QUANTUM TOMOGRAPHY, TELEPORTATION, AND CLONING

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Introduction

In this paper, in a simple unifying matrix framework, I will present general classification of all possible tomography methods, teleportation schemes, and optimal quantum cloning maps. We will see how every tomographic method or teleportation scheme corresponds to a choice of operator spanning sets, and how this framework also leads to methods for engineering new Bell measurements. On the other hand, the classification of all possible covariant cloning maps (that are optimal for a given criterion) includes all known types of cloning, and leads to methods for engineering new cloning machines, which can be physically realized through unitary transformation with ancilla, and/or via probabilistic quantum operations. Fidelity criteria for POVM's can be exploited to achieve joint POVM's via cloning. I will give concrete physical realizations in the paper.

The matrix formalism

Monopartite quantum systems. In the following $\mathcal{H}$ and $\mathcal{K}$ will denote two Hilbert spaces with $\dim(\mathcal{K}) = n \geq m = \dim(\mathcal{H})$, for which we fix orthonormal *standard basis* (SB) $\{|f(i)\rangle\} \in \mathcal{H}$ and $\{|e(j)\rangle\} \in \mathcal{K}$, respectively [when there is no ambiguity we'll also use the loose notation $\{|i\rangle\}$ for either SB]. By the same matrix symbol M we'll denote: a) the $m \times n$ matrix itself $M = [M(j)] \equiv [M(1)\, M(2)\, \ldots, M(n)\,] \equiv \{M_{ij}\}$, $M(j)$ column vectors; b) the operator $\hat{M} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ from $\mathcal{K}$ to $\mathcal{H}$: $\hat{M} = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} |f(i)\rangle\langle e(j)|$ [when identifying the operator with its matrix—i.e. dropping the "hat"—we must remember that the one-to-one correspondence needs keeping the SB as fixed. Hence, when considering basis e', f' different from the SB, the operator must be written in outer-product form as $M' = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} |f'(i)\rangle\langle e'(j)|$]; c) the vector set (VS) $|M(j)\rangle \doteq \sum_{i=1}^{m} M_{ij} |f(i)\rangle \in \mathcal{H}$, in term of which the operator writes equivalently as

$$M = \sum_{j=1}^{n} \sum_{j=1}^{n} |M(j)\rangle\langle e(j)| = \sum_{i=1}^{m} \sum_{j=1}^{n} |f(i)\rangle\langle M^\dagger(i)| \, .$$

The VS M is also a new basis if $n = m = \text{rank}(M)$. When $n > m = \text{rank}(M)$ strictly, we call the (complete) VS M a *spanning set* (SS). The relative scalar product between two VS X, Y is given by $\langle x(l)|y(s)\rangle = (X^\dagger Y)_{ls}$. Orthonormal basis correspond to unitary matrices. For a generic SS C the inner product is the positive matrix $P^2 = C^\dagger C$ called *Gram matrix*. If C is a (non orthogonal) basis, one can find a *biorthogonal* or *dual* basis B, such that $\langle b(l)|c(s)\rangle = (B^\dagger C)_{ls} = \delta_{ls}$, by matrix inversion $B = (C^\dagger)^{-1}$, shortly: $|b(l)\rangle = |c^{-1\dagger}(l)\rangle$ (orthonormal sets are trivially selfdual). The completeness relation writes $C^\dagger B = \sum_l = |c(l)\rangle\langle b(l)| = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity operator in $\mathcal{L}(\mathcal{H})$. From a biorthogonal couple (C, B) one can obtain another biorthogonal couple (C', B') with the same Gram matrix as $C' = UCV^\dagger$ and $B' = UBV^\dagger$, U and V being unitary matrices. Finally, by the QR algorithm (which is based on the Gram-Schmidt orthogonalization procedure), one achieves the factorization $C = QR$, where Q is unitary and R is upper triangular (with positive diagonal), and in this way the orthogonal basis Q is extracted from the nonorthogonal SS C.

