle a^2\rangle$. Analogous cancellations are found for other operators, as for the quadrature X_ϕ and the field operator a,[7, 10], where, remarkably, the ideal heterodyne noise is achieved for coherent states. Another noticeable feature is that, differently from the representative (2), the optimized estimator is peaked and symmetrically distributed around the mean value.[10] In Ref. 10 the method has been numerically implemented for estimating the matrix elements $\rho_{n,m}$ in the number representation, proving that a noise reduction up to 60% can be achieved, especially for low n and m. However, it is likely that a much better noise reduction can be achieved by a suitable choice of the basis of null functions.

Multimode homodyne tomography with one local oscillator

For $M+1$ radiation modes the method is easily generalized by using estimators for tensor product operators which are just the products of their relative estimators, i. e. $\mathcal{E}_\eta[\otimes_{n=0}^M O_n](\{x_n\};\{\phi_n\}) = \prod_{n=0}^M \mathcal{E}_\eta[O_n](x_n;\phi_n)$. The case of a general operator is then obtained by linearity. However, this method needs a separate measurement—whence a separate LO—for each mode. In Ref. 11 it is shown that it is possible to estimate the expectation value of any multimode observable using a single LO, scanning all possible linear combinations of modes on it. Here I don't give the derivation of the method, but just present the final results. The estimator is given by

$$\mathcal{E}_\eta[O](x;\theta,\psi) = \frac{\kappa^{M+1}}{M!}\int_0^\infty \mathrm{d}t\, e^{-t+2i\sqrt{\kappa t}x}\, t^M\, \mathrm{Tr}\{O\!:\exp[-2i\sqrt{\kappa t}X(\theta,\psi)]:\}\,, \tag{20}$$

where $:\ldots:$ denotes normal ordering, $\kappa = \frac{2\eta}{2\eta-1}$, and the quadrature operator $X(\theta,\psi)$ is the following linear combination of single-mode quadratures

$$X(\theta,\psi) = \frac{1}{2}\left[A^\dagger(\theta,\psi) + A(\theta,\psi)\right]\,, \qquad A(\theta,\psi) = \sum_{l=0}^M e^{-i\psi_l}u_l(\theta)a_l\,, \tag{21}$$

a_l and $a_l^\dagger$ ($l = 0, \ldots, M$) being the annihilation and creation operators of the $M+1$ independent modes with $[a_l, a_{l'}^\dagger] = \delta_{ll'}$, $\theta = (\theta_0, \ldots, \theta_M)$ and $\psi = (\psi_0, \ldots, \psi_M)$ denoting hyper-polar angles with ranges $\psi_l \in [0, 2\pi]$ and $\theta_l \in [0, \pi/2]$, whereas $u_l(\theta)$ are hyperspherical coordinates, such that $\sum_{l=0}^M u_l^2(\theta) = 1$, with $u_0(\theta) = \cos\theta_1$, $u_1(\theta) = \sin\theta_1 \cos\theta_2$, $u_2(\theta) = \sin\theta_1 \sin\theta_2 \cos\theta_3$, $\ldots$, $u_{M-1}(\theta) = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{M-1} \cos\theta_M$, $u_M(\theta) = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{M-1} \sin\theta_M$. The ensemble average $\langle O \rangle$ is obtained by averaging the estimator (20) as follows

$$\langle O \rangle = \int \mathrm{d}\mu[\psi] \int \mathrm{d}\mu[\theta]\, p(x, \theta, \psi)\, \mathcal{E}_\eta[O](x; \theta, \psi) \,, \tag{22}$$

where $\int \mathrm{d}\mu[\psi] \doteq \prod_{l=0}^M \int_0^{2\pi} \frac{\mathrm{d}\psi_l}{2\pi}$, and $\int \mathrm{d}\mu[\theta] \doteq 2^M M! \prod_{l=1}^M \int_0^{\pi/2} \mathrm{d}\theta_l \, \sin^{2(M-l)+1} \theta_l \cos\theta_l$.

In particular, one can estimate the matrix element $\langle \{n_l\} | R | \{m_l\} \rangle$ of the joint density matrix of modes. This will be obtained by averaging the following estimator

$$\mathcal{E}_\eta[|\{m_l\}\rangle\langle\{n_l\}|](x; \theta, \psi) = e^{-i\sum_{l=0}^M (n_l - m_l)\psi_l} \frac{\kappa^{M+1}}{M!} \prod_{l=0}^M \left\{ [-i\sqrt{\kappa} u_l(\theta)]^{\mu_l - \nu_l} \sqrt{\frac{\nu_l!}{\mu_l!}} \right\}$$
$$\times \int_0^\infty \mathrm{d}t\, e^{-t + 2i\sqrt{\kappa t} x}\, t^{M + \frac{1}{2}\sum_{l=0}^M (\mu_l - \nu_l)} \prod_{l=0}^M L_{\nu_l}^{\mu_l - \nu_l}[\kappa u_l^2(\theta) t] \,, \tag{23}$$

