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Coherent States and Infinite-Dimensional Lie Algebras: an Outlook (*) (**).

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Summary. — The possible extension of the notion of generalized coherent state to the case of infinite-dimensional affine Lie algebras is discussed with special attention to the resulting topological structure of the coherent states manifold, and to its connection with the structure of the algebra. The relevance for the solution of nonlinear dynamical systems equations of motion is briefly reviewed.

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The generalized coherent states associated with an arbitrary Lie group G [1,2] constitute an overcomplete set of quantum states, labelled by a point in a Kählerian manifold $\mathscr{M}$ homogeneous under the action of G.

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(**) To speed up publication, the authors have agreed not to receive proofs which have been supervised by the Scientific Committee.
(1) M. RASETTI: *Int. J. Theor.*, 13, 425 (1973).
(2) A. M. PERELOMOV: *Commun. Math. Phys.*, 26, 222 (1972).

Typically $\mathscr{M}$ is the homogeneous factor space quotient of G by the stability subgroup K leaving some vector $|\omega\rangle$ in the Hilbert space of states $\mathscr{H}$ fixed.

If G is a connected, simply connected nilpotent Lie group, its representations can all be obtained by Kirillov's coadjoint orbit method (3), and $\mathscr{M}$ can be identified with the symplectic manifold constituted by an orbit of the coadjoint representation.

In fact, if $\mathfrak{g}^*$ is the space dual to the Lie algebra $\mathfrak{g}$ of G, namely the space of linear functionals on $\mathfrak{g}$, then G—which acts in $\mathfrak{g}$ by the adjoint representation $\mathrm{Ad}(g)$, $g \in G$—acts in $\mathfrak{g}^*$ by the coadjoint representation $\mathrm{Ad}^*(g)$, and $\mathfrak{g}^*$ is foliated into orbits under such an action.

It was shown by Kirillov that any homogeneous symplectic manifold homogeneous with respect to the action of G is locally isomorphic with an orbit of the coadjoint representation of G; moreover, it derives from the Borel-Weil-Bott theory that these orbits are, for compact semi-simple Lie groups, in one-to-one correspondence with nonequivalent unitary representations of the group itself.

Let us further recall that generalized coherent states are defined by

$$|\zeta\rangle \equiv |\zeta_g\rangle = \pi^{-1}(g)\,T(g)\,|\omega\rangle\,, \tag{1}$$

where $T(g)$, $g \in G$, are the holomorphic representations of the complexification of G, and $\pi(g)$ is a holomorphic character for all g's in the coset labelled by the point $\zeta \in G/K \sim \mathscr{M}$; and that $\mathscr{M}$ is locally isomorphic to $\boldsymbol{C}^N$ for some integer $N \geq 1$.

On the other hand, if G is the dynamical group of a physical system, whose space of states is $\mathscr{H}$, then the time evolution of a state initially represented by a point $\zeta_0 \in \mathscr{M}$ is a path entirely embedded in $\mathscr{M}$ (more precisely, G needs not to be strictly a dynamical group; what is required is that the system Hamiltonian H be coherence preserving (4) for the coherent states of G).

The quantum evolution path in $\mathscr{M}$ is defined by the Feynman propagator

$$\langle \zeta'' t'' | \zeta' t' \rangle = \langle \zeta'' | \exp\left[-\frac{i}{\hbar}(t''-t')H\right] | \zeta' \rangle\,, \tag{2}$$

which can be written in the path integral form

$$\langle \zeta'' t'' | \zeta' t' \rangle = \int \mathscr{D}[\zeta(t)] \exp\left[\frac{i}{\hbar} S[\zeta(t)]\right] \tag{3}$$

with the action functional given by

$$S[\zeta(t)] \equiv \int_{t'}^{t''} L\,\mathrm{d}t = \int_{t'}^{t''} \langle \zeta(t) | (i\hbar\,\partial_t - H) | \zeta(t) \rangle\,\mathrm{d}t\,. \tag{4}$$

(3) A. A. KIRILLOV: *Elements of the Theory of Representations* (Springer-Verlag, Berlin, 1975), Chapt. 15.

(4) G. D'ARIANO, M. RASETTI and M. VADACCHINO: *J. Phys. A*, 18, 1295 (1985).

