IL NUOVO CIMENTO Vol. 108 B, N. 7 Luglio 1993

A Quantum Wavelet for Quantum Optics.

G. M. D'Ariano (*) and B. DeFacio
Department of Physics and Astronomy, Missouri University - Columbia, MO 65211, USA

(ricevuto il 5 Agosto 1992; approvato il 22 Gennaio 1993)

Summary. — A *quantum*, or operator-valued, wavelet is defined for a general density operator $\hat{\rho}$, in a basis generated by a general observable $\hat{\Theta}$ by defining an operator-valued dilation. The scale changing part of the dilation is shown to correspond to the Yuen squeeze operator. The wavelet gives a family of operator-valued coefficients which represent a given density operator in the eigenbasis of $\hat{\Theta}$, possibly a complete set of commuting observables. The wavelet is given in both the Heisenberg and Schrödinger pictures. Then an *inverse problem* is formulated which allows an unknown density operator to be calculated in terms of the family of all wavelet operators. It is interesting that a limiting process is required to obtain a unique inverse, when one exists. Then the Heisenberg-picture dilation is applied to two known examples: the unitary process of phase sensitive amplification and the irreversible process of number amplification.

PACS 42.50.Dv – Nonclassical photon states (including antibunched, squeezed, sub-Poissonian).
PACS 03.65 – Quantum theory; quantum mechanics.

1. – Introduction.

Wavelets were created a decade ago by Morlet for geophysical exploration [1]. They were immediately applied to mathematical physics [2], to classical analysis [3, 4, 5] and then to a variety of problems in signal processing and applied mathematics [6]. Inverse problems [7] are well suited to wavelet analysis [6, 8, 9] but all of the above works are c-number formulations. Here a quantum wavelet is given, which is tailored for quantum optics.

Many problems in quantum optics [10, 11] are naturally formulated using q-number coherent states [12, 13]. Because of the large number of degrees of freedom and the coherence, the von Neumann approach using a complete set of commuting observables is replaced by the overcomplete family of states (OFS) which are continuous in a set of labels [12, 14]. An introduction with references and reprints of

(*) Permanent address: Dipartimento di Fisica «Alessandro Volta», Università degli Studi di Pavia, via A. Bassi 6, I-27100 Pavia, Italy.

coherent states applied to problems including physics, chemistry and biology are available in Klauder and Skagerstam [12]. By using the group theory of the dilation group (but not the coherent states) Moses and Quesada [15] have computed the power spectrum of the Mellin transform and have given the all-important resolutions of the identity. Kaiser [16] recently gave an algebraic formulation of wavelets which showed both the algebraic structure and the relation to complex structure. There is an interesting relation between his work and the result presented here in the sense that a limit is required. In ref. [16] the limit is the completion of a sequence of spaces W_α which approach $L^2(R^1)$, an irreducible representation. Here a limit is required for the inverse to be unique.

The organization of the paper is the following: sect. 2 will define coherent states, squeezed states and wavelets and will present both c-number and q-number wavelet transforms; sect. 3 will give an inversion formula for the density operator; sect. 4 will give two examples of the wavelet, one a unitary phase sensitive detection and the other is the non-unitary number amplification. The conclusions are given in sect. 5.

2. – Quantum optics and wavelets.

The coherent states are an OFS defined for each complex z as

$$|z\rangle = \exp[za^\dagger - z^* a]\,|0\rangle \hat{D}(z)\,|0\rangle, \tag{1}$$

where a, $a^\dagger$ are the Bose creation and annihilation operators which satisfy the CCR,

$$[a, a^\dagger] = 1, \tag{2}$$

$|0\rangle$ is the Fock vacuum and $|z\rangle$ is continuous in z. For the real squeezing parameter y the operator $\hat{S}(y)$ of Yuen [17] is given by

$$\hat{S}(y) = \exp\left[\frac{1}{2}y((a^\dagger)^2 - a^2)\right]. \tag{3}$$

The squeezed state is then given by

$$|z, y\rangle = \hat{S}(y)\hat{D}(z)\,|0\rangle, \tag{4}$$

(more generally y is complex). The interpretation of $\hat{S}(y)$ follows from expressing $(a, a^\dagger)$ in terms of the position and momentum operators $(\hat{q}, \hat{p})$ and calculating that

$$\hat{S}(y)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\hat{S}^{-1}(y) = \begin{pmatrix}\exp[-y]\,\hat{q}\\ \exp[y]\,\hat{p}\end{pmatrix}. \tag{5}$$