Bipartite quantum systems. We now consider a bipartite quantum system with Hilbert space $\mathcal{H} \otimes \mathcal{K}$. The SB will be denoted by $|E(h)\rangle\!\rangle \equiv |E(ij)\rangle\!\rangle = |f(i)\rangle \otimes |e(j)\rangle$, $h \equiv (ij)$ polyindex. We introduce a matrix notation[1] which exploits the isomorphism $\mathcal{H} \otimes \mathcal{K} \simeq \mathcal{L}(\mathcal{K}, \mathcal{H})$. Every vector in $\mathcal{H} \otimes \mathcal{K}$ can be written in the matrix form $|A\rangle\!\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} |f(i)\rangle \otimes |e(j)\rangle$, where $A = [a(j)]$ is a $m \times n$ matrix. Also, we have $|A\rangle\!\rangle = \sum_{j=1}^{n} |a(j)\rangle \otimes |e(j)\rangle \equiv \sum_{i=1}^{m} |f(i)\rangle \otimes |a^T(i)\rangle$ (in the following, we'll use T for transposition and $*$ for complex conjugation, both with respect to the fixed SB, e.g. by O^* we denote the operator $O^* = \sum_{i=1}^{m} \sum_{j=1}^{n} O_{ij}^* |f(i)\rangle\langle e(j)|$. Notice the following simple rules: $A \otimes B|C\rangle\!\rangle = |ACB^T\rangle\!\rangle$, where $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$, $\langle\!\langle B|A\rangle\!\rangle = \text{Tr}[B^\dagger A]$. For $\mathcal{H} \simeq \mathcal{K}$ one can also write $|A\rangle\!\rangle = A \otimes I|I\rangle\!\rangle = I \otimes A^T|I\rangle\!\rangle$, and $\langle\!\langle B|A\rangle\!\rangle = \langle\!\langle I|A \otimes B^*|I\rangle\!\rangle$. We also have the rules for partial traces $\text{Tr}_{\mathcal{K}}[|A\rangle\!\rangle\langle\!\langle B|] = AB^\dagger \in \mathcal{L}(\mathcal{H})$ and $\text{Tr}_{\mathcal{H}}[|A\rangle\!\rangle\langle\!\langle B|] = A^T B^* \in \mathcal{L}(\mathcal{K})$. To a biorthogonal SS $\sum_l |c(l)\rangle\!\rangle\langle\!\langle b(l)| = I_{\mathcal{H}\otimes\mathcal{K}}$ it will correspond a biorthogonal SS of operators $\{b(l), c(l)\} \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that every operator $A \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ can be expanded as $A = \sum_l \text{Tr}[b^\dagger(l)A]c(l)$. The completeness and biorthogonality of the SS is equivalent to the so-called *orthogonality relations* $\sum_l b^\dagger(l)_{pq} c(l)_{rs} = \delta_{ps}\delta_{qr}$, i.e. $P = \sum_l b^\dagger(l) \otimes c(l)$ is the operator that permutes the two Hilbert spaces in the tensor product $\mathcal{H} \otimes \mathcal{K}$. The orthogonality relations lead to identities of the form $\sum_l b^\dagger(l) A c^T(l) = A^T$ and its transposed/dagger-ed, and, for $\mathcal{H} \simeq \mathcal{K}$ one also has $\sum_l b^\dagger(l)Ac(l) = \mathrm{Tr}[A]\,I$, and $\sum_l b^\dagger(l) \otimes c^T(l) \equiv |I\rangle\rangle\langle\langle I|$ and transposed/dagger-ed relations. Analogously to the monopartite case, changes of basis are in correspondence with matrices (tensors), e.g. $|c(l)\rangle\rangle = \sum_h c_{hl}|E(h)\rangle\rangle = \sum_{ij} c(l)_{ij}|E(ij)\rangle\rangle$, $h \equiv (ij)$ polyindex. One can see that finding the dual SS $\{b(l)\}$ of $\{c(l)\}$ resorts to the matrix-tensor inversion $B = (C^\dagger)^{-1}$, where $C = [c(l)] \equiv \{c_{hl}\}$, h polyindex [here, we consider $\mathcal{H} \simeq \mathcal{K}$ for simplicity: the generalization to different spaces is trivial, i.e. inverting the rank=m square part of the rectangular matrix and/or adding orthogonal vectors]. As in the monopartite case one can use the QR algorithm to factorize the matrix C and find an orthogonal basis Q of operators. Notice that an orthogonal basis $\{|q(l)\rangle\rangle$ will correspond to unitary matrix-tensors $Q = [q(l)]$, however the operators $q(l)$ are not unitary. >From the partial traces $\mathrm{Tr}_{\mathcal{H}}[|q(i)\rangle\rangle\langle\langle q(j)|] = q(i)q(j)^\dagger$ and $\mathrm{Tr}_{\mathcal{K}}[|q(i)\rangle\rangle\langle\langle q(j)|] = (q(j)^\dagger q(i))^T$, we see that choosing *maximally entangled* vectors (i.e. with partial trace proportional to the identity), gives $\sqrt{m}q(i)$ unitary (obviously for $\mathcal{H} \simeq \mathcal{K}$), with the additional orthogonality condition for the set $\mathrm{Tr}[q(i)q(j)^\dagger] = \delta_{ij}$. We'll call such operator set *unitary spanning sets*. Notice that in the present matrix formalism the *Schmidt form* $|A\rangle\rangle = \sum_{l=1}^k \lambda_l |v(l)\rangle|w^*(l)\rangle$, with $\lambda_l > 0$, $\sum_{l=1}^k \lambda_l^2 = 1$, $\{|v(l)\rangle\}$, $\{|w^*(l)\rangle\}$ orthonormal sets, is noting but the so-called *singular value decomposition* $A = V\Sigma W^\dagger$ of the matrix A, with V and W unitary, and $\Sigma = [diag(\lambda_l), 0]$, $k \equiv$rank(A) the *Schmidt number*. Other forms of the bipartite vector are related to other matrix decompositions, as, for example $|A\rangle\rangle = \sum_{i=1}^m |p(i)\rangle|u^T(i)\rangle$, with $P \geq 0$, $P^2 = AA^\dagger$, $UU^\dagger = I_m$ is just the *polar decomposition* $A = PU$ of the matrix A.