Since $\mathcal{M}$ is symplectic, a set of local charts of canonical coordinates for the orbit can be constructed, $\zeta := \{\zeta_i | i = 1, ..., N\}$, and the Lagrangian L defined by the above equation reads

(5) $$L = \frac{1}{2} i\hbar \sum_{i=1}^{N} \{(\zeta_i \partial_{\zeta_i} - \bar{\zeta}_i \partial_{\bar{\zeta}_i}) \ln \langle \zeta | \zeta \rangle - \langle \zeta | H | \zeta \rangle\},$$

where $F(\zeta, \bar{\zeta}) := \frac{1}{2} \ln \langle \zeta | \zeta \rangle$ is the Kähler potential.

$\mathfrak{h}(\zeta, \bar{\zeta}) = \langle \zeta | H | \zeta \rangle$ can then be viewed as the Hamiltonian generating a classical Lagrangian flow on a phase space which is $\mathcal{M}$ itself, equivalent to the quantum dynamics in the following sense: the quantum time evolution described by the propagator $\langle \zeta'' t'' | \zeta' t' \rangle$ coincides with the classical flow induced by $\mathfrak{h}$ on $\mathcal{M}$ (stationary-phase approximation of the path-integral in the functional integral representation of the propagator leads in fact to the Euler-Lagrange equations for the representative point of the coherent state $\zeta(t)$ generated by $\mathfrak{h}(\zeta, \bar{\zeta})$). Notice that, in general, $\mathfrak{h}(\zeta, \bar{\zeta})$ is a nonlinear functional and the coherent state representative, which corresponds to the holomorphic section of the line bundle associated with the complexified principal fibre bundle $K \to G \to G/K \to \mathcal{M}$ by the character π induced by some element $g_0 \in \mathfrak{g}$, can be viewed as (a class of) solutions to the equations of motion for the nonlinear classical dynamical system such that $\exp[g_0]|\omega\rangle$ determines at which point of each fibre of the fibre bundle over G/K induced by the Lie subalgebra associated with K the state representative should be taken (5).

The whole scheme can be extended to the case of infinite-dimensional Lie algebras in different ways. We shall briefly discuss here the extension which can be most naturally surmised from the dynamical picture described above, for the case—particularly interesting in terms of physical applications—of affine Lie algebras.

Indeed, just in view of its geometrical and dynamical meaning, the group G can be thought of as the semi-direct product of an Abelian subgroup $\mathcal{A}$ (translations) by a diffeomorphism group D:

(6) $$G = \mathcal{A} \otimes D,$$

Since D operates faithfully on $\mathcal{A}$, $\mathcal{A}$ is a maximal Abelian subgroup of G, and the following extension exists:

(7) $$E: 0 \to \mathcal{A} \xrightarrow{i} G \xrightarrow{\pi} D \to 1$$

(where we denoted the group composition in 0 and $\mathcal{A}$ as addition, that in D and 1 as multiplication).

(5) G. D'ARIANO and M. RASETTI: *Phys. Lett. A*, 107, 291 (1985).

The exactness of the sequence E implies that the monomorphism $\varkappa$ maps $\mathcal{A}$ isomorphically onto a normal subgroup of G, and that σ induces an isomorphism $G/\varkappa\mathcal{A} \sim D$ of the corresponding quotient group. A thorough analysis of the cohomology that E induces(6) leads to the construction of a group $\mathcal{S}$ of automorphisms of $\mathcal{M}$ which can be embedded as subgroup into G. $\mathcal{S}$ is such that any other subgroup $\mathcal{S}'$, generating $\mathcal{A}$ as a real vector space isomorphic to $\boldsymbol{R}^{2N}$, is isomorphic to $\mathcal{S}$ and indeed conjugate to it in the affine group $A(2N)$.

We are thus led to affine Lie algebras as the natural candidates on which to perform the desired generalization.

There are several constructions available of the affine algebras. The central extension of a loop algebra (7) is the simplest and we review it, even though it yields only the direct (or untwisted) algebras.

