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Quantum Computations with Polarized Photons

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Abstract

We propose a fully optical method to perform any quantum computation by supplying the prescriptions for a universal set of quantum gates. We also give the methods for the generation of input states and the realization of final measurements. The apparatus is scalable, and relies on high Kerr non-linearities.

1. Introduction

In recent years quantum computation has developed very fast both from the theoretical and the experimental point of view. The experimental applications span from ion traps [1], to nuclear magnetic resonance [2], to cavity QED [3]. However, these kinds of setups are hardly scalable, so that it may be problematic to build a quantum computer with more than a few qubits. More promising from this point of view is the recent proposal that relies on neutral atoms trapped in an optical lattice [4]. The main problem of quantum computation which uses matter degrees of freedom to encode the qubits is decoherence, which is unavoidably high in such systems. Quantum computation on radiation modes, on the other hand, has the advantage of greatly reducing decoherence, at the cost of requiring more complicated means for driving interactions between qubits. Schemes of optical quantum gates have been proposed in the last years [5, 6, 7]. However, only after the recent experimental availability [8] of giant Kerr nonlinearities (that were theoretically analyzed in [9]) optical realizations of quantum computers have become appealing. In this paper we propose a fully optical method to perform quantum computations, by giving the prescriptions for the generation of the input states, for the analysis of the final states, and for the implementation of a universal set of quantum gates.

2. Optical quantum computation

It has been demonstrated [10] that single qubit gates (i.e. gates that operate only on one qubit) and the Controlled-NOT gate are a universal set for quantum computation. The Controlled-NOT gate is a two qubit gate where the first qubit, called control qubit, is left unchanged while the second one, called target, is flipped when the state of the control is $|1\rangle$, namely

$$|\epsilon_1\rangle\,|\epsilon_2\rangle \to |\epsilon_1\rangle\,|\epsilon_1 \oplus \epsilon_2\rangle\ , \tag{2.1}$$

where $\oplus$ denotes addition modulo two and $\epsilon_i = \{0, 1\}$. Hence it will be sufficient to give a prescription for these gates in order to be able to perform any quantum computation.

We propose to encode each qubit in a polarization state of a single photon. The zero logical state $|0\rangle$ is, thus, encoded into $|1\rangle_a|0\rangle_b$, while the logical one $|1\rangle$ is given by

$|0\rangle_a|1\rangle_b$, where a and b are the annihilation operators for two orthogonal polarization modes ($|0\rangle_a$ and $|1\rangle_a$ being the vacuum and one photon states respectively for mode a, and analogously for other modes).

Universal set of gates. With the proposed encoding, the single qubit gate resorts to a simple polarization rotator, which is described by an Hamiltonian of the form

$$H_{\text{Rotat}} = i\hbar\varphi(a^\dagger b\, e^{i\psi} - ab^\dagger\, e^{-i\psi})\,. \tag{2.2}$$

The Hadamard gate, for example, is obtained by a $\varphi = \frac{\pi}{4}$ rotating plate. The phase ψ in (2.2) is simply obtained introducing a phase shift between the two modes a and b.

The Controlled-NOT gate can be obtained by using the device depicted in Fig. 1. It is based on the optical Cross-Kerr effect Hamiltonian

$$H_{\text{CKerr}} \doteq i\hbar\, \frac{\pi}{2}\, a^\dagger a(c^\dagger d - cd^\dagger)\,, \tag{2.3}$$

where a and b are the annihilation operators for the two orthogonal polarization modes pertaining to the state of the control qubit, while c and d are the two polarization annihilators for the target qubit. We show that the Hamiltonian H_{CKerr} is actually a realization of a Controlled-NOT gate, by verifying that it induces the transformations given in Eq. (2.1): by defining $\vartheta \doteq \frac{\pi}{2}\, a^\dagger a$, we have