Group representations. A simple way to construct unitary SS is to consider a (generally projective) unitary irreducible representation (UIR) of a group $\mathbf{G}$. We'll denote by $[\mathbf{G}, U, \mathcal{H}]$ the unitary representation of $\mathbf{G}$ on $\mathcal{H}$, $U(g)$ being the unitary representative of the group element $g \in \mathbf{G}$. Then, an operator SS $\{u(g)\} \in \mathcal{L}(\mathcal{H})$ is simply $u(g) \propto U(g)$, with the proportionality constant depending on the group representation. This is just a consequence of the orthogonality relations $\int \mathrm{d}g\, U(g) \otimes U^\dagger(g) = P$, which follow from the Schur lemmas, $\mathrm{d}g$ denoting an invariant Haar measure.[1] Corresponding to the operator SS the vectors $|u(g)\rangle\rangle$ will constitute an orthonormal basis of maximally entangled vectors, e. g. a Bell POVM on $\mathcal{H} \otimes \mathcal{H}$ [in the infinite dimensional case we can keep the vectors $|u(g)\rangle\rangle$ maximally entangled with Dirac-delta normalization]. Such Bell POVM will obviously be covariant under $[\mathbf{G}, U, \mathcal{H}]$.

Completely positive maps. The matrix formalism for entangled vectors can be used to exploit the one-to-one correspondence $\mathcal{E} \leftrightarrow R_{\mathcal{E}}$ between completely-positive (CP) maps $\mathcal{E} \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$, and operators $0 \leq R_{\mathcal{E}} \in$

$\mathcal{L}(\mathcal{H}\otimes\mathcal{K})$[2]. The correspondence is given by $R_{\mathcal{E}} = \mathcal{E}\otimes\mathcal{I}(|I\rangle\!\rangle\langle\!\langle I|) \geq 0$, and inversely $\mathcal{E}(\rho) = \mathrm{Tr}_2[I\otimes\rho^T R_{\mathcal{E}}]$, $\mathcal{I}$ denoting the trivial CP-map. In the following we'll call $R_{\mathcal{E}}$ the "R-matrix" or "R-operator" of $\mathcal{E}$. In this fashion the Krauss decomposition of the CP-map $\mathcal{E}(\rho) = \sum_l K_l\rho K_l^\dagger$ is just the diagonal form of its R-operator $\mathcal{E} = \sum_l |K_l\rangle\!\rangle\langle\!\langle K_l|$, and the trace-preserving condition $\sum_l K_l^\dagger K_l = I_{\mathcal{K}}$ becomes the partial-trace normalization $\mathrm{Tr}_{\mathcal{K}}[R_{\mathcal{E}}] = I_{\mathcal{H}}$. Similarly, Stinespring dilations of $\mathcal{E}$ are nothing but purifications of the R-operator in an extended Hilbert space. In general, all various matrix forms for the R-matrix will lead to different forms for the CP-map. The dual CP-map $\mathcal{E}^\vee \in \mathcal{L}(\mathcal{L}(\mathcal{K}),\mathcal{L}(\mathcal{H}))$ e.g. $X\to\mathcal{E}^\vee(X)$, $X\in\mathcal{L}(\mathcal{K})$ is obtained as $\mathcal{E}^\vee(X) = \mathrm{Tr}_1[X\otimes IR_{\mathcal{E}}^{T_2}]$, T_2 denoting partial transposition on the second Hilbert space, and we'll consistently call $R_{\mathcal{E}}^{T_2}$ the "dual R-matrix".

