

Doubly special relativity from quantum cellular automata

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Abstract – It is shown how a doubly special relativity model can emerge from a quantum cellular automaton description of the evolution of countably many interacting quantum systems. We consider a one-dimensional automaton that spawns the Dirac evolution in the relativistic limit of small wave vectors and masses (in Planck units). The assumption of invariance of dispersion relations for boosted observers leads to a non-linear representation of the Lorentz group on the (ω, k) -space, with an additional invariant given by the wave vector $k = \pi/2$. The space-time reconstructed from the (ω, k) -space is intrinsically quantum, and exhibits the phenomenon of *relative locality*.

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The existence of a fundamental scale of length or mass, which can be identified with the Planck scale, is a ubiquitous feature of quantum gravity models [1–7]. The appearance of the minimum length $\ell_P = \sqrt{\hbar G/c^3}$ is the result of combining the fundamental constants that characterize physical theories describing different scales: \hbar (quantum mechanics), c (special relativity), and G (gravity). The so-called *Planck length* ℓ_P is commonly regarded as the threshold below which the intuitive description of space-time breaks down, and new phenomenology is expected. A natural hypothesis is that quantum features become crucial in determining the structure of space-time below the Planck scale, leading to a radical departure from the traditional geometric concepts. This perspective makes one wonder about the fate of Lorentz symmetry at the Planck scale. A possible way of tackling this question is to consider a theory with two observer-independent scales, the speed of light and the Planck length, as proposed in the models of doubly special relativity (DSR) [8–13]. All the DSR models share the feature of a non-linear deformation of the Poincaré symmetry that eventually leads to a modification of the quadratic invariant

$$E^2 = p^2 + m^2. \quad (1)$$

Such deformed kinematics are especially interesting since they provide new phenomenological predictions, *e.g.* wave-length dependence of the speed of light and a modified threshold for particle creation in collision processes. Evidences for a violation of the Lorentz energy-momentum dispersion relation (1) have recently been sought in astrophysics, see, *e.g.*, the thresholds for ultra-high-energy cosmic rays [14,15], and in cold-atom experiments [16].

A recent approach to a Planck scale description of physical kinematics is that of quantum cellular automata (QCAs). The QCA generalizes the notion of cellular automaton of von Neumann [17] to the quantum case, with cells of quantum systems interacting with a finite number of neighbors via a unitary operator describing the single-step evolution. The general theory of QCA is rigorously treated in refs. [18,19] while first attempts to model relativistic dynamics with QCA appears in [20,21]. Here, following the proposal of refs. [22–26], we assume that each cell x of the lattice corresponds to the local value $\psi(x)$ of a quantum field whose dynamics is described by a QCA. From this perspective the usual quantum field evolution should emerge as a large-scale approximation of the automaton dynamics occurring at an hypothetical discrete Planck scale. In ref. [22] a QCA —called *Dirac QCA* in the following— has been proposed for describing the Planck scale physics of the Dirac field in $d = 1$ space dimension, assuming the Planck length as the distance between the cells. In ref. [24] it has been shown that the dynamics of such QCA recovers the usual Dirac evolution in the

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relativistic limit of small wave vectors $k \ll 1$ and small masses $m \ll 1$ (everything expressed in Planck units). In ref. [25] it has been shown that in $d = 3$ space dimensions and for a minimal number of field components, only two QCAs satisfy locality, homogeneity, and isotropy of the quantum-computational network, the two QCAs being connected by CPT symmetry, and giving the Dirac evolution in the relativistic limit. In $d = 1, 2$ space dimensions there is instead only one QCA satisfying the above requirements.

Clearly the QCA theoretical framework cannot enjoy a continuous Lorentzian space-time, along with the usual Lorentz covariance, which must break down at the Planck scale. For this reason the notion itself of a boosted reference frame as based on an Einsteinian protocol has still to be refined. However, whatever the final physical interpretation of the relativity principle, it must include the invariance of the dispersion relation in any of its expressions, being at the core of the physical law. In this letter we explore this route and, assuming the invariance of the Dirac QCA dispersion relation, we find a non-linear representation of the Lorentz group which exhibits the typical features of a DSR with an invariant energy scale. For the present purpose without loss of generality we focus on the easiest $d = 1$ dimensional case. As for any DSR model, it turns out that due to the non-linearity of the representation of the Lorentz group, the Lie algebra of the representation of the full Poincaré group, including boosts and translations, is distorted, and the space-time emerging from the automaton exhibits *relative locality*, namely the extensively studied phenomenon according to which the coincidence of two events is observer-dependent [27–29]. Specifically, we will show that in the automaton case the coincidence of particles trajectories is no longer observer independent. Contrarily to the usual special relativity, where the Poincaré group acts linearly both in the position and in the momentum space, in the DSR scenario there are essentially no restrictions on the non-linear energy-momentum transformations, allowing for a variety of possible models. Since these models are generally inequivalent from the physical point of view, an open problem is to single out one of them via physical principles. The quantum cellular automaton provides a microscopic dynamical model which naturally introduces a DSR.