1. Transformation $|1\rangle\,|1\rangle \;\rightarrow\; |1\rangle\,|0\rangle$:

$$\begin{aligned} & e^{\vartheta(c^\dagger d - cd^\dagger)}\,|1\rangle_a\,|0\rangle_b\,|1\rangle_c\,|0\rangle_d \\ &= e^{-\tan\vartheta cd^\dagger}\,(\cos\vartheta)^{c^\dagger c - d^\dagger d}\,e^{\tan\vartheta c^\dagger d}\,|1\rangle_a\,|0\rangle_b\,|1\rangle_c\,|0\rangle_d \\ &= \cos\vartheta\,|1\rangle_a\,|0\rangle_b\,|1\rangle_c\,|0\rangle_d - \sin\vartheta\,|0\rangle_a\,|1\rangle_b\,|0\rangle_c\,|1\rangle_d = -|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d\,; \end{aligned} \tag{2.4}$$

2. Transformation $|1\rangle\,|0\rangle \;\rightarrow\; |1\rangle\,|1\rangle$:

$$\begin{aligned} & e^{\vartheta(c^\dagger d - cd^\dagger)}\,|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d \\ &= e^{-\tan\vartheta cd^\dagger}(\cos\vartheta)^{c^\dagger c - d^\dagger d}\,e^{\tan\vartheta c^\dagger d}\,|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d \\ &= \frac{1}{\cos\vartheta}\,|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d + \sin\vartheta\,|1\rangle_a\,|0\rangle_b\,|1\rangle_c\,|0\rangle_d - \tan\vartheta\,\sin\vartheta\,|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d \\ &= |1\rangle_a\,|0\rangle_b\,|1\rangle_c\,|0\rangle_d\,; \end{aligned} \tag{2.5}$$

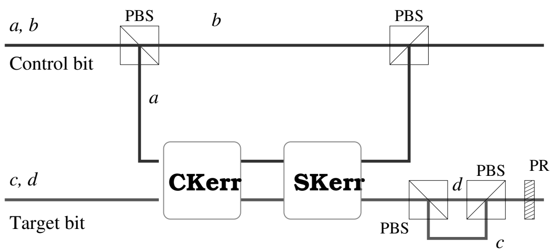

Fig. 1: Proposed Controlled-NOT gate. The device, whose core is described by the Cross-Kerr (CKerr) Hamiltonian (2.3), is composed of polarizing beam splitters (PBS) and two optical Kerr media. The modes a and b are the orthogonal polarization modes pertaining to the control qubit, while c and d are the orthogonal polarization modes pertaining to the target qubit. The Self-Kerr (SKerr) medium is introduced to obtain the correct phase dependance for the Controlled-NOT transformation.

3. Transformation $|0\rangle\,|1\rangle \to |0\rangle\,|1\rangle$:

$$e^{\frac{\pi}{2}a^\dagger a(c^\dagger d - cd^\dagger)}\,|0\rangle_a\,|1\rangle_b\,|1\rangle_c\,|0\rangle_d = |0\rangle_a\,|1\rangle_b\,|1\rangle_c\,|0\rangle_d\,; \tag{2.6}$$

4. Transformation $|0\rangle\,|0\rangle \to |0\rangle\,|0\rangle$:

$$e^{\frac{\pi}{2}a^\dagger a(c^\dagger d - cd^\dagger)}\,|0\rangle_a\,|1\rangle_b\,|0\rangle_c\,|1\rangle_d = |0\rangle_a\,|1\rangle_b\,|0\rangle_c\,|1\rangle_d\,. \tag{2.7}$$

The minus sign in Eq. (2.4) would lead to an incorrect phase relation between the input and the output states of the Controlled-NOT gate. Hence, it is necessary to correct it by adding a self Kerr medium (SKerr), described by the Hamiltonian

$$H_{\mathrm{SKerr}} \doteq \hbar\pi a^\dagger a d^\dagger d\,. \tag{2.8}$$

One can see immediately that $\exp\left[-\frac{i}{\hbar}H_{\mathrm{SKerr}}\right]|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d = -|1\rangle_a\,|0\rangle_b\,|0\rangle_c\,|1\rangle_d$, while all the other states of Eqs. (2.5), (2.6) and (2.7) are left unchanged.