The loop algebra construction starts with a finite-dimensional simple Lie algebra $\mathfrak{g}$, generalizes it to its loop algebra $L\mathfrak{g}$ and then adjoins to $L\mathfrak{g}$ a central element C: the result is an affine algebra $\hat{\mathfrak{g}}$. A basis for $L\mathfrak{g}$ is obtained by multiplying each basis element of $\mathfrak{g}$ by t^n, $t \in \boldsymbol{C}$ an indeterminate and n any integer. Thus $L\mathfrak{g}$ is an infinite-dimensional algebra with elements $t^n X$, $X \in \mathfrak{g}$. Direct affine Lie algebras are Lie algebras of the form $\hat{\mathfrak{g}} := L\mathfrak{g} \oplus \boldsymbol{C}C$, where $\boldsymbol{C}C$ is a one-dimensional space whose elements commute with any other element of the affine algebra. The multiplication in $L\mathfrak{g}$ (8) is redefined in $\hat{\mathfrak{g}}$ as

$$(8) \qquad [t^m X, t^n Y] = t^{m+n}[X, Y] + m(X|Y)\,\delta_{n+m,0}\,C\,,$$

where $(*\,|\,*)$ is some nonzero multiple of the Killing form on $\mathfrak{g}$:

$$(9) \qquad k(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))\,.$$

If we restricted our interest only to the adjoint representation, C would be trivial, since $(\alpha|\check{\alpha}_c) = 0$ for any α, where $\check{\alpha}_c$ denotes the co-root corresponding to C. However, for the highest-weight representations, which typically are physically more interesting (the state of highest weight is, *e.g.*, the state of lowest energy), the value of C is a positive integer. Notice that the adjoint representation is not a highest-weight representation.

The root-space decomposition of the affine algebra $\hat{\mathfrak{g}}$ is essentially the same as that of $L\mathfrak{g}$: in fact, they are exactly the same for nonzero affine roots, whereas the multiplicity of $\alpha = 0$ is increased by one because of C. If $\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$ is the root-space decomposition of $\mathfrak{g}$ (finite-dimensional), then the corresponding decom-

(6) V. PENNA and M. RASETTI: in preparation.

(7) H. GARLAND: *J. Algebra*, 53, 480 (1978).

(8) V. G. KAC: *Infinite Dimensional Lie Algebras* (Cambridge University Press, Cambridge, 1985).

position of $L\mathfrak{g}$ is

(10) $$L\mathfrak{g} = \bigoplus_{n \in Z} \bigoplus_{\alpha \in \Delta} t^n \mathfrak{g}^\alpha .$$

Thus a root β of the loop algebra consists of a pair, $\beta = (n, \alpha)$. One can consistently make $\delta = (1, 0)$ correspond to C if $(* \mid *)$ is extended in such a way that $(\delta \mid \check{\alpha}) = 0$. Then $(* \mid *)$ is positive definite on the dual $\hat{\mathfrak{h}}^*$ of the Cartan subalgebra $\hat{\mathfrak{h}} \lhd \hat{\mathfrak{g}}$, and one has $(\alpha + n\delta \mid \check{\alpha} + n\delta) = (\alpha \mid \check{\alpha})$ for all n's.

The reason for looking at loop algebras, even though they do not lead to all affine algebras, is that they allow an easy solution to the problem of identifying all diagonalizable operators implied by the structure of the affine algebra (9).

Since $(\delta \mid \check{\alpha}) = 0$, there exists no linear combination of the elements h_i of the Cartan subalgebra that measures the value of n in (10), even though n is an additive number in the adjoint representation. This has to be fixed by enlarging the subalgebra with an additional operator L_0 such that—denoting by $\check{\lambda}_0$ the corresponding co-root—$(\delta \mid \check{\lambda}_0) = -1$.

The explicit realization of L_0 on $L\mathfrak{g}$ is $-t\, d/dt$, so that $L_0(t^n X) = -n(t^n X)$. In other words, L_0 is a derivation of $L\mathfrak{g}$, and extends in a unique way to a derivation of $\hat{\mathfrak{g}}$ by requiring that $[L_0, C] = 0$. L_0 belongs to the algebra V_0 with basis $L_n = -t^{n+1}\, d/dt$ and commutation relations

(11) $$[L_m, L_n] = (m-n) L_{m+n}, \quad [L_m, (t^n X)] = -n(t^{n+1} X) .$$

In the central extension of the loop algebra also V_0 acquires a central extension, which is the Virasoro algebra (10). Virasoro and Heinsenberg algebras are the structural algebras induced by the Poisson bracket and the Hamiltonian functionals for the KdV and modified KdV symplectic structures (11,12).