A quantum field cellular automaton describes the one-step evolution $\psi(x) \rightarrow \mathbf{U}\psi(x)$ of a n -components field $\psi(x) := (\psi_1(x), \psi_2(x), \dots, \psi_n(x))^T$ defined on a discrete lattice $x \in \mathbb{Z}^d$ of quantum cells. As proved in ref. [21] for $d = 1$ and in ref. [25] for any d , in a non-interacting scenario all scalar fields exhibit trivial evolution, namely $\mathbf{U}\psi(x) = \psi(x)$, and the minimal dimension of the array $\psi(x)$ for the field to admit a non-trivial evolution is two. Our QCA then evolves a two-components field. In the following, we will restrict ourselves to the one-dimensional case $d = 1$, and x then denotes one site on a discrete array of cells. The two field components in this case can be labelled as $\psi(x) := (\psi_l(x), \psi_r(x))^T$, with ψ_r and ψ_l denoting

the *left* and *right* field modes. The one-dimensional Dirac QCA can be derived by imposing the invariance of the evolution with respect to the symmetries of the causal network [24,25], and is given by

$$\mathbf{U} = \begin{pmatrix} nS & -im \\ -im & nS^\dagger \end{pmatrix}, \quad n^2 + m^2 = 1, \quad (2)$$

with S the shift operator $S\psi(x) := \psi(x+1)$. The canonical basis of the Fock space of the field states are obtained by applying to the vacuum state $|\Omega\rangle$ the creation operators $\psi_s^\dagger(x)$, $s = r, l$. In the following we restrict ourselves to the one-particle sector for which an orthonormal basis is given by the states $|s\rangle|x\rangle := \psi_s^\dagger(x)|\Omega\rangle$. We write a generic one-particle state as $|\psi\rangle = \sum_{x,s} g_s(x)|s\rangle|x\rangle$ and eq. (2) defines a unitary matrix U on $\mathbb{C}^2 \otimes l_2(\mathbb{Z})$. In the Fourier-transformed basis $|s\rangle|\phi(k)\rangle$, with $|\phi(k)\rangle := (2\pi)^{-1/2} \sum_x e^{-ikx}|x\rangle$, $k \in B := [-\pi, \pi]$, the matrix U is written as

$$U = \int_B dk \hat{U}(k) \otimes |\phi(k)\rangle\langle\phi(k)|, \quad \hat{U}(k) = \begin{pmatrix} ne^{ik} & -im \\ -im & ne^{-ik} \end{pmatrix},$$

whose eigenvalues are $\exp[\pm i\omega(k)]$ where the function $\omega(k)$ is given by

$$\cos^2 \omega = (1 - m^2) \cos^2 k, \quad (3)$$

which is the dispersion relation of the Dirac automaton. A similar dispersion relation was obtained from a finite-difference version of the Maxwell and Klein-Gordon equations in refs. [30,31], with the same small-wavelength limit as in eq. (3). In the limit of small wave vectors and masses eq. (3) reduces to $\omega^2 = k^2 + m^2$, and we recover the Lorentz dispersion relation of eq. (1)¹. Disregarding the internal degrees of freedom, we consider the dispersion relation in eq. (3) as the core dynamics of the theory which should be independent of the reference frame. In one spatial dimension the Lorentz group consists in only the boost transformations which in the energy-momentum sector are represented by the linear map

$$L_\beta : (\omega, k) \mapsto (\omega', k') = \gamma(\omega - \beta k, k - \beta\omega), \quad (4)$$

with $\gamma := (1 - \beta^2)^{-1/2}$. It is immediate to check that the automaton dispersion relation of eq. (3) is not invariant under such standard boosts.

Following the DSR proposal of preserving the Lorentz group structure, the linear Lorentz boosts in eq. (4) should

¹For large wave vectors a thorough interpretation of k and ω in terms of the momentum and energy, respectively, would need an interacting theory.

