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A group-theoretical approach to the quantum damped oscillator

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Abstract

A group-theoretical approach to the quantum Liouville equation for the damped oscillator is presented. The method allows solutions of master equations with prescribed long-time behaviour. As an example, the master equation of a boson field relaxing toward a time-dependent coherent state is analyzed. The solution of more general master equations is sketched in the framework of a Liouville picture, the analogue of the Dirac picture in this context.

The typical case of a single quantum damped oscillator is that of a boson field mode a in a vacuum cavity with loss. The dynamical evolution of the density matrix $\hat{\rho}_t$ of the field is described by the Liouville equation

$$\frac{\mathrm{d}\hat{\rho}_t}{\mathrm{d}t}=\mathscr{L}\hat{\rho}_t\equiv-\tfrac{1}{2}\Gamma(\bar{n}+1)(a^\dagger a\hat{\rho}_t+\hat{\rho}_t a^\dagger a-2a\hat{\rho}_t a^\dagger)-\tfrac{1}{2}\Gamma\bar{n}(aa^\dagger\hat{\rho}_t+\hat{\rho}_t aa^\dagger-2a^\dagger\hat{\rho}_t a)\,, \tag{1}$$

where $\bar{n}$ denotes the mean number of thermal photons at the cavity temperature, and Γ is the cavity damping. Eq. (1) is obtained in the Dirac picture upon evaluating the second-order time evolution of the joint system–reservoir density matrix of the mode a in interaction with a Markovian thermal bath. The density matrix $\hat{\rho}_t$ is the reduced density matrix of the field, namely the joint photon–reservoir matrix partially traced over the Hilbert space of the reservoir [1]. The application of the Liouville equation (1) is not just restricted to the domain of quantum optics: very similar equations can be found in statistical mechanics, for modeling irreversible processes of either boson or fermion fields. Actually, Eq. (1) more generally describes the free dynamics of any open quantum system, the only restriction being that the thermal bath is Markovian. The method here presented could be used also to solve the fermion Liouville equation: however, for the sake of simplicity, we will focus our attention only on bosons.

The Liouville (super)operator $\mathscr{L}$ describes a nonunitary time evolution, which cannot trivially be integrated through standard Lie-algebraic techniques. Analytical solutions of Eq. (1) can be obtained upon resorting to quasiprobability representations of the density matrix [2], or, more generally, upon evaluating eigenvalues and eigenvectors (i.e. eigen density matrices) of $\mathscr{L}$ [3]. Here I show that Eq. (1) can be solved using a very simple algebraic ansatz, without any need of diagonalizing $\mathscr{L}$. The method is powerful and looks promising in view of a systematic search for solutions of more general dissipative equations for interacting systems (usually referred to as *master* equations).

In order to solve Eq. (1) one needs to consider an additional fictitious boson mode b which interacts with a

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through a bilinear time-dependent Hamiltonian, and which is traced out after the unitary evolution of the joint density matrix. In the formulas one has

$$\hat{\rho}_t \equiv e^{\mathscr{L}t}\hat{\rho} = \mathrm{Tr}_b[\hat{U}(t,0)\hat{\rho}\otimes\hat{\nu}\hat{U}(t,0)^\dagger] \,. \tag{2}$$

In Eq. (2) $\hat{\rho}'\otimes\hat{\rho}''$ denotes the direct product of density matrices in the Hilbert space $\mathscr{H}_a\otimes\mathscr{H}_b$ of the two modes a and b. The density matrix $\hat{\nu}$ represents the thermal state

$$\hat{\nu} = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{\hat{n}} , \tag{3}$$

with $\hat{n}$ denoting the number operator of the mode ($\hat{n}\otimes\hat{1} = a^\dagger a$, $\hat{1}\otimes\hat{n} = b^\dagger b$). The operator $\hat{U}(t,0)$ acting on $\mathscr{H}_a\otimes\mathscr{H}_b$ can be written in the form

$$\hat{U}(t,0) = \exp[-\arctan(e^{\Gamma t}-1)^{1/2}(ab^\dagger - a^\dagger b)] \,. \tag{4}$$

In the following I will shown that Eq. (2) with $\hat{U}(t,0)$ given by Eq. (4) and $\hat{\nu}$ by Eq. (3) provides the solution of Eq. (1) for any initial condition [#1] $\hat{\rho}_0 \equiv \hat{\rho}$.