The extended affine Cartan subalgebra is then finally defined as

(12) $$\hat{\mathfrak{c}} = \mathfrak{h} \oplus \boldsymbol{C}C \oplus \boldsymbol{C}L_0 ,$$

and the extended affine algebra as

(13) $$\hat{\mathfrak{a}} = \mathfrak{g} \oplus \boldsymbol{C}L_0 .$$

Its multiplication rule reads

(14) $$[t^m X + aC + bL_0, t^n Y + cC + dL_0] = \\ = t^{m+n}[X, Y] + m\delta_{m+n,0}(X \mid Y)C - nbt^n Y + mdt^m X ,$$

(9) I. B. Frenkel and V. G. Kac: *Invent. Math.*, 62, 23 (1980).
(10) P. Goddard and D. Olive: *Int. J. Mod. Phys. A*, 1, 303 (1986).
(11) B. L. Feigin and D. B. Fuks: *Funct. Anal. Appl.*, 16, 114 (1982).
(12) G. D'Ariano, A. Montorsi and M. Rasetti: in preparation.

whereas, written in terms of the Cartan decomposition, the commutation relations involving the extended Cartan subalgebra are

$$[h_i, e_{n\delta+\alpha}] = (\delta \mid \check{\alpha}_i)\, e_{n\delta+\alpha}, \qquad [L_0, e_{n\delta+\alpha}] = -\, n e_{n\delta+\alpha}. \tag{15}$$

As already mentioned, a more general construction of the affine algebras is that based on a Cartan matrix.

The Cartan matrix A of a simple Lie algebra of rank d is an integral $d \times d$ matrix satisfying the following conditions:

i) $A_{ii} = 2$;

ii) $A_{ij} \leq 0$ for $i \neq j$;

iii) if $A_{ij} = 0$, then $A_{ji} = 0$ (zeros appear symmetrically in A);

iv) there exists a nonsingular diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{D}A$ is a symmetric matrix;

v) A cannot be brought to block-diagonal form by rearranging rows and columns.

The symmetrizability condition iv) is related to the property that, due the duality, one can consistently choose, for all Kac-Moody algebras,

$$\check{\alpha}_i = \frac{2\alpha_i}{(\alpha_i \mid \alpha_i)}, \tag{16}$$

and define $\boldsymbol{D}$ in such a way that $(\boldsymbol{D}A)_{ij} = (\hat{\alpha}_i \mid \hat{\alpha}_j)$. Then conditions i) to v) are enough to get

$$A_{ij} = (\alpha_i \mid \check{\alpha}_j) = 2\frac{(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)}, \tag{17}$$

from which ensues the well-known interpretation of the Cartan matrix as scalar products in $\mathfrak{h}^*$.

A Lie algebra $\mathfrak{g}(A)$ can be constructed from a Cartan matrix A and a set of $(d+1)$ algebras $su(2)$ with commutation relations

$$[h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i, \qquad [e_i, f_i] = h_i \tag{18}$$

(where h_i is normalized in such a way that bosonic representations have even integer weights and fermionic representations have odd integer weights), by the method of generators and relations. The relations that generate $\mathfrak{g}(A)$ are

$$\begin{cases} [h_i, h_j] = 0, \qquad [h_i, e_j] = A_{ji}\, e_j, \\ [h_i, f_j] = -A_{ji} f_j, \qquad [e_i, f_j] = \delta_{ij}\, h_i. \end{cases} \tag{19}$$

The presentation of the Kac-Moody Lie algebra is completed by the Serre relations

$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0\,, \quad (\mathrm{ad} f_i)^{1-A_{ij}} f_j = 0\,. \tag{20}$$

This construction extends to any Cartan matrix and yields all Kac-Moody algebras in the same way. If $\boldsymbol{D}A$ is positive semi-definite (which, due to the axioms i) to v), allows only one zero eigenvalue of $\boldsymbol{D}A$) the corresponding algebra is affine. Also the twisted affine algebras, which could not be obtained by the loop algebra method, do follow from this construction.

