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Joint measurements via quantum cloning

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Abstract
We explore the possibility of achieving the optimal joint measurement of noncommuting observables on a single quantum system by performing conventional measurements of commuting self-adjoint operators on the clones of the original quantum system. We consider the case of both finite- and infinite-dimensional Hilbert spaces. In the former we study the joint measurement of three orthogonal components of a spin $\frac{1}{2}$; in the latter we consider the case of the joint measurement of any pair of noncommuting quadratures of one mode of the electromagnetic field. We show that universal covariant cloning is not ideal for joint measurements, and a suitable nonuniversal covariant cloning is needed.

Keywords: Quantum cloning, measurement of noncommuting observables

1. Introduction

The first scheme for the joint measurement of noncommuting observables performed on a single quantum system was introduced by Arthurs and Kelly [1]. The problem of evaluating the minimum added noise in the joint measurement of position and momentum, and more generally of a pair of observables whose commutator is not a c-number, was then solved by Yuen [2]. A similar approach to the problem has been followed in [3]. In the case of two quadratures of one mode of the electromagnetic field the problem can be phrased in terms of a coherent POVM whose Naimark extension introduces an additional mode of the field. This kind of measurement can be realized by means of a heterodyne detector [4].

The case of the angular momentum of a quantum system is more difficult, and no measurement scheme has appeared in the literature so far. Spin coherent states [5] can be introduced and interpreted as continuous (overcomplete) POVM, but the corresponding Naimark extension is unknown. It was shown that the spin coherent POVM minimizes suitably defined quantities that represent the precision and the disturbance of the measurement [6], but explicit realizations of such a POVM are not known (see [7] and [8][3]). The joint measurement of the three components J_x, J_y and J_z of the angular momentum could also be studied with a discrete spectrum, rather than continuous. This problem does not yet have a solution. Joint measurements are a crucial ingredient in general quantum teleportation schemes [9], and are essential in connecting the quantum with the classical meaning of the angular momentum itself. Therefore, it is of great interest to find schemes that realize them.

The idea of this paper is to use quantum cloning to achieve joint measurements. It is well known that perfect cloning of unknown quantum systems is forbidden by the laws of quantum mechanics [10]. The first universal cloning machine for spin-$\frac{1}{2}$ systems was proposed in [11], and later proved to be optimal in [12]. More general universal transformations were then proposed in [13] and proved to be optimal in [14, 15]. However, if we want to use quantum cloning to realize joint measurements, we may need to optimize it for a reduced covariance group, depending on the kind of desired joint measurement. The cloning transformations mentioned above were optimized by imposing total covariance, i.e. for all possible unitary transformations. In general a restriction of the covariance group leads to a higher fidelity of the cloning transformation, as for example in the case of phase covariant cloning [16], where, however, only the bounds for the fidelity of the optimal cloning are given, and not the form of the optimal map.

In the case of finite-dimensional systems we will study the joint measurement of the three components of spin-$\frac{1}{2}$ states by operating the $1 \to 3$ universal covariant cloning on the original state and then performing independent measurements of σ_x, σ_y and σ_z on the three output copies. We will show that the resulting POVM is not optimal with respect to the added noise.

For infinite-dimensional systems it is not clear how to find the universal transformations for cloning. The extension to infinite dimension of the maps given in [15] needs a regularization procedure, an example of which is given here

[3] For spin $\frac{1}{2}$ an Arthurs–Kelly scheme has been found.

in section 4. The infinite-dimensional $1 \rightarrow 2$ cloning machine proposed in [17] is universal for coherent states, with resulting fidelity equal to 2/3. In this paper we show that the cloning transformation proposed in [17] is optimal for the joint measurement of orthogonal quadratures, and the joint measurement can be generalized to any angle between two noncommuting quadratures.

The paper is organized as follows. In section 2 we consider the case of spin-$\frac{1}{2}$ systems. We first recall the universal $1 \rightarrow 3$ cloning transformation and then exploit it to achieve joint spin measurements. In section 3 we study the case of infinite-dimensional systems, first reviewing the $1 \rightarrow 2$ transformation of [17] and then applying it to the joint measurement of two quadratures of one mode of the electromagnetic field. In section 4 we present a regularization of the map in [15] in order to extend it to infinite-dimensional Hilbert spaces, and show that the universal cloning does not achieve the optimal joint measurement. We summarize the results in section 5.

2. The finite-dimensional case: joint spin measurements

In this section we analyse the case of spin-$\frac{1}{2}$ systems, by first reviewing the universal covariant cloning which produces three output copies from a single input, and then exploiting this procedure to achieve the joint measurement of the spin components. We will show that the joint measurement obtained in this way is only an approximation of the spin measurement POVM of [5].