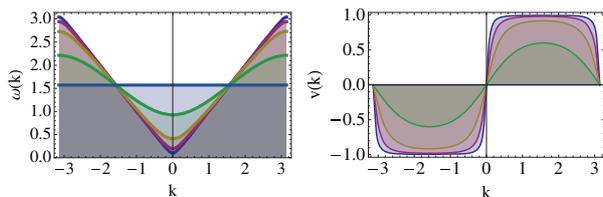


Fig. 1: (Color online) The automaton dispersion relation (left) and group velocity (right) for $m = 0.1, 0.2, 0.4, 0.8, 1$, from bottom to top at $k = 0$ (left), and at $k = \pi/2$ (right).

be replaced by a non-linear representation of the kind

$$L_{\beta}^D := \mathcal{D}^{-1} \circ L_{\beta} \circ \mathcal{D}, \quad (5)$$

where $\mathcal{D}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a non-linear map.

The specific form of \mathcal{D} gives rise to a particular energy-momentum Lorentz deformation. As pointed out in [13], in order to realize a DSR model, the non-linear map \mathcal{D} , has to satisfy the following constraints: i) the Jacobian matrix $J_{\mathcal{D}}(k, \omega)$ of \mathcal{D} evaluated in $k = \omega = 0$ must be the identity, ensuring that the non-linear transformations L_{β}^D recover the standard boosts in the regime of small momenta and energies; ii) the invertibility range of \mathcal{D} must be a Lorentz invariant region of the energy-momentum space; iii) the model will exhibit an invariant energy scale only if the map \mathcal{D} has a singular point, namely some energy ω_{inv} which is mapped to ∞ . Restating eq. (3) in the following way:

$$\frac{\sin^2 \omega}{\cos^2 k} - \tan^2 k = m^2,$$

the non-linear map \mathcal{D} in (5) can be taken to be

$$\mathcal{D}: (\omega, k) \mapsto \mathcal{D}(\omega, k) := (\sin \omega / \cos k, \tan k). \quad (6)$$

One can show that the map in eq. (6) automatically satisfies the aforementioned requirements i)–iii) with the invariant energy $\omega_{inv} = \pi/2$. By inserting the map (6) into eq. (5) we obtain the following deformed Lorentz transformations:

$$\begin{aligned} \omega' &= \arcsin[\gamma(\sin \omega / \cos k - \beta \tan k) \cos k'], \\ k' &= \arctan[\gamma(\tan k - \beta \sin \omega / \cos k)]. \end{aligned} \quad (7)$$

which leave the automaton dispersion relation of eq. (3) invariant.

The modified transformations have two symmetrical invariant momenta $k = \pm\pi/2$ corresponding to the invariant energy $\omega_{inv} = \pi/2$ independently of m . The fixed points split the domain $B = [-\pi, \pi]$ into two regions $B = B_1 \cup B_2$, with $B_1 := [-\pi/2, \pi/2]$ and $B_2 := [-\pi, -\pi/2] \cup [\pi/2, \pi]$, which remain separate under all possible boosts. The points $k = \pm\pi/2$ correspond to maxima of the group velocity $v := \partial_k \omega(k)$ (see fig. 1). While in region B_1 an increasing k corresponds to an increasing group velocity, in region B_2 we see the opposite behavior. However, as one can verify using the transformations (7), a boosted observer who sees an increased group velocity in B_1 also

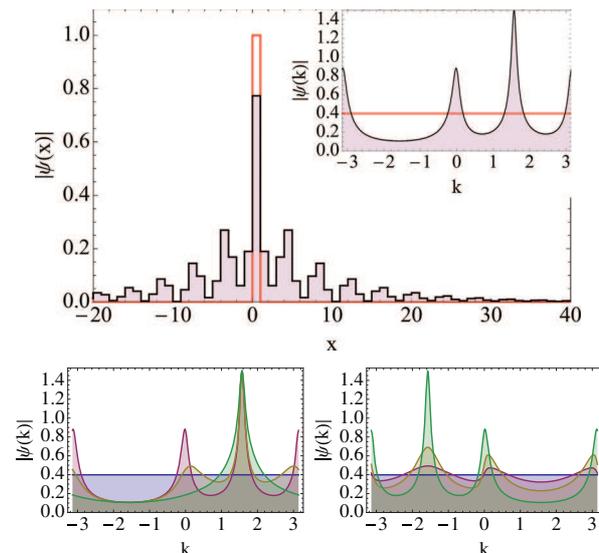


Fig. 2: (Color online) Top panel: delocalization of a state localized at $x = 0$ after a boost with $\beta = -0.99$ for mass $m = 0.1$. Bottom panels: left: momentum representation of a boosted localized state for different values of the mass $m = 0.1$ (red) 0.3 (orange) 0.8 (green) with $\beta = -0.99$. Right: momentum representation of a boosted localized state for different values of the boost $\beta = 0.4$ (red) 0.8 (orange) 0.99 (green) with $m = 0.1$.