From eq. (5) it is clear that phase space areas, including the uncertainty principle, are preserved under squeezing. Several kinds of squeezed states have been produced in the laboratory [18] so it is reasonable to hope that their precision will solve several problems including gravity wave detection via optical interferometry and optical-communication devices.

A good starting point for wavelets is the usual Fourier optics of finite-energy $L^2(R^d)$ signals which are invariant under translations of the origin [8]. The signals $f(t)$ can be real- or complex-valued and have windowed Fourier transforms given by

$$\Phi_{\mathrm{WFT}}(f;\, q,\, p) = \int_{-\infty}^{\infty} \mathrm{d}x \exp[ip\cdot x]\, g(x-q) f(x)\,, \tag{6}$$

where $g(\cdot)$ is a given, fixed windowing function which filters the signal f. The phase space variables (q, p) can either be continuous, $(q, p) \in R^2$, or discrete, $q = q_0 n$, $p = p_0 m$ $(n, m) \in Z^2$, with (q_0, p_0) positive and fixed scales of coordinate and wave number (or time and frequency). The uncertainty principle places some locality restrictions upon the pair (q, p) but it is far from ideal if $g(\cdot)$ is non-Gaussian and in that case other problems arise. One of the nice properties of wavelets is to improve locality while preserving the uncertainty principle.

The *direct* and *inverse* problems in signal detection are the following pair: i) In the *direct* problem f and g are given, the objective is to find $\Phi_{\mathrm{WFT}}(f;\, q,\, p)$. ii) In the *inverse* problem $\Phi_{\mathrm{WFT}}(f;\, q,\, p)$ and g are given and the objective is to find f. If a solution $f(x)$ exists to the inverse problem to eq. (6) some additional issues include uniqueness, stability and numerical construction must be solved.

The wavelet transform analogous to (6) is given by

$$W(f;\, a,\, b) = \frac{1}{a^{1/2}} \int_{-\infty}^{\infty} \mathrm{d}x \bar{\chi}\left(\frac{x-b}{a}\right) f(x)\,, \tag{7}$$

where $\bar{\chi}$ is the complex conjugate of a «mother wavelet» which is admissible, dilation invariant, mean zero and x is the space (or time) variable. The addition of the multiplicative scale change $1/a$ to the translation $x \to x - b$ is a key difference between eqs. (6) and (7). Thus, in the sense defined in ref. [8] it could be stated that Aslaken and Klauder [19] formulated and studied the first wavelet. The *admissibility condition* on χ for inversion is that

$$0 < c_\chi < \infty\,,$$

where

$$c_\chi \doteq 2\pi \int_{0+}^{\infty} \mathrm{d}\omega \frac{|F[\chi](\omega)|^2}{\omega}\,, \tag{8}$$

if $F[\chi]$ is the Fourier transform of χ. The resolution of the identity operator on L^2 also follows from eq. (8). The *Grossmann-Morlet* inverse formula [2]

$$f(x) = \frac{1}{c_\chi} \int_{0+}^{\infty} \frac{\mathrm{d}a}{a^2} \int_{-\infty}^{\infty} \mathrm{d}b\, a^{-1/2} \chi\left(\frac{x-b}{a}\right) W(f;\, a,\, b) \tag{9}$$

gives the inverse equation to eq. (7). Examples are given in refs. [1, 7] and [9].

All of the objects in eqs. (6)-(9) were ordinary functions in L^2. Next, a q-number version will be given where f is replaced by an arbitrary (pure or mixed) given density operator $\hat{\rho}$ and the scale change is shown to be implemented by a Yuen squeezing operator [17].