Let $\mathfrak{e}^*$ denote the dual of the Cartan subalgebra extended by L_0. The standard definitions of module for the finite-dimensional case (where a representation is a set of square matrices corresponding in a one-to-one fashion to the operators in the Lie algebra, and a module is a set of column matrices on which the representation matrices act, and—by Weyl theory—there exist representations for which a basis of the module consists of vectors labelled in the weight lattice Q, such that the highest-weight subspace has dimension one, and the highest weight completely specifies the representation) can be straightforwardly extended to the infinite-dimensional case. Just the same, a highest-weight module $M(\Lambda)$ is characterized by a highest-weight vector v^Λ, with

i) v^Λ is a weight vector for $M(\Lambda)$ with weight Λ in the extended lattice in $\mathfrak{e}^*$;

ii) v^Λ is annihilated by all raising operators $e_\alpha, \alpha > 0$;

iii) $M(\Lambda)$ is generated by successive applications of the lowering operators $e_\alpha, \alpha < 0$ on v^Λ.

For the case of affine $su(2)$, the weight λ is characterized by its components $\lambda(h_i) = (\lambda \mid \check{\alpha}_i)$ and the highest weight Λ has nonnegative integer components $\Lambda(h_i) > 0$, $i = 0,1$. Let us assume $\Lambda(h_0) = 1$ and $\Lambda(h_1) = 0$: then Λ carries the weight $(1, 0)$. The value of $\Lambda(L_0)$ has no effect on the representation structure, however, if one selects $\Lambda(L_0) = 0$, then

$$\Lambda(C) = \Lambda(h_0 + h_1) = (\Lambda \mid \check{\alpha}_0) + (\Lambda \mid \check{\alpha}_1) = (\Lambda \mid \check{\delta}) = 1\,. \tag{21}$$

Since $(\alpha \mid \check{\delta}) := \alpha(C) = 0$ for all affine roots, it follows that Λ is actually linearly independent of the roots. Moreover, $\lambda(C) = \Lambda(C)$.

Thus $\lambda(C)$ is constant on an irreducible representation. $\Lambda(C)$ is called the level of Λ and the above choice $\Lambda = (1,0)$ has level 1.

The correspondence $\check{\alpha}_i \to h_i$, $i = 0, 1, \ldots, d$, between $\hat{\mathfrak{h}}^*$ and $\hat{\mathfrak{h}}$ must now be extended to $\hat{\mathfrak{e}}^*$ and $\hat{\mathfrak{e}}$. Since L_0 is linearly independent of the h_i, a linearly independent vector Λ_0 has to be added to $\hat{\mathfrak{e}}^*$: a consistent choice is $(\Lambda_0 \mid \check{\delta}) = 1$ and $(\Lambda_0 \mid \omega_i) = 0$, $i = 0, 1, \ldots, d$, $\{\omega_i\}$ denoting a basis for the root lattice. The scalar product is symmetric only if $(\delta | -\check{\lambda}_0) := \Lambda_0(C)$, because if one further defines Λ_0 by $\Lambda_0(h_i) = \delta_{i,0}$, $i = 0, 1, \ldots, d$, then

(22)
$$(\check{\alpha}_i | \Lambda_0) = (\Lambda_0 | \check{\alpha}_i) = \delta_{i,0}.$$

Returning now to the general problem of constructing generalized coherent states, we observe the following. The algebra $\mathfrak{g}$ has a triangular decomposition

(23)
$$\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+,$$

where $n_+(n_-)$ is the subalgebra generated by the e_α with $\alpha > 0\,(<0)$. Once the highest-weight vector has been selected, the exponentiation of the subalgebra $b = \mathfrak{n}_+ \oplus \mathfrak{h}$ gives the Borel subgroup B, which is the minimal parabolic subgroup of G. We can then rephrase our definition above saying that a $\mathfrak{g}$-module M over $\boldsymbol{C}$ is a highest-weight module with highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a nonzero element $v^\Lambda \in M$ such that $\mathfrak{n}_+(v^\Lambda) = 0$; $\mathfrak{h}(v^\Lambda) = (\Lambda | \check{\alpha})\, v^\Lambda$ and $U(\mathfrak{n}_-)(v^\Lambda) = M$, U denoting the universal enveloping algebra.

This generalizes the role that K had in the conventional generalized coherent states.

We want now to introduce a completion of $U(\mathfrak{g})$, that we denote $\hat{U}_F(\mathfrak{g})$, allowing us to define a set of invariant functions separating the orbits of the Weyl group associated with G. The latter would manifestly provide the desired generalization of the notion of coherent states.