sees an increased group velocity in B_2 since in both cases the momentum k is mapped closer to the invariant point. Since the two physical regions B_1 and B_2 exhibit the same kinematics they are indistinguishable in a non-interacting framework. For massless particles the Dirac automaton dispersion relation (3) coincides with the undistorted one $\omega^2 = k^2$ and the group velocity no longer depends on k . Thus, the model we are considering does not exhibit a momentum-dependent speed of light.

The action of the boosts (7) on the states of the automaton (disregarding the internal degrees of freedom) reads

$$\begin{aligned} |\psi\rangle &= \int dk \mu(k) \hat{g}(k) |k\rangle \xrightarrow{L_{\beta}^D} \int dk \mu(k) \hat{g}(k) |k'\rangle = \\ &= \int dk \mu(k') \hat{g}(k(k')) |k'\rangle, \end{aligned} \quad (8)$$

where $\mu(k) = [2(1 - m^2) \tan \omega(k)]^{-1}$ is the density of the invariant measure in the k -space, k' is as in eq. (7), and $|k\rangle := (2(1 - m^2) \tan \omega(k))^{1/2} |\phi(k)\rangle$. One can verify that the transformation (8) is unitary. In fig. 2 we show how a perfectly localized state transforms under boosts.

Let us now deepen our analysis and consider how the features of the present framework affect the geometry of space and time. Under the action of the deformed boost L_{β}^D a function $\hat{f}(\omega, k)$ transforms as $\hat{f}'(\omega, k) = \hat{f}(\omega'(\omega, k), k'(\omega, k))$ and, following an ansatz due to Schützhold *et al.* [27], one can express the boosted function in the variables t, x by conjugating the boost L_{β}^D

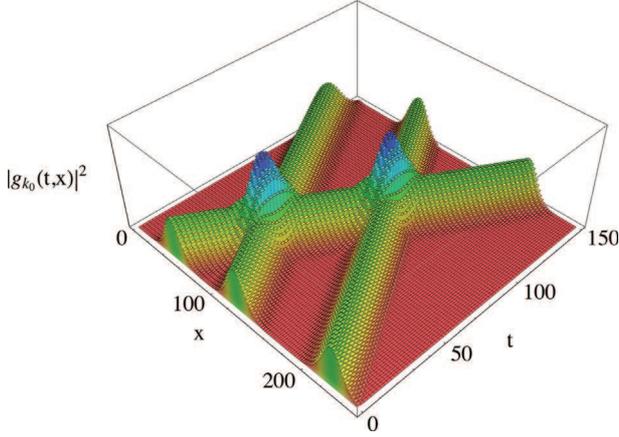


Fig. 3: (Color online) Two coincidences of travelling wave packets in the automaton evolution of eq. (2).

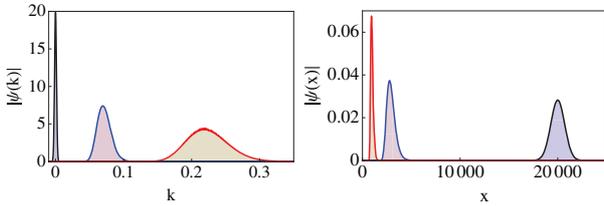


Fig. 4: (Color online) Transformation of a Gaussian state due to a boost for two different values of $\beta = -0.99, -0.999$ and $m = 0.1$ in the momentum (left) and the position (right) representations.

with the Fourier transform \mathcal{F} (see footnote ²), *i.e.*,

$$f' = \mathcal{F}^{-1} \circ L_{\beta}^D \circ \mathcal{F} f,$$

$$f'(t', x') = \sum_{x, t \in \mathbb{Z}} \int d\omega' dk' e^{-i\chi(\omega', k', x, t, x', t')} f(x, t),$$

$$\chi(\omega', k', x, t, x', t') = k(\omega', k')x - k'x' - \omega(\omega', k')t + \omega't'. \quad (9)$$