Let $\hat{\Theta}$ be an observable so that

$$\hat{\Theta}\,|\theta\rangle = \theta\,|\theta\rangle, \tag{10}$$

where the states $|\theta\rangle$ give the resolution of the unit operator $\hat{1}$ in terms of the measure $\mathrm{d}\mu(\theta)$,

$$\hat{1} = \int \mathrm{d}\mu(\theta)\,|\theta\rangle\langle\theta|\,. \tag{11}$$

The generating function of moments of the observable $\hat{\Theta}$ in the state whose density operator is $\hat{\rho}$ is given by

$$\langle \exp[i\alpha\hat{\Theta}]\rangle := \mathrm{Tr}[\exp[i\alpha\hat{\Theta}]\hat{\rho}] \tag{12}$$

and can easily be shown to be the Fourier transform of the probability distribution $\mathscr{P}(\hat{\rho};\theta)$,

$$\langle \exp[i\alpha\hat{\Theta}]\rangle = \int \mathrm{d}\mu(\theta)\exp[i\alpha\theta]\,\mathscr{P}(\hat{\rho};\theta)\,, \tag{13}$$

where

$$\mathscr{P}(\hat{\rho},\theta) = \mathrm{Tr}[\,|\theta\rangle\langle\theta|\hat{\rho}\,]. \tag{14}$$

If $\gamma(\theta)$ is a window, then the filtered version of (13) becomes

$$\langle \exp[i\alpha\hat{\Theta}]\rangle_\gamma = \int \mathrm{d}\mu(\theta)\exp[i\alpha\theta]\,\gamma(\theta)\,\mathscr{P}(\hat{\rho};\theta)\,. \tag{15}$$

Hence, a natural definition of a wavelet transform is

$$W(\hat{\rho};\eta,\varepsilon) := \frac{1}{\varepsilon^{1/2}}\int \mathrm{d}\mu(\theta)\,\chi\!\left(\frac{\theta-\eta}{\varepsilon}\right)\mathscr{P}(\hat{\rho};\theta), \tag{16}$$

which is, of course, still a c-number function as was eq. (7). Next, we will show that the scale change of multiplication of $1/\varepsilon$ corresponds to the squeezing of the quantum state, and therefore, also of the probability distribution. Consider the analytic function of the operator $\hat{\Theta}$ defined as

$$\chi(\hat{\Theta}) := \int \mathrm{d}\mu(\theta)\,\chi(\theta)\,|\theta\rangle\langle\theta|\,, \tag{17}$$

with

$$\chi(\hat{\Theta})\,|\theta\rangle = \chi(\theta)\,|\theta\rangle. \tag{18}$$

Using (17) and (18) it follows that

$$\int \mathrm{d}\mu(\theta)\,\chi(\theta)\,\mathscr{P}(\hat{\rho};\theta) = \mathrm{Tr}[\chi(\hat{\Theta})\hat{\rho}] = \langle\chi(\hat{\Theta})\rangle. \tag{19}$$

The action of a Heisenberg-picture dilation on $\hat{\Theta}$, with the translation parameter η

and the scale change parameter ε, is given by

$$\mathscr{D}^{\mathrm{H}}_{\eta\varepsilon}(\hat{\Theta}) = \frac{\hat{\Theta} - \eta}{\varepsilon}\,. \tag{20}$$

In the most favourable cases of $\hat{\Theta}$ this transformation is unitary. There are, however, other physically interesting cases for which this map has been shown *not* to be unitary but rather is a completely positive (CP for short) map [20]. See ref. [21] for a discussion of the role of CP maps in statistical mechanics. Examples will be given later in sect. 4. If $\mathscr{D}^H$ is either unitary or if it is CP and preserves products of observables, then it follows that

$$\mathscr{D}^{\mathrm{H}}_{\eta\varepsilon}(\chi(\hat{\Theta})) = \chi(\mathscr{D}^{\mathrm{H}}_{\eta\varepsilon}(\hat{\Theta})) = \chi\left(\frac{\hat{\Theta} - \eta}{\varepsilon}\right) \doteq \varepsilon^{1/2}\chi_{\eta\varepsilon}(\hat{\Theta}). \tag{21}$$