Apart from a phase factor, the limiting operator $\lim_{t\to\infty}\hat{U}(t,0)$ is equivalent to the permutation operator $\hat{P} = \hat{P}^\dagger$ between the two modes a and b. More precisely, the following limit holds,

$$\hat{U}(\infty,0) \equiv \lim_{t\to\infty}\hat{U}(t,0) = \exp(i\pi b^\dagger b)\hat{P} \,. \tag{5}$$

The last assertion follows from the evolution of the field operators,

$$\hat{U}(t,0)^\dagger\begin{pmatrix} a \\ b \end{pmatrix}\hat{U}(t,0) = \begin{pmatrix} e^{-\Gamma t/2} & (1-e^{-\Gamma t})^{1/2} \\ -(1-e^{-\Gamma t})^{1/2} & e^{-\Gamma t/2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} ,$$

$$\hat{U}(t,0)\begin{pmatrix} a \\ b \end{pmatrix}\hat{U}(t,0)^\dagger = \begin{pmatrix} e^{-\Gamma t/2} & -(1-e^{-\Gamma t})^{1/2} \\ (1-e^{-\Gamma t})^{1/2} & e^{-\Gamma t/2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} . \tag{6}$$

On the joint density matrix the permutation of modes reads as follows,

$$\hat{P}\hat{\rho}\otimes\hat{\nu}\hat{P} = \hat{\nu}\otimes\hat{\rho} \,. \tag{7}$$

Hence, from Eq. (5) one has that

$$\hat{U}(\infty,0)\hat{\rho}\otimes\hat{\nu}\hat{U}(\infty,0)^\dagger = \hat{\nu}\otimes e^{-i\pi b^\dagger b}\hat{\rho}e^{i\pi b^\dagger b} \,. \tag{8}$$

Eq. (8) guarantees that evolution (2) has $\hat{\nu}$ itself as the stationary solution, namely

$$\lim_{t\to\infty} e^{\mathscr{L}t}\hat{\rho} = \hat{\nu} \,. \tag{9}$$

Of course, the stationary solution of Eq. (1) is the thermal state (3): thus, it only remains to check that the chosen time dependence of $\hat{U}(t,0)$ in Eq. (4) provides exactly the time derivative (1). The derivative of $\hat{U}(t,0)$ (hereafter shortly denoted by $\hat{U}_t \equiv \hat{U}(t,0)$) can be written in the following ways,

$$\frac{d\hat{U}_t}{dt} = -\tfrac{1}{2}\Gamma(e^{\Gamma t}-1)^{-1/2}(ab^\dagger - a^\dagger b)\hat{U}_t = -\tfrac{1}{2}\Gamma(e^{\Gamma t}-1)^{-1/2}\hat{U}_t(ab^\dagger - a^\dagger b) \,. \tag{10}$$

Time differentiation of Eq. (2) leads to

[#1] The solution in Eqs. (2)–(4) is physically suggested by modeling the loss in form of a beam splitter with transmissivity $\eta = e^{-\Gamma t/2}$: the field mode a impinges into the beam splitter and the output (damped) mode is obtained by partially tracing the evolved joint density matrix over the mode b corresponding to the unused port.

$$\frac{\mathrm{d}\hat{\rho}_t}{\mathrm{d}t}=\tfrac{1}{2}\Gamma(\mathrm{e}^{\Gamma t}-1)^{-1/2}\{[a,\mathrm{Tr}_b(b^\dagger\hat{U}_t\hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger)]-[a^\dagger,\mathrm{Tr}_b(b\hat{U}_t\hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger)]\}\,, \tag{11}$$

where invariance of Tr_b under circular permutations of b-mode operators has been exploited. Using Eqs. (6) the following identities are obtained,

$$\mathrm{Tr}_b(b\hat{U}_t\hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger)=\mathrm{e}^{\Gamma t/2}\mathrm{Tr}_b(\hat{U}_t\hat{\rho}\otimes b\hat{\nu}\hat{U}_t^\dagger)-(\mathrm{e}^{\Gamma t}-1)^{1/2}a\hat{\rho}_t\,,$$
$$\mathrm{Tr}_b(b^\dagger\hat{U}_t\hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger)=\mathrm{e}^{\Gamma t/2}\mathrm{Tr}_b(\hat{U}_t\hat{\rho}\otimes b^\dagger\hat{\nu}\hat{U}_t^\dagger)-(\mathrm{e}^{\Gamma t}-1)^{1/2}a^\dagger\hat{\rho}_t\,, \tag{12}$$