2.1. Universal covariant 1 → 3 cloning

We consider the case of universal cloning, namely transformations whose efficiency does not depend on the form of the input state. General $N \rightarrow M$ universal cloning transformations, which act on N copies of a pure state $|\psi\rangle$ and produce M output copies as close as possible to the input state, were proposed in [13] and later proved to be optimal in [14, 15]. We consider here the form given in [15]. The output state ρ_M of the M copies for spin-$\frac{1}{2}$ systems is given by

$$\rho_M = \frac{N+1}{M+1} S_M(|\psi\rangle\langle\psi|^{\otimes N} \otimes \mathbb{I}^{\otimes(M-N)})S_M, \qquad (1)$$

where S_M is the the projection operator onto the symmetric subspace of the M output copies. The fidelity $F(N, M) = \langle\psi|\mathrm{Tr}_{M-1}[\rho_M]|\psi\rangle$ of each output copy with respect to the initial state $|\psi\rangle$ is given by

$$F(N, M) = \frac{M(N+1)+N}{M(N+2)}. \qquad (2)$$

Since the cloning transformation is universal it can be also viewed as a shrinking transformation of the Bloch vector of each copy, described by the shrinking factor $\eta(N, M)$ [12,14]: the density operator describing the state of the M output copies is given by $\rho_{\mathrm{out}} = \frac{1}{2}[\mathbb{I} + \eta(N, M)\vec{s}_{\mathrm{in}} \cdot \vec{\sigma}]$, where $\vec{s}_{\mathrm{in}}$ denotes the Bloch vector of the initial state $|\psi\rangle$ and $\{\sigma_\alpha, \alpha = x, y, z\}$ are the Pauli operators. For the optimal transformations (1) the shrinking factor is $\eta(N, M) = \frac{N}{M}\frac{M+2}{N+2}$. In the particular case of the $1 \rightarrow 3$ cloning the above map takes the form

$$\rho_3 = \tfrac{1}{2} S_3(|\psi\rangle\langle\psi| \otimes \mathbb{I}^{\otimes 2})S_3, \qquad (3)$$

where S_3 is the projector on the space spanned by the vectors $\{|s_i\rangle\langle s_i|, i = 0\text{–}3\}$, with $|s_0\rangle = |000\rangle$, $|s_1\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$, $|s_2\rangle = 1/\sqrt{3}(|011\rangle + |101\rangle + |110\rangle)$ and $|s_3\rangle = |111\rangle$, where $\{|0\rangle, |1\rangle\}$ is a basis for each spin-$\frac{1}{2}$ system. The value of the shrinking factor in this case is $\eta(1, 3) = 5/9$.

2.2. The joint spin measurement via cloning

We will now study a method to measure jointly the three components of a spin-$\frac{1}{2}$ system by first generating three approximate copies of the input state through the cloning transformation (3), and then performing independent measurements on the three copies, namely measuring a different spin component on each copy. The POVM corresponding to the usual projection on the α-component of the Bloch vector for one copy is given by the operator $[\mathbb{I}+m_\alpha\sigma_\alpha]/2$, where $\alpha = x, y, z$ and $m_\alpha = \pm 1$ corresponds to the outcome of the measurement. The POVM $\Omega(\vec{m})$ describing the measurement of the three components, each performed on a different copy, is then given by

$$\Omega(\vec{m}) = \tfrac{1}{8}(\mathbb{I} + m_x\sigma_x) \otimes (\mathbb{I} + m_y\sigma_y) \otimes (\mathbb{I} + m_z\sigma_z), \qquad (4)$$

where the triplet $\{m_x, m_y, m_z\}$ represents the outcomes of the measurement. We will now consider the sequence of the $1 \rightarrow 3$ cloning transformation followed by the measurement of a spin component on each of the three copies as a joint measurement on the initial input state of the original copy. In order to derive the corresponding POVM we first compute the probability distribution $p(\vec{m})$ as a function of the vector $\vec{m} = \{m_x, m_y, m_z\}$

$$p(\vec{m}) = \mathrm{Tr}[\Omega(\vec{m})\, \tfrac{1}{2}\, S_3(|\psi\rangle\langle\psi| \otimes \mathbb{I}^{\otimes 2})S_3]. \qquad (5)$$

This measurement, viewed as a joint measurement on the original copy $|\psi\rangle\langle\psi|$, can then be described in terms of the POVM $\Pi(\vec{m})$

$$\Pi(\vec{m}) = \tfrac{1}{2}\mathrm{Tr}_{2,3}[S_3\, \Omega(\vec{m})\, S_3], \qquad (6)$$

where Tr_i denotes the partial trace over the ith clone. A lengthy and straightforward matrix algebra gives

$$\Pi(\vec{m}) = \tfrac{1}{8}\,[\mathbb{I} + \tfrac{5}{9}\vec{m} \cdot \vec{\sigma}]. \qquad (7)$$

Notice that the 5/9 factor in front of the Pauli operators corresponds to the shrinking factor of the optimal $1 \rightarrow 3$ cloning transformation.

We will now compute the accuracy of this joint measurement. The POVM (7) leads to the following rescaling between the measured average value $\langle\sigma_\alpha\rangle_m$ and the theoretical one for all three spin components:

$$\langle\sigma_\alpha\rangle_m = \sum_{\vec{m}} m_\alpha\, \mathrm{Tr}[|\psi\rangle\langle\psi|\, \Pi(\vec{m})] = \tfrac{5}{9}\,\langle\psi|\sigma_\alpha|\psi\rangle. \qquad (8)$$