In practice, a careful study of an observable is required to establish if it is unitary or, if not, whether it is CP. The present study does not determine this, but this wavelet will be shown to apply to both cases later in this paper. The previous equation allows one to rewrite (16) as

$$W(\hat{\rho};\eta,\varepsilon) = \frac{1}{\varepsilon^{1/2}}\mathrm{Tr}\left[\chi\left(\frac{\hat{\Theta} - \eta}{\varepsilon}\right)\hat{\rho}\right] = \frac{1}{\varepsilon^{1/2}}\langle\mathscr{D}^{\mathrm{H}}_{\eta\varepsilon}(\chi(\hat{\Theta}))\rangle = \langle\chi_{\eta\varepsilon}(\hat{\Theta})\rangle. \tag{22}$$

It is possible to write the transformation in (21) in the Schrödinger picture as $\mathscr{D}^{\mathrm{S}}_{\eta\varepsilon}$, where

$$\mathrm{Tr}\,[\mathscr{D}^{\mathrm{H}}_{\eta\varepsilon}(\hat{\Theta})\,\hat{\rho}] = \mathrm{Tr}\,[\hat{\Theta}\,\mathscr{D}^{\mathrm{S}}_{\eta\varepsilon}(\hat{\rho})]. \tag{23}$$

In the Schrödinger picture, (22) becomes

$$W(\hat{\rho};\eta,\varepsilon) = \frac{1}{\varepsilon^{1/2}}\mathrm{Tr}\,[\chi(\hat{\Theta})\,\mathscr{D}^{\mathrm{S}}_{\eta\varepsilon}(\hat{\rho})] = \frac{1}{\varepsilon^{1/2}}\langle\chi(\hat{\Theta})\rangle_{\eta\varepsilon}\,, \tag{24}$$

which yields a «squeezed state» for every density operator $\hat{\rho}$. The choice of a scale $\varepsilon_{\max}$ corresponds to the «highest wave number» mode in the observable $\hat{\Theta}$. Heuristically, the limit of infinite wave number is required to obtain a unique inverse $k \to \infty$ or $\varepsilon \to 0$ provided one exists. This is somewhat analogous to the limit required in dequantization in ref. [22], although it is quite different in detail.

To properly apply eq. (23) or (24), the density operator $\hat{\rho}$ would be mapped according to an operator-valued dilation $\mathscr{D}^{\mathrm{H}}_{\eta\varepsilon}$ or $\mathscr{D}^{\mathrm{S}}_{\eta\varepsilon}$ composed with eigenfunctions through the operator $\chi(\Theta)$. The trace then yields an invariant (*i.e.* independent of the $|\theta\rangle$'s) operator-valued wavelet in the Heisenberg or Schrödinger time picture. The operators are sampled at equally spaced points in x (or t) and at intervals of $k/2^j$ for j taking positive integer values 0, 1, 2, ...

3. – An operator-valued wavelet inversion formula.

The inverse problem to eqs. (16) or (22) is to determine the density operator ρ from the family of all wavelet coefficients in the basis of $|\theta\rangle$'s. This means that the wavelet coefficients have been measured at all values of (η, ε) in phase space. Of course, the inversion cannot produce more information than the original system contained. Then, the values of the parameters η and ε become variable in the inverse theory. This involves an $\varepsilon \downarrow 0$ limit to obtain $\hat{\rho}$ uniquely; in the case that it exists and is unique. By using eq. (16), an operator-valued Grossmann-Morlet inversion formula will be derived in this section. To indicate where the objects live, the reader is referred to ref. [23] for the following facts.