A main virtue of the R-matrix representation resides in the possibility of classifying CP-maps via the Cholesky decomposition of R-matrices, and, in particular, classifying group-covariant CP-maps in terms of group-invariant R-matrices [3], thus resorting to a simple application of the Schur lemma. We call the CP-map $\mathcal{E}$ G-covariant, when $\mathcal{E}(U_g\rho U_g^\dagger) = V_g\mathcal{E}(\rho)V_g^\dagger$, U_g and V_g being unitary representatives of $g\in\mathbf{G}$ over $\mathcal{H}$ and $\mathcal{K}$, respectively. Then, $\mathcal{E}$ is G-covariant, iff $[R_{\mathcal{E}}, U_g\otimes V_g^*] = 0$. An operator $Q\in\mathcal{L}(\mathcal{K},\mathcal{H})$ is invariant under $[\mathbf{G},U,\mathcal{H}]$ iff $Q\equiv[Q_0]_U^{\mathbf{G}}$, where $[Q]_U^{\mathbf{G}} \doteq \int \mathrm{d}g\, U_g^\dagger Q U_g$ denotes group averaging. Then, one can prove that the general form of the R-matrix of the G-covariant CP-map is $R_{\mathcal{E}} = A_{\mathcal{E}}^\dagger A_{\mathcal{E}}$, with $A_{\mathcal{E}}$ (complex lower triangular) matrix itself invariant, which can be written in the block form $A_{\mathcal{E}} = [A_0]_{UV^*}^{\mathbf{G}} \equiv \oplus_\nu a_\nu$, each block a_ν comprising a full set of equivalent irreducible components of the representation.

The R-representation of CP maps turns out to be very useful for engineering new quantum channels and new POVM's. For example, approximating a new POVM $n(l)$ starting from a given one $o(l)$ via a CP-map, is equivalent to maximizing the POVM-fidelity $F = \mathrm{Tr}[(n(l)\otimes o(l))R_{\mathcal{E}}^{T_2}]$ (or other measure of distance between operators) over all possible R-matrices $R_{\mathcal{E}}$. In particular, for G-covariant POVM's, i.e. $n(g) = U_g^\dagger n(e)U_g$ and $o(g) = V_g^\dagger o(e)V_g$ (e the identity element of G) a CP-map covariant under $[\mathbf{G}, UV^*, \mathcal{H}]$ must be used, and the fidelity only between the POVM's "seeds" $F = \mathrm{Tr}[n(e)\otimes o(e)R_{\mathcal{E}}^{T_2}]$ needs to be optimized, which is perfectly analogous to the conventional fidelity for states.

Tomography, Teleportation, and Cloning

Quantum Tomography. Quantum tomography is a method to estimate the ensemble average $\langle A\rangle$ of any linear operator $A\in\mathcal{L}(\mathcal{H})$ by using only measurement outcomes of *quorum* of observables $\{c(l)\}$. One can immediately recognize that a *quorum* is nothing but a SS $\{c(l)\}$ made of observables[2]. Then, the tomographic estimation of the ensemble average $\langle A\rangle$ is obtained as the double average—over both the ensemble and the quorum—of the *unbiased estimator*[3] $\mathrm{Tr}[b^\dagger A]c(l)$, namely $\langle A\rangle = \sum_l \mathrm{Tr}[b^\dagger(l)A]\langle c(l)\rangle$. Hence, of the biorthogonal couple, the $c(l)$'s are the *quorum*, and the $b(l)$'s give the unbiased estimator.[4] Examples of applications are given in Table 1.