Therefore, the unbiased estimate $\langle\sigma_\alpha\rangle_{\mathrm{e}}$ for the spin components corresponds to rescaling the measured outcome variables to $m_\alpha = \pm 9/5$, such that $\langle\sigma_\alpha\rangle_{\mathrm{e}} = \frac{9}{5}\,\langle\sigma_\alpha\rangle_m$, and the second moment is also rescaled as follows: $\langle\Delta\sigma_\alpha^2\rangle_{\mathrm{e}} = \frac{81}{25}\,\langle\Delta\sigma_\alpha^2\rangle_m$. In order to study the uncertainty of this measurement we compute the sum of the variances corresponding to the three spin components $J_\alpha = \sigma_\alpha/2$. Since $\langle\sigma_\alpha^2\rangle_m = 1$ for all the components, the uncertainty in the estimate is given by

$$\langle\Delta J^2\rangle_e = \sum_{\alpha=x,y,z} \langle J_\alpha^2\rangle_e - \langle J_\alpha\rangle_e^2 = \tfrac{1}{4}(3\tfrac{81}{25} - 1) = \tfrac{109}{50}. \quad (9)$$

Let us evaluate the corresponding accuracy for the coherent measurement [5]. The coherent POVM is given by the projection onto spin coherent states $|\boldsymbol{n}\rangle\langle\boldsymbol{n}|$ [18], where $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector and $\boldsymbol{J}\cdot\boldsymbol{n}|\boldsymbol{n}\rangle = -j|\boldsymbol{n}\rangle$. Let us calculate, as an example, the uncertainty related to the component J_z. Since the measurement is unbiased, the measured mean values of the spin components $\langle J_\alpha\rangle_m$ coincide with the theoretical mean values, and we do not need to introduce rescaling factors as we did in the previous case. The estimated values therefore coincide with the measured ones. For the component J_z one has [18]

$$\langle J_z\rangle_m = \int \mathrm{d}\mu\,(\boldsymbol{n})(j+1)\cos\theta|\boldsymbol{n}\rangle\langle\boldsymbol{n}|, \quad (10)$$

where $\mathrm{d}\mu\,(\boldsymbol{n}) = \mathrm{d}\boldsymbol{n}\,(2j+1)/4\pi$. The measured mean value of J_z^2 is given by

$$\langle J_z^2\rangle_m = \int \mathrm{d}\mu\,(\boldsymbol{n})(j+1)^2\cos^2\theta|\boldsymbol{n}\rangle\langle\boldsymbol{n}|, \quad (11)$$

that can be written as [18]

$$\langle J_z^2\rangle_m = \frac{j+1}{j+3/2}\left[\langle\psi|J_z^2|\psi\rangle + \frac{1}{2}(j+1)\right]. \quad (12)$$

The measured mean values related to the x and y components can be calculated analogously and one has the same relation as equation (12) for all components $\alpha = x, y, z$. The total uncertainty in the spin measurement then takes the form

$$\langle\Delta J^2\rangle_e = \frac{j(j+1)^2}{j+3/2} + 3\frac{(j+1)^2}{2j+3} - \sum_{\alpha=x,y,z}\langle J_\alpha\rangle_e^2 \geqslant 2j+1, \quad (13)$$

where for $j = 1/2$ and pure states the bound is achieved, and is equal to 2. This value has to be compared with equation (9), obtained by three measurements on the three cloned copies. As we can see, the joint measurement via universal covariant cloning does not achieve the minimum added noise as does the optimal POVM; however, it provides a good approximation. Notice that the minimum added noise would be achieved by a discrete POVM of the form $\Pi(\vec{m}) = \frac{1}{8}[\mathbb{I} + \vec{m}\cdot\vec{\sigma}]$.

3. The infinite-dimensional case: joint quadrature measurements

In this section we study the cloning for infinite-dimensional systems proposed in [17]. We review such $1 \to 2$ transformation and then apply it to the joint measurement of two quadratures of one mode of the electromagnetic field. We will show that the cloning transformation is optimal for joint measurements of orthogonal quadratures, and the joint measurement can be generalized to any angle between two noncommuting quadratures by suitably changing the state of the ancilla.

3.1. $1 \to 2$ cloning for continuous variables

For the following, it is convenient to introduce the formalism of heterodyne eigenvectors. Consider the heterodyne-current operator [20] $Z = a + b^\dagger$, which satisfies the commutation relation $[Z, Z^\dagger] = 0$ and the eigenvalue equation $Z|z\rangle\rangle_{ab} = z|z\rangle\rangle_{ab}$, with $z \in \mathbb{C}$. The eigenstates $|z\rangle\rangle_{ab}$ are given by [21,22]

$$|z\rangle\rangle_{ab} \equiv D_a(z)|0\rangle\rangle_{ab} = D_b(z^*)|0\rangle\rangle_{ab}, \quad (14)$$

where $D_d(z) = \mathrm{e}^{zd^\dagger - z^*d}$ denotes the displacement operator for mode d and $|0\rangle\rangle_{ab} \equiv \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-)^n|n\rangle_a|n\rangle_b$. The eigenstates $|z\rangle\rangle_{ab}$ are a complete orthogonal set with Dirac normalization ${}_{ab}\langle\langle z|z'\rangle\rangle_{ab} = \delta^{(2)}(z - z')$, $\delta^{(2)}(z)$ denoting the delta function over the complex plane. For $z = 0$ the state $|0\rangle\rangle_{ab}$ can be approximated by a physical (normalizable) state, corresponding to the output of a nondegenerate optical parametric amplifier (NOPA)—a so-called twin beam—in the limit of infinite gain at the NOPA [21].