Let $\mathscr{H}$ be the complex, separable Hilbert space of the system, let $\mathscr{B}(\mathscr{H})$ denote the set of all bounded operators on $\mathscr{H}$, let $\mathscr{F}(\mathscr{H})$ denote the finite-rank operators on $\mathscr{H}$, $\mathscr{C}(\mathscr{H})$ denote the compact operators on $\mathscr{H}$ defined as

$$\mathscr{C}(\mathscr{H}) \doteq \mathscr{F}(\mathscr{H}^\nu), \tag{25}$$

where ν is the closure in the uniform norm. Let the Hilbert-Schmidt operators $\mathscr{L}(\mathscr{H})$ be given by

$$\mathscr{L}(\mathscr{H}) = \{A \,|\, \mathrm{Tr}(A^*A) < \infty\}, \tag{26}$$

where $\mathrm{Tr}(A^*A)$ is the Hilbert-Schmidt norm of A. Then Emch [23] has shown that the Hilbert-Schmidt operators are a two-sided ideal and form a Hilbert algebra with inner product $(\,,)$ given by

$$(A, B) = \mathrm{Tr}(A^*B). \tag{27}$$

The trace-class operators on $\mathscr{H}$ are written as $\mathscr{T}(\mathscr{H})$ and are defined as

$$\mathscr{T}(\mathscr{H}) \doteq \{\hat{\rho} \,|\, \mathrm{Tr}(\hat{\rho}) < \infty\}. \tag{28}$$

Emch also has shown that

$$\mathscr{F}(\mathscr{H}) \subset \mathscr{T}(\mathscr{H}) \subset \mathscr{L}(\mathscr{H}), \tag{29}$$

so that the generalized Grossmann-Morlet inversion formula will be proven in $\mathscr{L}(\mathscr{H})$. Fourier transforms, *e.g.* $F[\hat{\rho}](\omega)$, are included in this family of operators and the generalized Parseval's theorem preserves the Hilbert-Schmidt norm given in eq. (26). The Fourier transform can either be chosen as the usual Euclidean inner products or can have the symplectic structure which was used by Daubechies and Grossmann [22] in their study of path integrals and some of the associated integral transforms. Schematically, the wavelet *direct* problem for quantum optics was given in eq. (22) as an explicit expression for the map (in the eigenfunction $|\theta\rangle$) as

$$\mathscr{D}: \hat{\rho} \to W, \tag{30}$$

whereas the inverse problem requires the map

$$\mathscr{I}: W \to \hat{\rho}, \tag{31}$$

where W must be in the range of $\mathscr{D}$ if a unique inverse is to exist. If W is not in the range of $\mathscr{D}$, then at most some *generalized inverse* can exist and such objects cannot be unique (see ref. [24]). If the q-number wavelet $\hat{\chi}$ has mean zero, the operator $\hat{c}_\chi$,

$$\hat{c}_\chi = 2\pi \int_{0+}^{\infty} \mathrm{d}\omega \frac{|F[\chi](\omega)|^2}{\omega}, \tag{32}$$

must satisfy the admissibility condition

$$0 < \|\hat{c}_\chi\|_{\mathrm{HS}} < \infty \tag{33}$$

in the Hilbert-Schmidt norm for the present.

The idea is that a fixed $\hat{\rho}$ is sought from data including the wavelet coefficients at all values of (η, ε) with $\eta \in \boldsymbol{R}^1$ and $\varepsilon \in \boldsymbol{R}^1_+$, *i.e.* $\varepsilon > 0$. Since the Hilbert algebra is a Hilbert space, the Riesz representation theorem on the Hilbert space will be used with $\hat{\rho}$ as an arbitrary test operator in the algebra.

3·1. *Operator-valued Grossmann-Morlet inverse formula.* – If $\hat{\chi}_{\eta\varepsilon}$ is an admissible wavelet in the sense of eq. (33), then

$$\hat{\rho} = (\hat{c}_\chi)^{-1} \int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{\varepsilon^2} \int_{-\infty}^{+\infty} \mathrm{d}\eta W(\hat{\rho}, \eta, \varepsilon) \hat{\chi}_{\eta\varepsilon}, \tag{34}$$

in the inverse formula to eq. (22).