where $\mu_l = \max(m_l, n_l)$, and $\nu_l = \min(m_l, n_l)$. Using simple identities for Laguerre polynomials one can easily derive the estimator for the probability distribution of the total number of photons $N = \sum_{l=0}^M a_l^\dagger a_l$

$$\mathcal{E}_\eta[|n\rangle\langle n|](x; \theta, \psi) = \frac{\kappa^{M+1}}{M!} \int_0^\infty \mathrm{d}t\, e^{-t + 2i\sqrt{\kappa t} x}\, t^M L_n^M[\kappa t] \,, \tag{24}$$

where $|n\rangle$ denotes the eigenvector of N for eigenvalue n. Notice that the estimator in Eq. (24) does not depend on phases ϕ_l and angles θ_l, and thus their knowledge is not needed in this measurement. Other examples of two-mode estimators are

$$\mathcal{E}_\eta[z^{a^\dagger a + b^\dagger b}](x; \theta, \psi) = \frac{1}{(z + \frac{1-z}{\kappa})^2} \Phi\left(2, \frac{1}{2} - \frac{1-z}{z + \frac{1-z}{\kappa}} x^2\right) \,, \tag{25}$$

where a and b denote the annihilator operators of the two modes, and $\Phi(\alpha; \beta; z)$ denotes the customary confluent hypergeometric function. For the first two moments one obtains the simple expressions

$$\mathcal{E}_\eta[a^\dagger a + b^\dagger b](x; \theta, \psi) = 4x^2 + \frac{2}{\kappa} - 2 \,, \tag{26}$$

$$\mathcal{E}_\eta[(a^\dagger a + b^\dagger b)^2](x; \theta, \psi) = 8x^4 + \left(\frac{24}{\kappa} - 20\right) x^2 + \frac{6}{\kappa^2} - \frac{10}{\kappa} + 4 \,. \tag{27}$$

It is worth noticing that analogous estimators for the difference of photon numbers are singular, and in order to recover the correlation between modes a cutoff procedure is needed, analogous to the one used in Ref. 12.

GENERALIZATION TO SPINS AND OTHER DYNAMICAL SYSTEMS

The tomographic method can be easily generalized to other dynamical systems, looking for a quorum of observables that allow the estimation of any ensemble average.

The starting point is to generalize the identity (3) exploiting a unitary irreducible representation of a Lie group generated by the quorum.[4] Here, for the sake of simplicity, I consider only the case of a semisimple compact Lie group $\mathbb{G} = \{g\}$, with irreducible unitary representation R of dimension $d < \infty$. However, with some technicalities, the method can be extended to the cases of infinite dimensional representations, noncompact groups, and reducible groups with central extension.

The equivalent of Eq. (7) can be obtained by noticing that the integral over $\mathbb{G}$ $\int \mathrm{d}\mu(g) R^\dagger(g) O R(g)$ commutes with all unitary operators $R(h)$, $h \in \mathbb{G}$ (this can be checked by group composition $R(h)R(g) = R(hg)$, and by shifting the integration variable g using invariance of the Haar's measure $\mathrm{d}\mu(g)$). As the representation is irreducible, according to the Shur's lemma the integral is a multiple of the identity: this constant can then be obtained by evaluating the trace of the integral, using the trace invariance under cyclic permutation, and normalization of the Haar's measure over $\mathbb{G}$. In this way one obtains the identity

$$\int \mathrm{d}\mu(g) R^\dagger(g) O R(g) = \frac{1}{d} \mathrm{Tr}(O) , \tag{28}$$

Similarly, using orthogonality of characters, one can show that $\int \mathrm{d}\mu(g)\, \mathrm{Tr}[R(g)] R^\dagger(g) = d^{-1}$, and through the following steps

$$\begin{aligned} &\int \mathrm{d}\mu(g)\, \mathrm{Tr}[O R^\dagger(g)] R(g) = d \int \mathrm{d}\mu(g) \int \mathrm{d}\mu(h)\, R^\dagger(h) O R^\dagger(g) R(h) R(g) \\ &= \int \mathrm{d}\mu(h)\, R^\dagger(h) O\, \mathrm{Tr}[R(h)] = \frac{1}{d} O , \end{aligned} \tag{29}$$

one proves the general tomographic identity

$$\int \mathrm{d}\mu(g)\, \mathrm{Tr}[O R^\dagger(g)] R(g) = \frac{1}{d} O , \tag{30}$$