It is also useful to evaluate the expression ${}_{cb}\langle\langle z|z'\rangle\rangle_{ab}$ which is given by

$${}_{cb}\langle\langle z|z'\rangle\rangle_{ab} = \frac{1}{\pi}D_a(z')\mathcal{T}_{ac}D_c^\dagger(z), \quad (15)$$

where $\mathcal{T}_{ac} = \sum_n |n\rangle_a\,{}_c\langle n|$ denotes the *transfer* operator [9], i.e. $\mathcal{T}_{ac}|\psi\rangle_c = |\psi\rangle_a$ for any state $|\psi\rangle$. In the following we transpose the main results of the continuous variable cloning of [17], according to the formalism just introduced. The input state at the cloning machine can be written

$$|\phi\rangle = |\varphi\rangle_c \otimes \int_{\mathbb{C}} \mathrm{d}^2z\, f(z, z^*)|z\rangle\rangle_{ab} \quad (16)$$

where $|\varphi\rangle_c$ is the initial state to be cloned, belonging to the Hilbert space $\mathcal{H}_c$, whereas $\mathcal{H}_a$ is the Hilbert space pertaining to the cloned state, and $\mathcal{H}_b$ is an ancillary Hilbert space. We do not specify for the moment the explicit form of the function $f(z, z^*)$. The cloning transformation is realized by the unitary operator

$$U = \exp[c(a^\dagger + b) - c^\dagger(a + b^\dagger)] = \exp[2\mathrm{i}(Y_c\mathrm{Re}\,Z - X_c\mathrm{Im}\,Z)] \quad (17)$$

with X_c, Y_c denoting the conjugated quadratures for mode c, namely $X_c = (c + c^\dagger)/2$ and $Y_c = (c - c^\dagger)/2\mathrm{i}$.

The unitary evolution in equation (17) can be approached experimentally by means of a network of three NOPAs under suitable gain conditions [23]. Notice the simple relation $U|z\rangle\rangle_{ab} = D_c^\dagger(z)\,|z\rangle\rangle_{ab}$. The state after the cloning transformation is given by

$$|\phi_{\mathrm{out}}\rangle = U|\phi\rangle = \int_{\mathbb{C}} \mathrm{d}^2z\, f(z, z^*)\, D_c^\dagger(z)|\varphi\rangle_c \otimes |z\rangle\rangle_{ab}. \quad (18)$$

Let us evaluate the one-mode restricted density matrix ϱ_c and ϱ_a corresponding to the state $|\phi_{\mathrm{out}}\rangle$, for the Hilbert spaces $\mathcal{H}_c$ and $\mathcal{H}_a$ supporting the two clones. For ϱ_c one has

$$\varrho_c = \mathrm{Tr}_{ab}[|\phi_{\mathrm{out}}\rangle\langle\phi_{\mathrm{out}}|] = \int_{\mathbb{C}} \mathrm{d}^2z\,|f(z, z^*)|^2 D_c^\dagger(z)|\varphi\rangle_c\,{}_c\langle\varphi|D_c(z), \quad (19)$$

where we have evaluated the trace by using the completeness and the orthogonality relation of the eigenstates $|z\rangle\rangle_{ab}$. For ϱ_a, using equation (15), one has

$$\varrho_a = \mathrm{Tr}_{cb}[|\phi_{\mathrm{out}}\rangle\langle\phi_{\mathrm{out}}|] = \int_{\mathbb{C}} \mathrm{d}^2 w\, |\tilde{f}(w, w^*)|^2 D_a^\dagger(w)|\varphi\rangle_{a\,a}\langle\varphi|D_a(w), \tag{20}$$

where $\tilde{f}(w, w^*)$ denotes the Fourier transform over the complex plane

$$\tilde{f}(w, w^*) = \int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{wz^* - w^* z} f(z, z^*). \tag{21}$$

Hence, for $f(z, z^*) = \tilde{f}(z, z^*)$ one has $\varrho_c = \varrho_a$, namely the two clones are identical. In the following we will show that the choice of the function $f(z, z^*)$ determines a criterion of optimality in terms of joint measurement of noncommuting quadratures of the original system, through independent measurements of commuting quadratures over the two clones.

3.2. The joint measurement of quadratures via cloning

Quantum cloning allows one to engineer new joint measurements of a quantum system, by suitably measuring the cloned copies. In the case of $1 \to 2$ copies just introduced, we will show that measuring two quadratures on the two clones is equivalent to the joint measurement of conjugated quadratures on the original, similarly to a heterodyne measurement. Consider the simplest case

$$f(z, z^*) = \sqrt{\frac{2}{\pi}} \exp(-|z|^2) \tag{22}$$

in equations (16), (19) and (20). One obtains $\varrho_c = \varrho_a$, namely the two clones are identical, and their state is given by the original state $|\varphi\rangle$ degraded by Gaussian noise. The state preparation $|\chi\rangle$ pertaining to the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ is given explicitly by

$$\begin{aligned} |\chi\rangle &= \sqrt{\frac{2}{\pi}} \int_{\mathbb{C}} \mathrm{d}^2 z\, \mathrm{e}^{-|z|^2} |z\rangle\rangle_{ab} \\ &= \sqrt{2\pi}\, \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{2}{3}\right)^n a^{\dagger n} a^n |0\rangle\rangle_{ab} \\ &= \frac{2\sqrt{2}}{3} \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n |n\rangle_a \otimes |n\rangle_b \\ &= \mathrm{e}^{\mathrm{atanh}\frac{1}{3}(ab - a^\dagger b^\dagger)} |0\rangle_a \otimes |0\rangle_b. \end{aligned} \tag{23}$$