	l	$\mathrm{d}\mu(l)$	$\mathcal{D}$	$c(l), b(l)$
1a	α	$\mathrm{d}^2\alpha$	$\mathbb{C}$	$\pi^{-1/2}D(\alpha)$
1b	(k,ϕ)	$\mathrm{d}k\|k\|\mathrm{d}\phi/(4\pi)$	$\mathbb{R}\times[0,\pi]$	$\exp(ikX_\phi)$
1c	(x,ϕ)	$\mathrm{d}x\mathrm{d}\phi/\pi$	$\mathbb{R}\times[0,\pi]$	$E_x^\phi, -\frac{1}{2}\mathrm{P}\frac{1}{(x-X_\phi)^2}$
2	α	$\mathrm{d}^2\alpha$	$\mathbb{C}$	$\pi^{-1/2}(-)^{a^\dagger a}D(2\alpha)$
3	(x,t)	$\mathrm{d}x\,\mathrm{d}t$	$\mathbb{R}\times\mathbb{R}$	$e^{-ip^2t}\|q\rangle\langle q\|e^{ip^2t}$
4	$(\psi,\vec{n})$	$\frac{2J+1}{8\pi}\mathrm{d}\vec{n}\,\mathrm{d}\psi\,\sin^2\frac{\psi}{2}$	$S^2\times S^1$	$\exp(i\psi\vec{J}\cdot\vec{n})$
5	ν	1	$\{x,y,z,t\}$	σ_ν
6	(n,ϕ)	$\mathrm{d}\phi/(2\pi)$	$\mathbb{Z}\times S^1$	$e_{s(n)}^{\|n\|}e^{ia^\dagger a\phi}$
7	(ψ,ϕ)	$\mathrm{d}\phi\mathrm{d}\psi/(4\pi^2)$	$S^1\times S^1$	$e^{-ia^\dagger a\psi}\|e^{i\phi}\rangle\langle e^{i\phi}\|e^{-ia^\dagger a\psi}$

Table 1 Examples of spanning sets used in different tomographic schemes. $\mathcal{D}$ denotes the domain of the index l, $\mathrm{d}\mu(l)$ the integration measure for continuous l. The dual basis $b(l)$ is not specified when $c(l)$ is self adjoint or unitary. Examples 1 are various versions of quantum homodyne tomography[5]. E_x^ϕ denotes the projector over the quadrature eigenvector at phase ϕ, with eigenvalue x. Example 2 is the parity photon-counting tomography over displaced states or observables [6, 7]. Example 3 is the tomography of a free particle[7], p and q denoting momentum and position respectively, and t the time. Example 4 is the angular momentum tomography[4]. Example 5 is the Pauli tomography[4]. Example 6 and 7 are the nonlinear and the nonunitary phase-tomographies of Ref. [7] (both have a quorum of generalized observables, i.e. nonorthogonal POVM's). $|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$ denotes the Susskind-Glogower phase vector, $e_\pm$ the raising/lowering photon operators, $s(n)$ is the sign of n. Other examples, as the multimode homodyne tomography, can be found in Ref. [4]. For more details the reader is addressed to the extensive literature on the subject.