which is the equivalent of Eq. (3). Identity (30) is used to obtain the estimator for $\langle O \rangle$, by taking the ensemble average of both sides of the identity. As an example, here I report the estimator for the measurement of the matrix element of the density operator of a spin J. Upon denoting by $|m\rangle$ the eigenvector of J_z with eigenvalue $m \in [-J, J]$, one has[13]

$$\mathcal{E}[|l\rangle\langle i|](m; \vec{n}) = (2J+1) \left(\lambda_{i,m} \lambda^*_{l,m} - \frac{\lambda_{i,m+1}\lambda^*_{l,m+1} + \lambda_{i,m-1}\lambda^*_{l,m-1}}{2} \right) , \tag{31}$$

where

$$\begin{aligned} \lambda_{l,m} &= e^{i\varphi(m-l)} \sqrt{(J+m)!(J-m)!(J+l)!(J-l)!} \\ &\times \sum_\nu \frac{(-1)^\nu (\cos\frac{\vartheta}{2})^{2J+m-l-2\nu} (-\sin\frac{\vartheta}{2})^{l-m+2\nu}}{(J-l-\nu)!(J+m-\nu)!(\nu+l-m)!\nu!} , \end{aligned} \tag{32}$$

$\vec{n}$ being a unit vector with polar angles θ and ϕ, and the average being performed over all possible measurement results m and over all possible spin rotations around $\vec{n}$ according to the formula

$$\langle O \rangle = \sum_{m=-J}^{J} \int \frac{d\vec{n}}{4\pi}\, p(m; \vec{n})\, \mathcal{E}[O](m; \vec{n}) , \tag{33}$$

$p(m; \vec{n})$ denoting the probability of outcome m for the measurement of $\vec{J} \cdot \vec{n}$. One can easily recognize the correspondence with conventional optical quantum tomography: $x \leftrightarrow m$, $\phi \leftrightarrow \vec{n}$. On the basis of Eq. (33) an experimental setup can be devised for tomography of spin observables, which is a simple modification of the Stern-Gerlach experiment.[13]

The quantum tomographic technique opens the possibility of a new type of experiments and tests of quantum mechanics. In Ref. 14 a test of Bell's inequality is proposed that is based on two-mode homodyne tomography, with the possibility of achieving very good detection quantum efficiencies. Using three-mode homodyne tomography, in principle it is now possible to make a complete test of the preparation of a Greenberger-Horne-Zeilinger state,[15] which cannot be checked by simple coincidence measurements. Finally, it is now possible a direct test of nonclassicality on various one-mode and two-modes states, by tomographically measuring some special observables of the field.[16]

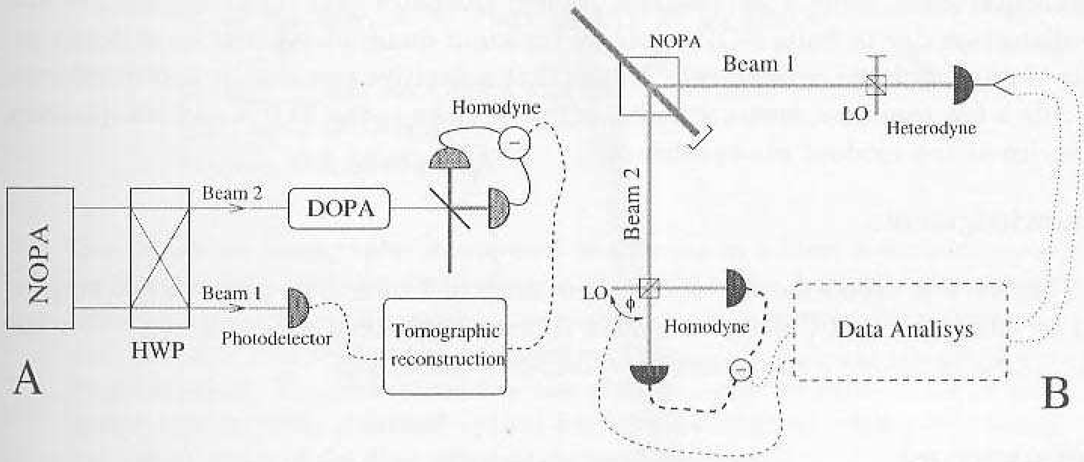

Figure 1. Two examples of *conditioned tomography*. The general scheme involves a nondegenerate optical parametric amplifier (NOPA) that produces a couple of correlated twin beams. A quantum measurement is performed on one beam, and a tomographic reconstruction is made on the second beam, conditioned on the result of the first measurement. Figure A: Experimental scheme for generation and tomographic detection of Schrödinger cats (see text).[18] Scheme for testing the state reduction for heterodyne detection.[19]. After heterodyning the beam 1 of the twin couple, the reduced state of beam 2 is tomographically reconstructed conditioned by the heterodyne outcome. In place of the heterodyne detector one can put any other kind of detector for testing the state reduction on different observables.