One recognizes in equation (23) the twin-beam state at the output of a NOPA with total number of photons $N = \langle\chi|a^\dagger a + b^\dagger b|\chi\rangle = 1/4$, corresponding to a gain $G = 9/8$ [23]. More generally, notice that

$$\sqrt{\frac{2}{\pi\Delta^2}} \int_{\mathbb{C}} \mathrm{d}^2 z\, \mathrm{e}^{-\Delta^2|z|^2} |z\rangle\rangle = \mathrm{e}^{\mathrm{atanh}\lambda(ab - a^\dagger b^\dagger)} |0\rangle_a \otimes |0\rangle_b, \tag{24}$$

with $\lambda = (\Delta^2 - 1/2)/(\Delta^2 + 1/2)$.

Now let us evaluate the entangled state ϱ at the output of the cloning machine. After tracing over the ancillary mode b, one has

$$\varrho = \mathrm{Tr}_b[|\phi_{\mathrm{out}}\rangle\langle\phi_{\mathrm{out}}|] = \tfrac{1}{2} P_{c,a}(|\varphi\rangle_{c\,c}\langle\varphi| \otimes \mathbb{I}_a) P_{c,a}, \tag{25}$$

where $P_{c,a}$ is the projector given by

$$\begin{aligned} P_{c,a} &= \int_{\mathbb{C}} \mathrm{d}^2 z\, \frac{2}{\pi}\, \mathrm{e}^{-|z|^2} D_c^\dagger(z) \otimes D_a(z) \\ &= V \left(\int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{-\frac{1}{2}|z|^2} D_c^\dagger(z) \otimes \mathbb{I}_a \right) V^\dagger \\ &= V \left(\int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi}\, \mathrm{e}^{-zc^\dagger} \mathrm{e}^{z^* c} \otimes \mathbb{I}_a \right) V^\dagger \\ &= V(|0\rangle_{c\,c}\langle 0| \otimes \mathbb{I}_a) V^\dagger, \end{aligned} \tag{26}$$

and $V = \exp[\frac{\pi}{4}(c^\dagger a - c a^\dagger)]$ realizes the unitary transformation

$$V \begin{pmatrix} c \\ a \end{pmatrix} V^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix}. \tag{27}$$

In the last line of equation (26) a derivation similar to equation (23) has been followed. Measuring the quadratures X_c and Y_a over the two clones is then equivalent to performing a measurement on the original state $|\varphi\rangle_c$, with the measurement described by the following POVM:

$$F(x, y) = \tfrac{1}{2} \mathrm{Tr}_a[P_{c,a}|x\rangle_{c\,c}\langle x| \otimes |y\rangle_{a\,a}\langle y|P_{c,a}], \tag{28}$$

where $|x\rangle_c$ and $|y\rangle_a$ denote the eigenstates of X_c and Y_a, respectively. From the relations [22]

$$V^\dagger|x\rangle_{c\,c}\langle x|\otimes|y\rangle_{a\,a}\langle y|V = 2|\sqrt{2}(x - \mathrm{i}y)\rangle\rangle_{ca\,ca}\langle\langle\sqrt{2}(x - \mathrm{i}y)|, \tag{29}$$

$$_c\langle 0|z\rangle\rangle_{ca} = \frac{1}{\sqrt{\pi}} |z^*\rangle_a, \tag{30}$$

$$V|\alpha\rangle_c \otimes |\beta\rangle_a = |(\alpha + \beta)/\sqrt{2}\rangle_c \otimes |(\beta - \alpha)/\sqrt{2}\rangle_a \tag{31}$$

(in equations (30) and (31) the single-mode states denote coherent states) one obtains

$$F(x, y) = \frac{1}{\pi} |x + \mathrm{i}y\rangle_{c\,c}\langle x + \mathrm{i}y|, \tag{32}$$

namely the coherent-state POVM, which is the well known optimal POVM for the joint measurement of the conjugated quadratures X_c and Y_c. In fact, from equations (18)–(20), one has the following relations between the quantum expectation values $\langle\phi_{\mathrm{out}}|\cdots|\phi_{\mathrm{out}}\rangle$ over the output state $|\phi_{\mathrm{out}}\rangle$ with respect to the values $\langle\varphi|\cdots|\varphi\rangle$ over the original input state:

$$\langle\phi_{\mathrm{out}}|g(c, c^\dagger)|\phi_{\mathrm{out}}\rangle = \int_{\mathbb{C}} \mathrm{d}^2 z\, |f(z, z^*)|^2 \langle\varphi|g(c - z, c^\dagger - z^*)|\varphi\rangle, \tag{33}$$

$$\langle\phi_{\mathrm{out}}|g(a, a^\dagger)|\phi_{\mathrm{out}}\rangle = \int_{\mathbb{C}} \mathrm{d}^2 z\, |\tilde{f}(z, z^*)|^2 \langle\varphi|g(c - z, c^\dagger - z^*)|\varphi\rangle, \tag{34}$$

which holds for any function g. In particular, for $f(z, z^*)$ given by equation (22), one has

$$\langle\phi_{\mathrm{out}}|\Delta X_c^2|\phi_{\mathrm{out}}\rangle = \langle\varphi|\Delta X_c^2|\varphi\rangle + \tfrac{1}{4}, \tag{35}$$

$$\langle\phi_{\mathrm{out}}|\Delta Y_a^2|\phi_{\mathrm{out}}\rangle = \langle\varphi|\Delta Y_c^2|\varphi\rangle + \tfrac{1}{4}, \tag{36}$$

namely one achieves the simultaneous measurement of conjugated quadratures over the input state with minimum added noise [2], thus proving the optimality of the joint measurement.