Proof. – By showing that for arbitrary $\hat{\rho}_1, \hat{\rho}_2 \in \mathscr{L}(\mathscr{H})$ it is possible to write

$$(\hat{\rho}_1, \hat{\rho}_2) = (\hat{c}_\chi)^{-1} \int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{\varepsilon^2} \int_{-\infty}^{+\infty} \mathrm{d}\eta (\hat{\rho}_1, \hat{\chi}_{\eta,\varepsilon})(\hat{\chi}_{\eta,\varepsilon}, \hat{\rho}_2), \tag{35}$$

then by the Riesz representation theorem eq. (34) follows. The Fourier transforms in the proof, which is to follow, can either be the ordinary Euclidean inner product or the symplectic inner product used by Daubechies and Grossmann [22]. Taking the right-hand side of eq. (35) and calculating:

$$\int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{\varepsilon^2} \int_{-\infty}^{+\infty} \mathrm{d}\eta (\hat{\rho}_1, \hat{\chi}_{\eta,\varepsilon})(\hat{\chi}_{\eta,\varepsilon}, \hat{\rho}_2) = \int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{\varepsilon^2} \int_{-\infty}^{+\infty} \mathrm{d}\eta \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 \int_{-\infty}^{+\infty} \mathrm{d}\omega_2 \cdot$$

$$\cdot F[\hat{\rho}_1)]^*(\omega_1) F[\hat{\chi}_{\eta\varepsilon}](\omega_2) F[\hat{\chi}_{\eta\varepsilon}]^*(\omega_2) F[\hat{\rho}_2](\omega_2) = \int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{\varepsilon^2} \int_{-\infty}^{+\infty} \mathrm{d}\eta \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 \int_{-\infty}^{+\infty} \mathrm{d}\omega_2 \cdot$$

$$\cdot \exp[i\eta(\omega_1 - \omega_2)] \varepsilon F[\hat{\chi}](\varepsilon\omega_1) F[\hat{\chi}]^*(\varepsilon\omega_2) F[\hat{\rho}_1]^*(\omega_1) F[\hat{\rho}_2](\omega_2) =$$

$$= 2\pi \int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{|\varepsilon|} |F[\hat{\chi}](\varepsilon)|^2 \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 F[\hat{\rho}_1]^*(\omega_1) F[\hat{\rho}_2](\omega_1) = \hat{c}_\chi(\hat{\rho}_1, \hat{\rho}_2).$$

Equating the first expression to the last, one verifies eq. (35). Then the Riesz representation theorem yields eq. (34). □

Remarks. – 1) In order to calculate a unique operator $\hat{\rho}$ from eq. (35), the limit $\varepsilon \to 0$ (or, equivalently, the wave number $k \to \infty$) is required for uniqueness.

2) The *dequantization* mentioned by Daubechies and Grossmann [22] in their study of *quantization* also involves a limit for uniqueness.

3) If $\hat{\rho}_0$ is chosen with Hilbert-Schmidt norm one, a wavelet representation of the identity on $\mathcal{L}(\mathcal{H})$ can be obtained from eq. (35) as

$$1_{\mathcal{L}} = (\hat{c}_\chi)^{-1} \int_{0+}^{\infty} \frac{\mathrm{d}\varepsilon}{|\varepsilon|^2} \int_{-\infty}^{+\infty} \mathrm{d}\eta \, \hat{W}(\cdot\,; \eta, \varepsilon) \hat{\chi}_{\eta\varepsilon} , \tag{36}$$

where $1_{\mathcal{L}} f(x)$ is obtained by substituting $f(x)$ into $W(\cdot\,; \eta, \varepsilon)$. The role of χ is clear.

4) The fact that the quantum wavelet given in eqs. (22) and (34) is very different from Aslaken and Klauder [19], Kaiser [25, 26, 16], Moses and Quesada [15], are evidence of the power and versatility of the coherent states and their continuous representations [12, 13]. One of us (BDF) is investigating the connection of this work to the work by Kaiser [16] because the Gel'fand-Neumark-Segal, GNS, representation on $\mathcal{L}(\mathcal{H})$, as a *-algebra, is also based upon a cyclic vector.