and with the aid of the commutation rules

$$b\hat{\nu}=\frac{\bar{n}}{\bar{n}+1}\,\hat{\nu}b,\qquad b^\dagger\hat{\nu}=\frac{\bar{n}+1}{\bar{n}}\,\hat{\nu}b^\dagger\,, \tag{13}$$

one gets the intermediate steps

$$\mathrm{Tr}_b(b\hat{U}_t\hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger)=(\mathrm{e}^{\Gamma t}-1)^{1/2}[\bar{n}\hat{\rho}_t a-(\bar{n}+1)a\hat{\rho}_t],$$
$$\mathrm{Tr}_b(b^\dagger\hat{U}_t\hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger)=(\mathrm{e}^{\Gamma t}-1)^{1/2}[\bar{n}a^\dagger\hat{\rho}_t-(\bar{n}+1)\hat{\rho}_t a^\dagger]\,. \tag{14}$$

The desired result (1) finally follows from Eqs. (11) and (14).

The explicit form of transformation (2) in terms of a-mode operators is given by

$$\hat{\rho}_t=\sum_{n,m}^{\infty}\hat{V}_t^{(n,m)}\hat{\rho}\hat{V}_t^{(n,m)\dagger}\,, \tag{15}$$

where the operators $\hat{V}_t^{(n,m)}$ acting on $\mathscr{H}_a$ are defined in terms of matrix elements of $\hat{U}_t$ evaluated on $\mathscr{H}_b$,

$$\hat{V}_t^{(n,m)}={}_b\langle n|\hat{U}_t|m\rangle_b\sqrt{\frac{\bar{n}^m}{(\bar{n}+1)^{m+1}}}\,. \tag{16}$$

One of the following alternative expressions can be used,

$${}_b\langle p+s|\hat{U}_t|p\rangle_b=\frac{(-1)^s(1-\mathrm{e}^{-\Gamma t})^{s/2}\mathrm{e}^{-\Gamma tp/2}}{s!}\sqrt{\frac{(p+s)!}{p!}}\,\mathrm{e}^{\Gamma ta^\dagger a/2}\mathscr{N}[\Phi(p+s+1,s+1;(\mathrm{e}^{-\Gamma t}-1)a^\dagger a]a^s$$
$$=(-1)^s(\mathrm{e}^{\Gamma t}-1)^{s/2}\mathrm{e}^{\Gamma tp/2}\sqrt{\frac{p!}{(p+s)!}}\,a^s\mathrm{e}^{-\Gamma ta^\dagger a/2}\mathscr{A}[L_p^{(s)}((1-\mathrm{e}^{-\Gamma t})a^\dagger a)]\,. \tag{17}$$

Here $\mathscr{N}$ and $\mathscr{A}$ denote the normal and anti-normal ordering, $\Phi(\alpha,\beta;x)$ is the customary degenerate hypergeometric function and $L_p^{(s)}(x)$ is the generalized Laguerre polynomial [4]. The upper-diagonal matrix elements can be recovered through the identity

$${}_b\langle p|\hat{U}_t|p+s\rangle_b={}_b\langle p+s|\hat{U}_t^\dagger|p\rangle_b^\dagger\,. \tag{18}$$

The case of zero thermal photons corresponds to the simple solution

$$\mathrm{e}^{\mathscr{L}t}\hat{\rho}=\sum_{n=0}^{\infty}\frac{(\mathrm{e}^{\Gamma t}-1)^n}{n!}\,a^n\mathrm{e}^{-\Gamma ta^\dagger a/2}\hat{\rho}\mathrm{e}^{-\Gamma ta^\dagger a/2}(a^\dagger)^n\,. \tag{19}$$

By construction, it is clear that the method can also be used as a tool for generating new analytically-solvable Liouville equations with prescribed stationary solutions $\hat{\nu}$ different from the thermal state (3). For example, for coherent $\hat{\nu}=|\alpha\rangle\langle\alpha|$ the time derivative of Eq. (2) leads to

$$\frac{\mathrm{d}\hat{\rho}_t}{\mathrm{d}t}=-\tfrac{1}{2}\Gamma(a^\dagger a\hat{\rho}_t+\hat{\rho}_t a^\dagger a-2a\hat{\rho}_t a^\dagger)-\tfrac{1}{2}\Gamma(1-\mathrm{e}^{-\Gamma t})^{-1/2}[\bar{\alpha}a-\alpha a^\dagger,\hat{\rho}_t]\,, \tag{20}$$

which is the master equation of a forced harmonic oscillator in a zero temperature thermal bath, with time-dependent Hamiltonian

$$\hat{H}_t = -\tfrac{1}{2}\mathrm{i}\Gamma(1-\mathrm{e}^{-\Gamma t})^{-1/2}(\bar{\alpha}a - \alpha a^\dagger) . \qquad (21)$$