The condition in order to obtain identical clones $f(z,z^*) = \tilde{f}(z,z^*)$ can be satisfied also by a bivariate Gaussian of the form

$$f(z,z^*) = \sqrt{\frac{2}{\pi}}\,\exp\left(-\frac{\mathrm{Re}^2 z}{\sigma^2} - \sigma^2\,\mathrm{Im}^2 z\right). \qquad (37)$$

In such a case the cloning transformation becomes optimal for the joint measurement of noncommuting quadratures at angles which depend on the parameter σ in equation (37). In fact, equation (25) is replaced by

$$\varrho = \tfrac{1}{2} P_{c,a}(\sigma)(|\,\varphi\rangle_{c\,c}\langle\varphi\,| \otimes \mathbb{I}_a) P_{c,a}(\sigma), \qquad (38)$$

where the projector $P_{c,a}(\sigma)$ is evaluated as follows:

$$\begin{aligned} P_{c,a}(\sigma) &= \int_{\mathbb{C}} \mathrm{d}^2 z\,\frac{2}{\pi}\,\exp\left(-\frac{\mathrm{Re}^2 z}{\sigma^2} - \sigma^2\,\mathrm{Im}^2 z\right) \\ &\quad \times D_c^\dagger(z) \otimes D_a(z) \\ &= V\,S_c(\ln\sigma)\left(\int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi}\,\mathrm{e}^{-\frac{1}{2}|z|^2}\,D_c^\dagger(z) \otimes \mathbb{I}_a\right) S_c^\dagger(\ln\sigma)\,V^\dagger \\ &= V\,S_c(\ln\sigma)\,(|\,0\rangle_{c\,c}\langle 0\,| \otimes \mathbb{I}_a)\,S_c^\dagger(\ln\sigma)\,V^\dagger \\ &= S_c(\ln\sigma) \otimes S_a(\ln\sigma)\,P_{c,a}\,S_c^\dagger(\ln\sigma) \otimes S_a^\dagger(\ln\sigma), \end{aligned} \qquad (39)$$

and $S_d(r) = \exp[r(d^{\dagger 2} - d^2)/2]$ denotes the squeezing operator for mode d

$$S_d^\dagger(r)\,d\,S_d(r) = (\cosh r)\,d + (\sinh r)\,d^\dagger. \qquad (40)$$

As in equation (28), one can evaluate the POVM that is obtained upon measuring the quadratures X_c and Y_a over the clones. From the relations for quadrature projectors

$$\begin{aligned} S_c^\dagger(\ln\sigma)\,|\,x\rangle_{c\,c}\langle x\,|\,S_c(\ln\sigma) &= \frac{1}{\sigma}|\,x/\sigma\rangle_{c\,c}\langle x/\sigma\,|, \\ S_a^\dagger(\ln\sigma)\,|\,y\rangle_{a\,a}\langle y\,|\,S_a(\ln\sigma) &= \sigma|\,x\sigma\rangle_{c\,c}\langle x\sigma\,|, \end{aligned} \qquad (41)$$

and from equations (29)–(31), one has

$$\begin{aligned} F_\sigma(x,y) &= \tfrac{1}{2}\mathrm{Tr}_a[P_{c,a}(\sigma)|\,x\rangle_{c\,c}\langle x\,| \otimes |\,y\rangle_{a\,a}\langle y\,|P_{c,a}(\sigma)] \\ &= \frac{1}{\pi} D_c(x+\mathrm{i}y)\,S_c(\ln\sigma)|\,0\rangle_{c\,c}\langle 0\,|S_c^\dagger(\ln\sigma)\,D_c^\dagger(x+\mathrm{i}y). \end{aligned} \qquad (42)$$

Equation (42) shows that the POVM is formally a squeezed state. Such a kind of POVM is optimal [2] for the joint measurement of the two noncommuting quadrature operators X_ϕ, $X_{-\phi}$, with $\phi = \mathrm{arctg}(\sigma^2)$. In fact, one has the relations

$$\int \mathrm{d}x \int \mathrm{d}y\,(x\cos\phi \pm y\sin\phi)\,F_\sigma(x,y) = X_{\pm\phi}, \qquad (43)$$

$$\int \mathrm{d}x \int \mathrm{d}y\,(x\cos\phi \pm y\sin\phi)^2\,F_\sigma(x,y) = X_{\pm\phi}^2 + \tfrac{1}{4}|\,\sin(2\phi)\,|, \qquad (44)$$

namely the outcomes $x\cos\phi \pm y\sin\phi$ trace the expectation values of the observables $X_{\pm\phi}$ respectively, with minimum added noise [2].