Quantum Teleportation. In a general teleportation scheme, Alice and Bob share the entangled pure state $|A\rangle\!\rangle\langle\!\langle A|$ on the Hilbert space $\mathcal{H}_2\otimes\mathcal{H}_3$, while Alice performs a joint measurement on $\mathcal{H}_1\otimes\mathcal{H}_2$, corresponding to the POVM $|q(h)\rangle\!\rangle\langle\!\langle q(h)|$. The state that emerges at $\mathcal{H}_3$ at the Bob side after the Alice measurement is given by $\rho_l = \mathrm{Tr}_{12}[(\rho)_1\;(|A\rangle\!\rangle\langle\!\langle A|)_{23}\;(|q(h)\rangle\!\rangle\langle\!\langle q(h)|)_{12}] = A^* q^\dagger(h)\,\rho\, q(h) A^T$, for measurement outcome l. Alice transmits l to Bob, and if the quantum operation $A^* q^\dagger(l)$ is invertible[8]—as if unitary, when both $|A\rangle\!\rangle\langle\!\langle A|$ and $|q(h)\rangle\!\rangle\langle\!\langle q(h)|$ are maximally-entangled (the POVM is Bell)—then Bob can apply the inverse operation of $A^* q^\dagger(l)$ and recover the original state. *E voilà*: that's all! In Ref. [9] the teleportation schemes corresponding to SS from UIR of groups were presented, showing that all known teleportation schemes correspond to abelian groups. The original teleportation scheme [10] for one qubit corresponds to the projective UIR of the dihedral group D_2. The generalization to d dimension of the same Ref. [10] is the projective UIR of the group $\mathbb{Z}_d \times \mathbb{Z}_d$.[5] The continuous variables teleportation of Ref. [11] is an example of infinite dimensional teleportation, with the group being the abelian group of translations over the complex plane, with projective UIR given by the Weyl-Heisenberg group of displacement operators $D(z)$.[6] This last case is particularly interesting, since from it we can learn a general procedure to engineer Bell measurements from local measurements. Here the shared resource (from parametric downconversion) cannot be maximally entangled due to the infinite dimension of the Hilbert space, and the output state comes out distorted as $D_A^\dagger(z)\,\rho\,D_A(z)$, where $D_A(z) = D(z)\,A\,D^\dagger(z)$. The Bell measurement is achieved through the global unitary transformation $V = \exp\left(a^\dagger b - ab^\dagger\right)$ performed by the beam splitter over local (i.e. single-mode) homodyne joint measurements, resulting in $\frac{1}{\sqrt{\pi}}|D(x+iy)\rangle\rangle = V|2^{-1/2}x\rangle_0 \otimes |2^{-1/2}y\rangle_{-\pi/2}$, where $|v\rangle_\phi$ denotes an eigenvector of the quadrature operator $X_\phi = \frac{1}{2}\left(a^\dagger e^{i\phi} + ae^{-i\phi}\right)$ with eigenvalue v. Perfect Teleportation is achieved in the limit of vanishing distortion for $A_{ij} = \lim_{\xi\to 1}(1-|\xi|^2)^{1/2}\xi^j\delta_{ij}$. The present Bell measurement scheme maybe generalized to arbitrary quantum system by devising a single global unitary operator which achieves the transformation $V|i\rangle \otimes |j\rangle = |q(ij)\rangle\rangle$ from the local observable $|i\rangle \otimes |j\rangle$ to the Bell one $|q(ij)\rangle\rangle$, where $\{q(h)\}$ $h = (ij)$ is a unitary SS (i.e. maximally entangled). Modulo local unitary transformations, this resort to find the solution V of the factorization equation $V^\dagger\,q(ij) \otimes I\,V = m^{-1/2}U(i) \otimes U(j)$, where $U(ij) = \sqrt{m}q(ij)$, m =dim($\mathcal{H}$) (we choose $U(0,0) = I$), and the unitary local operators $U(j)$ connect the local POVM to a fixed reference vector $|0\rangle$ as $|j\rangle = U(j)|0\rangle$. For a single qubit a solution is given by the phase-shift operator $V = \exp\left(i\frac{\pi}{4}\sigma_y \otimes \sigma_y\right)$. For arbitrary quantum system, the general solution is unknown.