5) It would be interesting to study nuclear magnetic resonance, NMR, and magnetic-resonance imaging, MRI, using these ideas. This investigation is in progress.

4. – Applications.

Two examples which can be treated using the wavelet transform (22) will be given next. These include a unitary dilation (squeezing composed with translation) of one of the quadratures of the electric field. The second is a dilation map of the particle number operator, which is not unitary but rather a CP map.

Consider the quadrature of the electric field, which is defined in terms of a real phase ϕ and the creation and destruction operators as

$$\hat{\mathcal{E}}_\phi = \frac{1}{2}(\exp[i\phi]\, a + \exp[-i\phi]\, a^\dagger) . \tag{37}$$

The dilation map for $\hat{\mathcal{E}}_\phi$ is given by

$$\frac{\hat{\mathcal{E}}_\phi - \eta}{\varepsilon} = \hat{D}^\dagger_{\eta\varepsilon} \hat{\mathcal{E}}_\phi \hat{D}_{\eta\varepsilon} , \tag{38}$$

where $\hat{D}_{\eta\varepsilon}$ is a unitary operator which is the product of the squeezing operator $\hat{S}_\varepsilon$ and the translation $\hat{T}_\eta$:

$$\hat{D}_{\eta\varepsilon} = \hat{S}_\varepsilon \hat{T}_\eta , \tag{39}$$

where

$$\hat{S}_\varepsilon := \exp\left[\frac{1}{2} \ln \varepsilon [\exp[2i\phi]\, a^2 - \exp[-2i\phi]\, (a^\dagger)^2]\right], \tag{40}$$

and

$$\hat{T}_\eta := \exp\left[(\exp[i\phi]\,a - \exp[-i\phi]\,a^\dagger)\right]. \tag{41}$$

These objects were first given by Yuen [17], and correspond to *phase sensitive amplification* [27].

For a very different example, consider the case of number amplification, when the operator $\hat{\Theta}$ is taken as the number operator $\hat{n}$. The dilation of $\hat{n}$ can be written as the composition of the translation map $\mathscr{T}_\eta$ and the scale change $\mathscr{S}_\varepsilon$. The translation of the number corresponds to a unit-preserving CP map. The form of a CP map is

$$\mathscr{T}(\hat{\Theta}) = \sum_\alpha \hat{V}_\alpha^\dagger \hat{\Theta} \hat{V}_\alpha\,, \tag{42}$$

where

$$\sum_\alpha \hat{V}_\alpha^\dagger \hat{V}_\alpha = 1 \tag{43}$$

is the completeness relation. The map $\mathscr{T}_\eta$ has the form of eq. (42),

$$\mathscr{T}_\eta(\hat{n}) = \hat{n} + \eta = (\hat{E}_-)^\eta \hat{n} (\hat{E}_+)^\eta\,, \tag{44}$$

where $\hat{E}_\pm$ denote the shift operators given by

$$\hat{E}_- = (a^\dagger a + 1)^{-1/2} a\,, \qquad \hat{E}_+ = (\hat{E}_-)^\dagger\,, \tag{45}$$

$$\hat{E}_\pm\,|n\rangle = |n \pm 1\rangle. \tag{46}$$

The shift operator are not unitary although

$$\hat{E}_- \hat{E}_+ = 1 \tag{47}$$

because

$$\hat{E}_+ \hat{E}_- = 1 - |0\rangle\langle 0|\,. \tag{48}$$

As a consequence of (48), the translation map $\mathscr{T}_\eta$ does not preserve operator products (namely $\mathscr{T}_\eta(ab) \neq \mathscr{T}_\eta(a)\cdot\mathscr{T}_\eta(b)$). However, the product is preserved on a subclass of operators which annihilate the vacuum. In order to use the preceding development it is necessary to require that the map $\chi(\hat{n})$ satisfy

$$\chi(\hat{n})\,|0\rangle = 0\,. \tag{49}$$

Here η must be a positive integer in the number amplification process.