Eq. (20) is obtained by means of Eqs. (12) along with the following identity,

$$\mathrm{Tr}_b(b^\dagger \hat{U}_t \hat{\rho}\otimes\hat{\nu}\hat{U}_t^\dagger) = \mathrm{e}^{\Gamma t/2}\mathrm{Tr}_b(\hat{U}_t\hat{\rho}\otimes\hat{\nu}b^\dagger\hat{U}_t^\dagger) - (\mathrm{e}^{\Gamma t}-1)^{1/2}\hat{\rho}_t a^\dagger . \qquad (22)$$

It is easy to check that Eq. (20) has the stationary state $\hat{\rho}_\infty \equiv \hat{\nu} = |\alpha\rangle\langle\alpha|$.

The present method can also be used for Liouville equations with nonstationary long-time solution, namely when $\hat{\rho}_t \to \hat{\nu}_t$ for large t and nonconstant $\hat{\nu}_t$. This follows from the observation that Eq. (8) trivially holds also for time-dependent $\hat{\nu} \equiv \hat{\nu}_t$. Generalizing the last example to the case of a time-dependent coherent state $\hat{\nu}_t = |\alpha_t\rangle\langle\alpha_t|$ an additional term in Eq. (20) is obtained, due to the explicit time derivative of $\hat{\nu}_t$. The forcing Hamiltonian (21) then becomes

$$\hat{H}_t = -\mathrm{i}\left(\tfrac{1}{2}\Gamma(1-\mathrm{e}^{-\Gamma t})^{-1/2} + (1-\mathrm{e}^{-\Gamma t})^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\right)(\bar{\alpha}_t a - \alpha_t a^\dagger) . \qquad (23)$$

I end this Letter with some remarks on the time evolution of the field operators (Heisenberg picture), and with some notes on the solution of the time-reversed Liouville equation, and of the more general master equation with given interaction Hamiltonian $\hat{H}$.

Schrödinger and Heisenberg pictures. Corresponding to the Schrödinger-picture evolution (2) of the density matrix $\hat{\rho}$ one can evaluate the Heisenberg evolution of any field operator $\hat{O}$. The latter can be written in the form

$$\hat{O}_t = \mathrm{e}^{\mathscr{L}^\vee t}\hat{O} , \qquad (24)$$

where $\mathscr{L}^\vee$ denotes the dual of $\mathscr{L}$ in the following sense,

$$\mathrm{Tr}_a(\hat{O}\mathscr{L}\hat{\rho}) = \mathrm{Tr}_a(\hat{\rho}\mathscr{L}^\vee\hat{O}) . \qquad (25)$$

The Schrödinger evolution in Eq. (2) can always be recast in the form

$$\mathrm{e}^{\mathscr{L}t}\hat{\rho} = \sum_{\iota\in \mathrm{I}} \hat{V}_t^{(\iota)}\hat{\rho}\hat{V}_t^{(\iota)\dagger} , \qquad (26)$$

where I denotes a numerable set of (poly)indices ι. Invariance of Tr_a under cyclic permutations allows evaluation of the Heisenberg evolution in terms of the Schrödinger map (26), namely [#2]

$$\mathrm{e}^{\mathscr{L}^\vee t}\hat{O} = \sum_{\iota\in \mathrm{I}} \hat{V}_t^{(\iota)\dagger}\hat{O}\hat{V}_t^{(\iota)} . \qquad (27)$$

For example, from Eqs. (19) and (27) one obtains the evolution of the field operator a,

$$\mathrm{e}^{\mathscr{L}^\vee t}a = \sum_{n=0}^{\infty}\frac{(\mathrm{e}^{\Gamma t}-1)^n}{n!}\mathrm{e}^{-\Gamma t a^\dagger a/2}(a^\dagger)^n a a^n \mathrm{e}^{-\Gamma t a^\dagger a/2} = \mathrm{e}^{-\Gamma t/2}a . \qquad (28)$$

Notice that the time evolution (28) does not preserve the commutation relation $[a, a^\dagger]=1$, because the present Heisenberg picture is not unitary.