Quantum Optimal Cloning[3]. A quantum cloning map $\mathcal{C}$ from N to M copies is just a CP-map in $\mathcal{L}(\mathcal{L}(\mathcal{H}^{\otimes N}), \mathcal{L}(\mathcal{H}^{\otimes M}))$ which outputs a permutation-invariant state, i.e., invariant under S_M (which doesn't necessarily mean that the state is Bose or Fermi). A G-covariant cloning is a quantum cloning with R-matrix invariant under $U_g^{\otimes N} V_g^{*\otimes N}$. We call the quantum cloning optimal when it satisfies some optimality condition, typically it minimizes a given cost-function. The cloning which is covariant under the full unitary group $U(m)$, m =dim($\mathcal{H}$), and minimizes the fidelity $F = \mathrm{Tr}[\sigma^{\otimes M}\mathcal{C}(\sigma^{\otimes N})]$ for pure states σ is the universal cloning of Werner[12]. While proving optimality, one can also see that here the output state $\mathcal{C}(\sigma^{\otimes N})$ is bosonic. A $N = 1 \to M = 2$ cloning which achieves the optimal joint measurement of two conjugated quadratures of radiation (analogous of position and momentum of a particle) is the cloning of Cerf[13], which is covariant under the WH group. It minimizes the seed POVM fidelity $F = \mathrm{Tr}[(E_0^0 \otimes E_0^{\pi/2} \otimes v) R_C^{T_3}]$, where the vacuum state v "seeds" the coherent state POVM (optimal joint measurement), which is WH-covariant, and $E_0^0 \otimes E_0^{\pi/2}$ seeds the joint local measurements of commuting quadratures over clones under the joint displacement $D^{\otimes 2}$. Stinespring dilations of the cloning map $\mathcal{C}$ in the form $\mathcal{C}(\rho) = S^\dagger(\rho \otimes I_\mathcal{A})S$ for $\rho \in \mathcal{L}(\mathcal{H}^{\otimes N})$ for some ancillary Hilbert space $\mathcal{A}$, with $S \in \mathcal{L}(\mathcal{H}^{\otimes N}, \mathcal{H}^{\otimes N} \otimes \mathcal{A})$ generally not unitary, naturally provide quantum operations which physically achieve the cloning. For the Cerf cloning map[13] ($N = 1 \to M2$) one has $\mathcal{C}(\rho) = S(\rho \otimes I)S$, with $S = \frac{1}{\sqrt{2}} V(v \otimes I)V^\dagger$, V being the beam splitter transformation given above. This map can be achieved either probabilistically or unitarily (on a further extended Hilbert space). The probabilistic way is mimicked by just a couples of beam splitters in a row [14], with an ancillary chaotic input radiation (which imitates the identity in the Stinespring extension). A unitary realization is given in Ref. [15] through a network of three parametric amplifiers. Extensions of the probabilistic map to generic N and M can be obtained using multi-splitters and chaotic ancillary radiation, however the efficiency is going down exponentially vs M (for example, for $N = 1$ and large M it goes asymptotically as $M\overline{n}^{1-M}$ [14], $\overline{n}$ being the thermal photon number of the ancillary state).

Notes

1. Throughout the paper, $\int dg$ will represent a discrete sum for G discrete.

2. We call *observable* a generally complex operator with orthonormal spectral resolution, or, in other words, $c(l)$ is a generally complex function of a single self adjoint operator

3. Notice that the general method of noise deconvolution given in Ref. [4]—where the deconvolved estimator is achieved by evaluating the inverse CP-map of the noise over the estimator—is just equivalent to finding the biorthogonal basis of the noisy quorum.

4. The *quorum* obtained with the method of operator SS can be used also for quantum tomographic strategies different from the averaging one, e.g. the maximum-likelihood strategy. This method works only for the estimation of the density matrix itself (or for any set of unknown parameters of the density matrix), and it is restricted to finite dimensions. The likelihood function is just $L(\rho) = \sum_{j=1}^N \log \mathrm{Tr}[\rho p(l_j)(x_j)] - N\mathrm{Tr}(\rho)$, where j runs over the N measurements in the sample, $p(l)(x)$ denotes the (noisy) POVM corresponding to the l-th element of the quorum, with outcome x, and the search of the maximum is made over the parameters θ which parameterize the density matrix $\rho = \rho_\theta$. A full reconstruction of the density matrix can be achieved with the Cholesky parametrization $\rho = \tau^\dagger \tau$, searching over matrix elements (real on the diagonal) of the lower-triangular matrix τ, through a downhill simplex method.

5. $\mathbb{Z}_d \times \mathbb{Z}_d$ denotes the abelian group of discrete translations over a lattice embedded in a torus. The m-dimensional projective UIR is given by: $Z(j,l) = \sum_k e^{2\pi i k j/m} |k\rangle\langle k \oplus l|$, $Z(l,j)\, Z(l',j') = e^{2\pi i j l'/m} Z(l \oplus l', j \oplus j')$; The particular case for $m = 2$ is the dihedral group D_2 of π-rotations around three perpendicular axes, with the projective representation isomorphic to the nonabelian group of the three Pauli matrices plus identity. More explicitly, one has $U(0,0) = I$, $U(0,1) = \sigma_x$, $U(1,0) = \sigma_z$, $U(1,1) = i\sigma_y$.

6. The abelian group of translations over the complex plane, with projective UIR given by the Weyl-Heisenberg group (WH) of displacement operators $(z \in \mathbb{C})$: $D(z) = \exp(za^\dagger - z^* a)$, $D(z)D(w) = D(z+w)e^{\frac{1}{2}(zw^* - z^* w)}$.

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