The squeezing of the number operator

$$\mathscr{S}_\varepsilon(\hat{n}) = \varepsilon\hat{n} \tag{50}$$

must be restricted to scaling where ε is a positive integer. However, it is possible to generalize this scaling to non-integer ε by using the largest integer contained in the bracket as $[\![\varepsilon\hat{n}]\!]$. It can be shown [27] that (50) is obtained by the following CP map:

$$\hat{S}_0^{(\varepsilon)}\,\hat{n}\,(\hat{S}_0^{(\varepsilon)})^\dagger = \varepsilon\hat{n}\,, \tag{51}$$

where, in general,

$$\tilde{S}_\lambda^{(\varepsilon)} = \sum_{n=0}^{\infty} |n\rangle\langle n\varepsilon + \lambda| . \tag{52}$$

The operators $\tilde{S}_\lambda^{(\varepsilon)}$ satisfy the relations [20, 27]

$$\sum_{\lambda=0}^{\varepsilon-1} (\tilde{S}_\lambda^{(\varepsilon)})^\dagger \tilde{S}_\lambda^{(\varepsilon)} = 1 , \tag{53}$$

$$\tilde{S}_\lambda^{(\varepsilon)} (\tilde{S}_\mu^{(\varepsilon)})^\dagger = \delta_{\lambda\mu} , \tag{54}$$

$$\tilde{S}_\lambda^{(\varepsilon)} \tilde{S}_\mu^{(\varepsilon')} = \tilde{S}_{\lambda\varepsilon'+\mu}^{(\varepsilon\varepsilon')} . \tag{55}$$

The de-amplification for ε non-integer can be obtained as a CP map, only if ε is the inverse of an integer [20]. Thus, if ε is denoted as $1/r$, where r is a positive integer, the de-amplification CP map is given by

$$\mathcal{S}_{1/r}(\hat{n}) = [\![\hat{n}/r]\!] = \sum_{\lambda=0}^{r-1} (\tilde{S}_\lambda^{(r)})^\dagger \hat{n} \tilde{S}_\lambda^{(r)} . \tag{56}$$

This transformation preserves operator products as a consequence of the orthogonality conditions (54). In contrast, general operator products are not preserved by amplification (51) but products of number operators are preserved.

The composition law for the squeeze maps

$$\mathcal{S}_\varepsilon \cdot \mathcal{S}_{\varepsilon'} = \mathcal{S}_{\varepsilon\varepsilon'} , \tag{57}$$

which follows from (55), implies the $2^{\pm k}$ scaling of Daubechies wavelets [4]

$$\underbrace{\mathcal{S}_2 \cdot \mathcal{S}_2 \ldots \mathcal{S}_2}_{k \text{ copies}} = \mathcal{S}_{2^k} \tag{58}$$

and, analogously,

$$\mathcal{S}_{1/2} \cdot \mathcal{S}_{1/2} \ldots \mathcal{S}_{1/2} = \mathcal{S}_{1/2^k} . \tag{59}$$

5. – Conclusions.

Two operator-valued wavelet transforms were given for quantum optics, both in the Heisenberg and in the Schrödinger pictures: i) the squeezing for one quadrature of the electric field and ii) the number amplification and de-amplification processes. The first example is implemented by a unitary squeeze operator; the second with a non-unitary but completely positive map. An inverse problem was formulated giving the density operator $\hat{\rho}$ in terms of the wavelet transform W and a fixed q-number wavelet $\hat{\chi}$. Work continues on these operators.

* * *

This was initiated at the *Workshop on Squeezed States and the Uncertainty Relations* at the University of Maryland, March 28-30, 1991. It was supported in part by US AFOSR grant 90-307 USA, and Ministero dell'Università e della Ricerca

Scientifica e Tecnologica, Italy. One of us (BDF) thanks Dr. C. R. Thompson, Profs. G. V. Welland and Bjorn Jawerth for useful conversations on wavelets and Thor for much company during this and other works.

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