Time-reserved Liouville equation. Eq. (1) describes an irreversible process which leads to a unique stationary solution $\hat{\nu}$ independently of the initial state $\hat{\rho}=\hat{\rho}_0$. The irreversible nature of the process is reflected by the

[#2] The map (27) mathematically is classified as a *normal unit preserving completely positive* (CP) *map*. It represents the most general dynamical map for quantum dynamics of open systems. For the theory of these maps see Ref. [5]. For application of CP maps to quantum optics see also Ref. [6].

nonanalytic form of the operator $\hat{U}(t,0)$ in Eq. (4) which has nonunitary prolongation for $t\to -t$. However, despite $\hat{U}(t,0)$ is nonunitary Eqs. (15)–(17) can be analytically continued, thus providing a solution $\hat{\rho}_t=\exp(-\mathscr{L}t)\hat{\rho}$ for the time-reversed Liouville equation. On the other hand, the inverse unitary operator $\hat{U}(0,t)\equiv\hat{U}(t,0)^\dagger$ after trace (2) does not provide the time-reversed solution, but again solves the original equation (1) (a whole family of operators of a form similar to Eq. (4) could be used as well, as it is evident upon exploiting the invariance of Eq. (1) over phase changes of a). In order to make the transition from the "damped" to the "amplified" oscillator, operator (4) should be modified in the following fashion,

$$\hat{U}(t,0)=\exp[-\operatorname{arctanh}(1-e^{-\Gamma t})^{1/2}(a^\dagger b^\dagger-ab)]\,, \tag{29}$$

namely the analytic continuation must be performed within the whole dynamical group GL(2, C), from the compact real form SU(2) for the damped oscillator, to the noncompact form SU(1, 1) for the amplified one. For the thermal state $\hat{\nu}$ operator (29) provides the solution of the equations

$$\frac{d\hat{\rho}_t}{dt}=-\tfrac{1}{2}\Gamma\bar{n}(a^\dagger a\hat{\rho}_t+\hat{\rho}_t a^\dagger a-2a\hat{\rho}_t a^\dagger)-\tfrac{1}{2}\Gamma(\bar{n}+1)(aa^\dagger\hat{\rho}_t+\hat{\rho}_t aa^\dagger-2a^\dagger\hat{\rho}_t a)\,, \tag{30}$$

which models a phase-insensitive amplifier. Notice that Eq. (30) becomes formally identical to the time-reversed equation of (1) after the substitution $\bar{n}\to-\bar{n}-1$: physically this means that the reservoir of the amplifier should be considered as having negative temperature $T=\hbar\omega/K_B\log[\bar{n}/(\bar{n}+1)]$.

Liouville picture. The solution of the general master equation

$$\frac{d\hat{\rho}_t}{dt}=\mathscr{L}\hat{\rho}_t-i[\hat{H},\hat{\rho}_t] \tag{31}$$

for a given Hamiltonian $\hat{H}$ can be written in terms of the solution of the Liouville equation (1) as follows,

$$\hat{\rho}_t=e^{\mathscr{L}t}\left(\hat{\rho}-i\int_0^t d\tau\, e^{-\mathscr{L}\tau}\,\mathrm{ad}\,\hat{H}\,\hat{\rho}_\tau\right), \tag{32}$$

where ad $\hat{H}$ denotes the commutator with $\hat{H}$ and all superoperators in Eq. (32) act on the right. The exponential $e^{-\mathscr{L}t}$ has to be considered as the inverse transformation of (2), after analytic continuation of Eqs. (15)–(17) for $t\to -t$. Iteration of Eq. (32) leads to

$$\hat{\rho}_t=e^{\mathscr{L}t}\mathscr{P}\exp\left(-i\int_0^t d\tau\, e^{-\mathscr{L}\tau}\,\mathrm{ad}\,\hat{H}\,e^{\mathscr{L}\tau}\right)\hat{\rho}\,, \tag{33}$$

where $\mathscr{P}$ denotes the chronological ordering operator. Eq. (33) can be viewed as the solution of the master equation (31) in the "Liouville picture", the analogue of the interaction Dirac picture in this context.

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References

[1] P. Meystre and M.S. Sargent III, Elements of quantum optics (Springer, Berlin, 1991).
[2] B. Daeubler, H. Risken and L. Schoendorff, Phys. Rev. A 48 (1993) 3955.
[3] H.-J. Briegel and B.-G. Englert, Phys. Rev. A 47 (1993) 3311.
[4] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series, and products (Academic Press, New York, 1980).
[5] E.B. Davies, Quantum theory of open systems (Academic Press, New York, 1976).
[6] G.M. d'Ariano, Int. J. Mod. Phys. B 6 (1992) 1292.