We notice that, due to the non-linearity of \mathcal{D} , the map (9) does not correspond to a change of coordinates from (t, x) to (t', x') and therefore we cannot straightforwardly interpret the variables t and x as the coordinates of points in a continuum space-time interpolating the automaton cells: this may be regarded as manifestation of the quantum nature of space-time. One can then adopt the heuristic construction of ref. [27], interpreting physically the coordinates (x, t) in terms of the mean position x at time t of a restricted class of states that can be interpreted as moving particles, namely narrow-band Gaussian wave packets moving at the group velocity. In this construction points in space-time are regarded as crossing points of the trajectories of two particles as in fig. 3 (such points have an “extension” due to the Gaussian profile). For a function $g_{k_0}(t, x)$ peaked around k_0 , the map (9) can be

²Here \mathcal{F} denotes the Fourier transform in both time and position variables: $\mathcal{F}(f)(\omega, k) = \sum_{t, x} e^{i(\omega t - kx)} f(t, x)$, $\mathcal{F}^{-1}(f)(t, x) = \int d\mu e^{-i(\omega t - kx)} f(t, x)$.

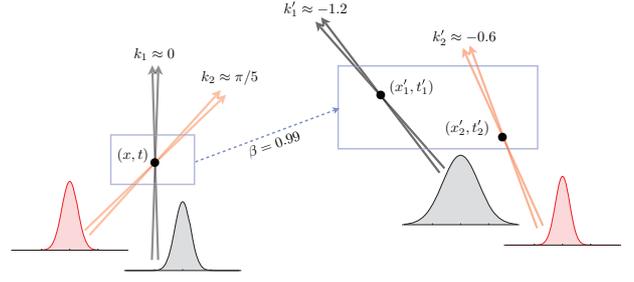


Fig. 5: (Color online) Relative locality. In the left reference frame, the joint intersection of four wave packets, the first couple having wave vector close to 0 and the second couple close to $\pi/5$, locates the point with coordinates (x, t) . In the boosted reference frame on the right, by applying the transformation of eq. (8), the four wave packets no longer intersect at the same point.

approximated by taking the first-order Taylor expansion of $k(\omega', k')$ and $\omega(\omega', k')$ respectively around k'_0 and $\omega'(k'_0)$ in the function χ . One can indeed verify that a narrow wave packet remains narrow under a boost (see fig. 4), thus confirming the validity of the linear approximation. Expanding the function χ to first order in $\omega' - \omega_0$ and $k' - k_0$ gives an expression of the following type:

$$\chi(\omega', k', x, t, x', t') \approx (x, t)J(k_0)(k', \omega')^T - (x', t')(k', \omega')^T$$

and after performing the integral in $d\omega' dk'$ one obtains the following transformations:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} \approx \begin{pmatrix} -\partial_{\omega'} k & \partial_{k'} k \\ \partial_{\omega'} \omega & -\partial_{k'} \omega \end{pmatrix}_{k'=k'_0} \begin{pmatrix} t \\ x \end{pmatrix}. \quad (10)$$

Since eq. (10) defines a linear transformation of the variables x and t and the wave packets move along straight lines, we can interpret (10) as the transformation of the coordinates x_p and t_p of a point p in space-time, namely of the intersection of the trajectories of two particles having k 's close to some common k_0 . However, the k -dependence of the transformations (10) makes the geometry of space-time observer-dependent in the following sense. Consider a point p which is given by the intersection of four wave packets, the first pair peaked around k_1 and the second pair peaked around k_2 ($k_1 \neq k_2$). Because of the k -dependence in (10), a boosted observer will actually see the first pair intersecting at a point which is different from the one where the second pair intersects (see fig. 5). This effect, first noticed in ref. [27] is the characteristic trait of the so-called *relative locality* [28,29]. The space-time resulting in such a way from the automaton dynamics is not “objective”, in the sense that events that coincide for one observer may not for another boosted observer. The above heuristic construction is in agreement with the assertion of ref. [29] that relative locality appears as a feature of all models in which the energy-momentum space has a non-flat geometry. This can be easily seen by requiring that the transformation (10) does not depend on k_0 and

remembering that for $k_0 = 0$ one must recover the usual Lorentz transformations.

In this letter we have shown that the quantum cellular automaton of refs. [24,25] provides a microscopic kinematical model compatible with the recent proposals of DSR. We obtained the non-linear representation of the Lorentz group in the energy-momentum space by assuming the invariance of the dispersion relation of the automaton. Using the arguments of ref. [27] we heuristically derived a space-time that exhibits the phenomenon of relative locality. Our analysis has been carried out in the easiest case of one space dimension, which, however, is sufficient to the analysis of the present letter. The same arguments can be easily generalized to three space dimensions using the results of ref. [25], leading to additional symmetry violations, *e.g.* rotational covariance.

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