We now give a study of the conditions for the practical implementation of the Kerr device based on the the Hamiltonian $H_{\mathrm{C\,Kerr}}$. In order to obtain a useful device, some preliminary assumptions are needed. In the first place, all optical frequencies must be equal, in order to permit the cascading of more gates in a QC network and to make the device scalable. In the second place, modes a and b and modes c and d must have orthogonal polarization vectors $\vec{e}$ (i.e. $\vec{e}_a \perp \vec{e}_b$ and $\vec{e}_c \perp \vec{e}_d$). Moreover, the atomic/molecular transition frequencies involved must be close (but not equal) to the mode frequencies, in order to enhance the susceptibility (remaining in a perturbation approach). Finally, one must require modes a and b to have a wave vector $\vec{k}$ different from modes c and d, in order to keep the target and the control qubit spatially separated. It is indeed possible to meet these conditions in a physical medium: assume the electric dipole approximation, and consider the medium as an assembly of distinguishable, independent and similarly oriented atoms/molecules. It is possible to obtain the following form for the third order susceptibility tensor $\chi^{(3)}$ [11]

$$\chi^{(3)}(-\omega_\sigma,\omega_1,\omega_2,\omega_3) = \frac{N}{\epsilon_0}\,\frac{e^4}{3!\,\hbar^3}\,S\sum_{lmno}\rho_0(l) \tag{2.9}$$

$$\left[\frac{\vec{e}_\sigma^{\,*}\cdot\vec{r}_{lm}\vec{e}_1\cdot\vec{r}_{mn}\vec{e}_2\cdot\vec{r}_{no}\vec{e}_3\cdot\vec{r}_{ol}}{(\Omega_{ml}-\omega_\sigma)\,(\Omega_{nl}-\omega_2-\omega_3)\,(\Omega_{ol}-\omega_3)} + \frac{\vec{e}_1\cdot\vec{r}_{lm}\vec{e}_\sigma^{\,*}\cdot\vec{r}_{mn}\vec{e}_2\cdot\vec{r}_{no}\vec{e}_3\cdot\vec{r}_{ol}}{(\Omega_{ml}+\omega_1)\,(\Omega_{nl}-\omega_2-\omega_3)\,(\Omega_{ol}-\omega_3)}\right.$$
$$\left. + \frac{\vec{e}_1\cdot\vec{r}_{lm}\vec{e}_2\cdot\vec{r}_{mn}\vec{e}_\sigma^{\,*}\cdot\vec{r}_{no}\vec{e}_3\cdot\vec{r}_{ol}}{(\Omega_{ml}+\omega_1)\,(\Omega_{nl}+\omega_1+\omega_2)\,(\Omega_{ol}-\omega_3)} + \frac{\vec{e}_1\cdot\vec{r}_{lm}\vec{e}_2\cdot\vec{r}_{mn}\vec{e}_3\cdot\vec{r}_{no}\vec{e}_\sigma^{\,*}\cdot\vec{r}_{ol}}{(\Omega_{ml}+\omega_1)\,(\Omega_{nl}+\omega_1+\omega_2)\,(\Omega_{ol}+\omega_\sigma)}\right],$$

where l, m, n and o are the energy levels of the Kerr medium, $\omega_{\sigma,1,2,3}$ are the optical frequencies of the four radiation modes, $\Omega_{lm,mn,no,ol}$ are the frequencies of the dipole transitions, $\vec{e}_j$ are the polarization vectors of the radiation fields, $e\vec{r}$ are the dipole moments of the atoms/molecules, $\rho_0(l)$ is the diagonal thermal equilibrium density–matrix element for the state l (i.e. the statistical fraction of the total molecular population which, in thermal equilibrium, occupies the l energy level), N is the density of dipoles, and the intrinsic symmetrization operation S requires that the expression following it is to be summed over all the possible permutations of the pairs $(\vec{e}_1,\omega_1)$, $(\vec{e}_2,\omega_2)$, and $(\vec{e}_3,\omega_3)$. In order to obtain a Hamiltonian operator of the form (2.3) it is possible to choose, for example, the energy level structure and the mode correspondence depicted in Fig. 2. Other choices are, of course, possible. With any choice of the modes from the susceptibility given in Eq. (2.9)

σ	$\leftrightarrow$	$a^\dagger$
1	$\leftrightarrow$	a
2	$\leftrightarrow$	c
3	$\leftrightarrow$	$d^\dagger$
Mode correspondence		