The root lattice Q has a natural decomposition $Q = Q_+ \oplus Q_-$, where $Q_+ = \sum_i \boldsymbol{Z}^{(>)} \alpha_i$, $\boldsymbol{Z}^{(>)}$ denoting the set of nonnegative intergers, and $\Delta = \{\alpha_i \in \mathfrak{h}^* | \alpha_i \neq 0,\ \mathfrak{g}^{(\alpha_i)} \neq \{0\}\}$ is the set of roots of $\mathfrak{g}$. One calls $\Delta_+ := \Delta \cap Q_+$ the set of positive roots. We denote by $\tilde{\Delta}_+$ the set of the same elements, in which every positive root appears with its multiplicity dim $\mathfrak{g}^{\alpha_i}$.

For $\beta \in Q_+$ let $\mathscr{P}_\beta$ be the set of maps $k: \tilde{\Delta}_+ \to \boldsymbol{Z}^{(>)}$; $\alpha \mapsto k_\alpha$ such that $\beta = \sum_{\alpha \in \tilde{\Delta}_+} k_\alpha \alpha$, and set $\mathscr{P} = \bigcup_{\beta \in Q_+} \mathscr{P}_\beta$. Moreover, put $|k| = \sum_\alpha k_\alpha$.

For each $\alpha \in \tilde{\Delta}_+$ it is now possible to choose a basis $\{f_\alpha^{(i)}\}$ of $\mathfrak{g}^{-\alpha}$, and fix (arbitrarily) an order for the unions of these bases, that we denote $\{F_k | k = 1, 2, \ldots\}$. For $k \in \mathscr{P}_\beta$ write

(24)
$$F^{(k)} = \prod_j F_j^{k_j}.$$

It is known ([13]) that for $k, m \in \mathscr{P}$ elements of the form $F^{(k)} \psi_{km}\, \iota(F^{(m)})$, where $\psi_{k,m}$ is an element of the basis of the symmetric algebra $S(\mathfrak{h})$ and ι is the involutive

([13]) V. G. Kac: *Laplace Operators of Infinite-Dimensional Lie Algebras and Theta Functions*, MIT preprint (1987).

automorphism of $U(\mathfrak{g})$ determined by $\iota(e_i)=f_i$; $\iota(f_i)=e_i$; $\iota|_{\mathfrak{h}}=\mathbf{1}_{\mathfrak{g}}$, constitute a basis for $U(\mathfrak{g})$. We define $U_F(\mathfrak{g})$ by assuming that elements of $\hat{U}_F(\mathfrak{g})$ are expressions of the form

(25)
$$g_c = \sum_{k,m\in\mathscr{P}} F^{(k)}\psi_{k,m}\,\iota(F^{(m)})\Big|_{||k|-|m||<c},$$

where c is a given constant. With this definition the multiplication in $\hat{U}_F(\mathfrak{g})$ is directly obtained by simple extension of that in

(26)
$$u_F(\mathfrak{g}) := U(\mathfrak{g}) \underset{C}{\otimes} F/\mathscr{S},$$

where F is the algebra of complex-valued functions on $\mathfrak{h}^*$ (more precisely on $\mathfrak{h}^*$ minus a suitable union of affine hyperplanes where the $\mathfrak{g}$-module $M(\Lambda)$ contains a submodule isomorphic to $M(\Lambda-n\alpha)$) and $\mathscr{S}$ the two-sided ideal which generates the canonical embedding of $\hat{\mathfrak{h}}$ into F. This makes $\hat{U}_F(\mathfrak{h})$ an associative algebra.

Let $\mathscr{C}$ be the category of $\mathfrak{g}(A)$-modules M which are $\mathfrak{h}$ diagonalizable (*i.e.* factorizable in the form $M=\bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$; the action of $U(\mathfrak{g})$ on a module M from the category $\mathscr{C}$ extends to the action of the completion algebra $\hat{U}_F(\mathfrak{g})$ by the property $\psi(v^\lambda)=\psi(\lambda)\,v^\lambda$ for $\psi\in F$ and $v^\lambda\in M^\lambda$. In particular, if Λ lies in $\mathfrak{h}^*$ minus the singular affine hyperplanes mentioned above, then $\hat{U}_F(\mathfrak{g})$ acts on $M(\Lambda)$.

It was shown by Kac [13] that, for any given function $\psi\in F$, there exists a unique element

(27)
$$\mathscr{L}_\psi := \sum_{\beta\in Q_+}\ \sum_{k,m\in\mathscr{P}_\beta} F^{(k)}\psi_{k,m}\,\iota(F^{(m)})$$

in the centre $\mathscr{Z}_F$ of the algebra $\hat{U}_F(\mathfrak{g})$. Otherwise stated, the map $\mu\colon \mathscr{Z}_F\to F$, referred to as Harish-Chandra homomorphism, defined by $\mathscr{L}_\psi\mapsto\psi$, is an isomorphism.