4. Regularization of the universal covariant cloning

In this section we give a procedure to extend the completely positive (CP) map for the universal cloning of Werner's paper [15] in the case of infinite-dimensional Hilbert space. The procedure is based on a suitable regularization in order to achieve a trace-preserving map. In particular, we will show that the universal $1 \to 2$ cloning does not provide a tool to obtain the joint measurement of noncommuting observables. Hence, we prove that Werner-type cloning and the cloning of [17] used in the previous section are different, and they are optimal for different purposes.

We rewrite here the CP map for $N \to M$ cloning given in [15]

$$T(\varrho) = \frac{\mathrm{d}[N]}{\mathrm{d}[M]}\,S_M\,(\varrho \otimes \mathbb{I}^{\otimes(M-N)})\,S_M, \qquad (45)$$

where $d[N] = \binom{d+N-1}{N}$, d being the dimension of a single-copy Hilbert space; S_M is the projector on the symmetric subspace, as mentioned in section 2, and $\varrho = |\,\psi\rangle\langle\psi\,|^{\otimes N}$ is the initial state of N identical copies in the state $|\,\psi\rangle\langle\psi\,|$. The projector S_M can be written in terms of two-site permutation operators $\Pi_{(ij)}$ (transposition), by using recursively the relation [24]

$$S_M = \frac{1}{M}\left(\mathbb{I} + \sum_{i=1}^{M-1} \Pi_{(iM)}\right) S_{M-1}. \qquad (46)$$

The permutation operator $\Pi_{(ij)}$ can be expressed on the Hilbert space $\mathcal{H}_i \otimes \mathcal{H}_j$ as [25] $\Pi_{(ij)} = \sum_n A_n \otimes A_n^\dagger$, where $\{A_n\}$ are a generic set of operators satisfying the completeness relation $B = \sum_n \mathrm{Tr}[A_n^\dagger B] A_n$. The map in equation (45) can be formally extended to infinite-dimensional Hilbert space upon using the transposition operator

$$\tilde{\Pi}_{(ij)} = \int \frac{\mathrm{d}^2\alpha}{\pi}\,D_i(\alpha) \otimes D_j^\dagger(\alpha); \qquad (47)$$

however, the trace-preserving condition on physical CP maps imposes replacement of the identity operator in equation (45) with a normalizable state. Here we suggest a regularization of $1 \to 2$ cloning in $\mathcal{H}_c \otimes \mathcal{H}_a$ by using equation (47) along with the normalizable (thermal) state $\lambda^{a^\dagger a}$, and then we write

$$\tilde{T}(\varrho) = K\,\tilde{S}_2\,(\varrho \otimes \lambda^{a^\dagger a})\,\tilde{S}_2, \qquad (48)$$

where K is a constant and

$$\tilde{S}_2 = \tfrac{1}{2}(\mathbb{I}_c \otimes \mathbb{I}_a + \tilde{\Pi}_{(ca)}). \qquad (49)$$

From the identities

$$\begin{aligned} \mathrm{Tr}_a[\tilde{\Pi}_{(ca)}] = \mathbb{I}_c, \qquad \mathrm{Tr}_c[\tilde{\Pi}_{(ca)}] = \mathbb{I}_a, \\ \tilde{\Pi}_{(ca)}\,(A \otimes B) = (B \otimes A)\,\tilde{\Pi}_{(ca)}, \end{aligned} \qquad (50)$$

and the trace-preserving condition $\mathrm{Tr}\,\tilde{T}(\varrho) = 1$, one obtains the value of K

$$K = 2\{\mathrm{Tr}[(\mathbb{I} + \varrho)\,\lambda^{c^\dagger c}]\}^{-1}. \qquad (51)$$

Notice that the dependence of K on ϱ makes the transformation in equation (48) nonlinear; however, such a nonlinear character is vanishing for $\lambda \to 1$. The regularization indeed consists in taking the limit $\lambda \to 1$. In this case the one-site restricted density matrix is given by

$$\mathrm{Tr}_1\,[\tilde{T}(\varrho)] = \mathrm{Tr}_2\,[\tilde{T}(\varrho)] = \frac{1}{2}\left(\varrho + \frac{\lambda^{c^\dagger c}}{\mathrm{Tr}[\lambda^{a^\dagger a}]}\right), \qquad \lambda \to 1, \qquad (52)$$

which generalizes the depolarizing Pauli channel to the infinite-dimensional case.

In the following we will show that, in contrast to the cloning of section 3, our regularization of Werner-type cloning does not allow us to achieve the optimal joint measurement of conjugated quadratures. In fact, similarly to equation (28), one can evaluate the POVM that corresponds to separate quadrature measurements over the two clones as follows:

$$G(x, y) = \mathrm{Tr}_c[K\, \lambda^{a^\dagger a}\, \tilde{S}_2\, |\, x\rangle_{c\,c}\langle x\, | \otimes |\, y\rangle_{a\,a}\langle y\, |\, \tilde{S}_2]. \quad (53)$$

Asymptotically, in the limit $\lambda \to 1$, one rewrites

$$\begin{aligned} G(x, y) \simeq \frac{1-\lambda}{2}(&{}_a\langle y|\lambda^{a^\dagger a}|y\rangle_a\, |\, x\rangle_{c\,c}\langle x\, | \\ &+{}_a\langle x|\lambda^{a^\dagger a}|x\rangle_a\, |\, y\rangle_{c\,c}\langle y\, | + {}_a\langle x|\lambda^{a^\dagger a}|y\rangle_a\, |x\rangle_{cc}\langle y| \\ &+{}_a\langle y|\lambda^{a^\dagger a}|x\rangle_a\, |y\rangle_{cc}\langle x|). \end{aligned} \quad (54)$$