Using parametric downconversion, a new set of experiment is now possible, which we can categorize as *conditioned tomography*. The general scheme is the following. A nondegenerate optical parametric amplifier (NOPA) produces a couple of correlated twin beams 1 and 2 from vacuum downconversion. A quantum measurement is performed on beam 1, and a tomographic reconstruction is made on beam 2, conditioned on the result λ of the first measurement, namely using an estimator $\mathcal{E}_\eta[O](x;\phi;\lambda)$ which depends on the outcome λ of the measurement on beam 1.

An example of conditioned tomographic measurement scheme is depicted in Fig. 1A, which represents a tomographic improvement of a scheme proposed in Ref. 17 in order to generate and detect *Schrödinger-cat* states. The experiment consists in feeding the twin beams—here two orthogonally polarized modes of radiation, the *signal* and the *readout*—into a half-wave plate (HWP) which rotates the polarization direction. The rotation angle and the gain of the NOPA are related by the back-action-evading condition. When a number of photons is detected at the readout mode, a Schrödinger-cat appears on the signal mode (an additional squeezer (DOPA) is inserted on the signal mode to "stretch" the cat). In Ref. 18 it is shown that the tomographic technique tolerates very realistic values for quantum efficiency at the readout photodetector, and a precise reconstruction of the cat at the signal mode is possible, recovering the visibility of the homodyne probability oscillation, which otherwise would have been completely washed out by the low quantum efficiency at the readout detector.

Another new experiment based on the conditioned tomographic scheme is the proposal of Ref. 19 for a test of state reduction. The scheme of the experiment is depicted in Fig. 1B. Again a couple of twin beams is generated by a NOPA. After heterodyning beam 1, the reduced state of beam 2 is tomographically reconstructed conditioned by the heterodyne outcome. In place of the heterodyne detector one can put any other kind of detector for testing the state reduction on different observables: for heterodyne detection the reduced state is a coherent state, whereas, for example, for photodetection it is a number eigenstate. The state reduction can be tested by a direct measurement of the fidelity between the theoretically expected reduced state and the experimental state, using a suitable conditioned estimator that takes into account also state distortion due to finite NOPA gain and nonunit quantum efficiencies at detectors. Monte Carlo simulated experiments[19] show that a decisive test can be performed even with only a few thousand measurements, with low gains at the NOPA and low quantum efficiencies at the readout photodetector.

Acknowledgments

This work is supported by Istituto Nazionale di Fisica della Materia and cosponsored by Ministero dell'Università e della Ricerca Scientifica e Tecnologica under the project *Amplificazione e rivelazione di radiazione quantistica.*

REFERENCES

1. G. M. D'Ariano, in Quantum Optics and Spectroscopy of Solids, ed. by T. Hakioglu and A. S. Shumovsky, (Kluwer Academic Publisher, Amsterdam 1997), p. 175.
2. G. M. D'Ariano, in Quantum Communication,Computing, and Measurement, ed. by O. Hirota, A. S. Holevo, and C. M. Caves, (Plenum Publishing, New York 1997), p. 253.
3. G. M. D'Ariano, and N. Sterpi, *J. Mod. Opt.* 44:2227 (1997).
4. M. Paini, Thesis, University of Pavia (1999)
5. G. M. D'Ariano, C. Macchiavello, and N. Sterpi, in this volume.
6. G.M. D'Ariano, U. Leonhardt, and H. Paul, *Phys. Rev. A* 52:R1801 (1995).
7. G. M. D'Ariano and M. G. A. Paris, *Acta Physica Slovaca* 48:191 (1998)
8. G. M. D'Ariano, unpublished.
9. Th. Richter, *Phys. Lett.* A221:327 (1996).
10. G. M. D'Ariano and M. G. A. Paris, unpublished.
11. G. M. D'Ariano, M. F. Sacchi, M. Vasilyev, and P. Kumar, unpublished.
12. G. M. D'Ariano, M. Vasilyev, and P. Kumar, *Phys. Rev. A* 58:636 (1998).
13. G. M. D'Ariano, L. Maccone, and M. Paini, unpublished
14. G. M. D'Ariano, L. Maccone, M. F. Sacchi, and A. Garuccio, in this volume.
15. G. M. D'Ariano, M. F. Sacchi, and Y. Shih, unpublished
16. G. M. D'Ariano, M. F. Sacchi, and P. Kumar, *Phys. Rev. A* (1999), in press.
17. S. Song, C.M. Caves and B. Yurke, *Phys. Rev. A* 41:5261 (1990).
18. G.M. D'Ariano, C. Macchiavello and L. Maccone, unpublished.
19. G. M. D'Ariano, P. Kumar, C. Macchiavello, and N. Sterpi, unpublished