Define now the operation in the group of automorphisms of the root space

(28)
$$r_i\circ\beta := \beta-(\beta|\check{\alpha}_i)\,\alpha_i\,,\qquad i=0,1,\dots,d\,,$$

where β is a real positive root, a fundamental reflection. The group W of fundamental reflections, generated by the r_i, $i=0,\dots,d$, is the Weyl group associated with $\mathfrak{g}(\hat{\mathfrak{g}})$. The weight system of a highest-weight module is the union of infinitely many Weyl group orbits of weights, each itself infinite. A convenient representative of an orbit is the weight in the fundamental chamber

(29)
$$\Gamma := \{\lambda\in\mathfrak{h}^*\,|\,(\lambda|\check{\alpha}_i)\geq 0,\ i=0,1,\dots,d\}\,.$$

The set $T=W\circ\Gamma$ is a solid convex cone in $\hat{\mathfrak{h}}^*$, called the Tits cone. One complexifies it by setting $T_{\mathbf{C}}=\{x+iy\,|\,x\in T;\ y\in\mathfrak{h}^*\}$. We denote by $\mathscr{H}$ the interior of T_C in metric topology and consider the set of elements in the centre of $\hat{U}_F(\mathfrak{g})$:

$$Z := \{\mathscr{L}_\psi \in \mathcal{Z}_F \,|\, \mathscr{L}_\psi \text{ holomorphic on } \mathscr{H}\}. \tag{30}$$

The elements of Z are called Laplace operators for $\mathfrak{g}$.

If ρ is an element of $\mathfrak{h}^*$ such that $(\rho|\check{\alpha}_i) = 1$, $i = 0, \ldots, d$, and γ a map on functions f over $\mathfrak{h}^*$, defined by $(\gamma \circ f)(\lambda) = f(\lambda - \rho)$; then the algebra of functions $(\gamma \circ \mu)(Z)$ contains all the functions defined over $\mathfrak{h}^*$ which are holomorphic and W-invariant on $\mathscr{H}$. Among such functions the theta series

$$\Theta_{h_i,\xi}(\lambda) = \sum_{h \in W \circ h_i} \exp[\xi(\lambda|h)], \tag{31}$$

where ξ is a real number, $h_i \in \mathfrak{h}$ is such that $\forall(\alpha_k\,|\,\check{\alpha}_i)$ is a nonnegative integer and $\lambda \in \mathfrak{h}^*$ is such that $\forall(\lambda|\check{\alpha}_k)$ is a nonnegative integer, converge [14] to holomorphic functions over $\mathscr{H}$ which are well defined if the series is extended by 0 at $\mathfrak{h}^* \setminus \mathscr{H}$: the theta functions of $\mathfrak{g}$.

$\Theta_{h_i,\xi}(\lambda)$ is invariant under the action of $\hat{W}$ and separate the orbits of W; $\hat{W} := W \otimes (i(2\pi/\xi)\,Q))$.

Thus the generalization proposed extends the notion, known for conventional (Glauber) coherent states, of coherent states as theta-functions [15,16]. The remaining step is of course the construction of further generalized states squeezing the latter by the action of the local group of diffeomorphisms D of $\mathscr{M}$, in a way similar to what happens in the case of standard squeezed states [17,18].

Work is in progress along these lines.

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● RIASSUNTO

Si discute la possibile estensione del concetto di stato coerente generalizzato al caso di algebre di Lie a infinite dimensioni, con particolare riferimento alla struttura topologica di stati multipli coerenti che ne deriva. È pure discussa l'importanza di ciò per le soluzioni delle equazioni di moto di sistemi dinamici non lineari.

Когерентные состояния и бесконечно-мерные алгебры Ли: Перспектива.

Резюме (*). — Обсуждается возможное расширение обобщённого когерентного состояния на случай бесконечно-мерных аффинных алгебр Ли. Особое внимание обращается на топологическую структуру множества когерентных состояний и на его связь со структурой алгебры. Анализируется решение уравнений движения для нелинейных динамических систем.

(*) *Переведено редакцией.*