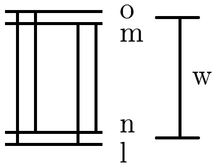

Fig. 2: Possible mode correspondence and energy level structure for the Kerr device needed for the implementation of the Controlled-NOT gate.

we obtain the Hamiltonian

$$H_{\mathrm{Kerr}} \propto i\left[(a^\dagger a + aa^\dagger)\,(c^\dagger d - cd^\dagger)\right], \tag{2.10}$$

from which it is possible to obtain the Hamiltonian (2.3) by compensating the term with $aa^\dagger$ in Eq. (2.10) through a polarization rotator (PR), as depicted in Fig. 1. The conditions given above impose also some constraints on the wave vectors of the modes, which must satisfy the relation $\vec{k}_\sigma \equiv \vec{k}_1 \neq \vec{k}_2 = \vec{k}_3$, and on the optical frequencies which must satisfy $\omega_\sigma = -\omega_1 = \omega_2 = -\omega_3 \doteq \omega$. Moreover the dipole orientations must be $\vec{r}_{lm} \parallel \vec{r}_{mn}$ and $\vec{r}_{no} \perp \vec{r}_{ol}$, while the transition frequencies must satisfy $\Omega_{ml} \simeq \omega$, $\Omega_{nl} \simeq 0$, $\Omega_{ol} \simeq \omega$, etc. Notice that the frequencies can't be perfectly matched, otherwise the perturbative approach which was used in deriving Eq. (2.9) would be lost.

Analogous considerations hold for the derivation of the Hamiltonian $H_{\mathrm{S\,Kerr}}$ of Eq. (2.8). By using the expression (2.9) for the susceptibility $\chi^{(3)}$, we can also reach an Hamiltonian of the form

$$H \propto \left[a^\dagger a d^\dagger d\right]_{\mathrm{sym}}, \tag{2.11}$$

where sym denotes symmetrization. The Hamiltonian (2.11) is equivalent to (2.8) apart from free evolution terms, which may be compensated using two polarizing beam splitters which introduce a phase shift between the two modes at the output signal as depicted in Fig 1.

Input states. The present method to perform quantum computations requires the use of polarized single photon states. One way to generate such states is the following. A coherently pumped non-degenerate optical parametric amplifier (NOPA) generates a twin beam state

$$|\psi\rangle = (1 - |\gamma|^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \gamma^n \, |n\rangle_a \, |n\rangle_b \,, \tag{2.12}$$

where γ is the gain parameter of the NOPA. By placing a photodetector at one of the two exiting beams, in the limit of unit detector efficiency, the other beam can be made to collapse in the Fock state $|n\rangle$ pertaining to the number n of photons that the photodetector counted. In the case of non-unit quantum efficiency, one has to keep the NOPA gain γ low, in order to have a very low probability of having more than one photon in each of the twin beams. Now, when the detector clicks, we can assume that a one photon state $|1\rangle_a$ is present in the other beam. Less naive and more reliable setups for the generation of one photon input states can be obtained by making use of the Fock Filter device proposed in [12].

Final measurements. The measurements of the output states, to be performed as the final step of a quantum computation, consist of polarization measurements. These may be easily achieved by means of a polarization beam splitter, followed by two highly efficient avalanche photodetectors (as it is only necessary to measure the presence or absence of the field).

3. Conclusions

We have presented a method to perform any quantum computation in a fully optical way, by encoding each qubit in a single photon polarization state. A universal set of gates has been proposed, along with the prescriptions for the generation of the input states and for the retrieval of the final results at the end of the quantum computation. A study of the practical feasibility of the Hamiltonian of the device needed to implement the Controlled-NOT gate was given. The proposed universal set of gates is fully scalable, since it is possible to increase the network size, without having an exponential increase in its physical resources. The decoherence in this setup is very low, and it is mostly due to losses in the Kerr medium. The major limitation in the practical realization of the present proposal is the very high Kerr nonlinearities that are needed. However, since methods for achieving Giant Kerr shifts by using electromagnetically induced transparency have been recently proposed we think that the present scheme may open new perspectives for experimental quantum computation.

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