Notice that one has

$$\begin{aligned} \int \mathrm{d}x \int \mathrm{d}y\, x\, G(x, y) &= \frac{1}{2}X_c + \frac{1-\lambda}{2}(\mathrm{Tr}[X_a\lambda^{a^\dagger a}] \\ &+\lambda^{c^\dagger c}X_c + X_c\lambda^{c^\dagger c}) \to \frac{1}{2}X_c, \end{aligned} \quad (55)$$

$$\begin{aligned} \int \mathrm{d}x \int \mathrm{d}y\, x^2\, G(x, y) &= \frac{1}{2}X_c^2 + \frac{1-\lambda}{2}(\mathrm{Tr}[X_a^2\lambda^{a^\dagger a}] \\ &+\lambda^{c^\dagger c}X_c^2 + X_c^2\lambda^{c^\dagger c}) \to \frac{1}{2}X_c^2 + \frac{1}{8}\left(1 + \frac{2\lambda}{1-\lambda}\right), \end{aligned} \quad (56)$$

and analogous expressions for integration on y. Hence, the average values of the variables x and y provide the expectation values of the quadratures X_c and Y_c (apart from the scaling factor 1/2, similar to the shrinking factor of section 2). However, one can see that the statistical error for such variables diverges for $\lambda \to 1$ since the the second moment goes to infinity.

The symmetrizer in equation (49) can be rewritten as follows:

$$\begin{aligned} \tilde{S}_2 &= \frac{1}{2}\, V\left[\mathbb{I}_c \otimes \mathbb{I}_a + \int \frac{\mathrm{d}^2\alpha}{\pi}\, D_c(\sqrt{2}\,\alpha) \otimes \mathbb{I}_a\right] V^\dagger \\ &= \tfrac{1}{2}\, V\, [\mathbb{I}_c \otimes \mathbb{I}_a + (-)^{c^\dagger c} \otimes \mathbb{I}_a]\, V^\dagger \\ &= V\left[\sum_{n=0}^{\infty} |2n\rangle_{cc}\langle 2n| \otimes \mathbb{I}_a\right] V^\dagger. \end{aligned} \quad (57)$$

This expression can be more easily compared with the projector of equation (26) that achieves the cloning transformation for the optimal joint measurement. The different action of the two projectors is clear on the basis of coherent states. One has

$$\begin{aligned} &\tilde{S}_2|\alpha\rangle_c|\beta\rangle_a \propto |\alpha\rangle_c|\beta\rangle_a + |\beta\rangle_c|\alpha\rangle_a \\ &\quad\times P_{c,a}|\alpha\rangle_c|\beta\rangle_a \propto |(\alpha+\beta)/2\rangle_c|(\alpha+\beta)/2\rangle_a, \end{aligned} \quad (58)$$

hence the operator $P_{c,a}$ indeed projects on a space that is smaller than the symmetric subspace. In fact the cloning map $\mathcal{T}(\varrho) = \frac{1}{2}P_{ca}(\sigma)(\varrho \otimes \mathbb{I}_a)P_{ca}(\sigma)$ is not universally covariant, but is covariant only under the group of unitary displacement operators, namely

$$\mathcal{T}(D(\alpha)\,\varrho\, D^\dagger(\alpha)) = D(\alpha)^{\otimes 2}\, \mathcal{T}(\varrho)\, D^\dagger(\alpha)^{\otimes 2}. \quad (59)$$

5. Conclusions

In this paper we have investigated the possibility of achieving the joint measurement of noncommuting observables on a single quantum system by means of quantum cloning. We have shown that the universal covariant cloning is not optimal for joint measurements, and a suitable noncovariant cloning is needed. Hence, different measures of quality should be used for quantum cloning, depending on what final use is to be made of the output copies. This is also indicated by recent studies of different copying machines for information transfer [26]. If we want to use quantum cloning to realize joint measurements, we need to optimize it for a suitable reduced covariance group, depending on the kind of desired joint measurement. For spin $\frac{1}{2}$—a finite-dimensional example—the universal cloning optimized by imposing total covariance [13, 15] is not optimal for the joint measurement. Also in the infinite-dimensional case, the suitably regularized universal covariant cloning does not allow us to achieve the ideal joint measurement of noncommuting observables.

A restriction of the covariance group in general leads to a higher fidelity of the cloning transformation, as in the case of phase covariant cloning [16] or for the cloning map of [17], which is not universal. The last case indeed provides a tool to perform the ideal joint measurement, as we have shown in section 3.

Regarding the experimental feasibility of the schemes of measurement presented in this paper, we want to stress that a way to implement the universal cloning was proposed in [19], with clones as indistinguishable photons, and the final measurement of the three spin components on the three output copies would correspond to nonlinear observables of radiation, whose measurement is not currently feasible. On the contrary, the infinite-dimensional case is more realistic, since the $1 \to 2$ cloning considered in section 3 can be achieved experimentally by means of a sequence of parametric amplifiers [23], and the quadrature measurements are obtained by customary homodyne detectors.

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