

# How much a Quantum Measurement is Informative?

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**Abstract.** The informational power of a quantum measurement is the maximum amount of classical information that the measurement can extract from any ensemble of quantum states. We discuss its main properties. Informational power is an additive quantity, being equivalent to the classical capacity of a quantum-classical channel. The informational power of a quantum measurement is the maximum of the accessible information of a quantum ensemble that depends on the measurement. We present some examples where the symmetry of the measurement allows to analytically derive its informational power.

**Keywords:** informational power, quantum measurement

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The information stored in a quantum system is accessible only through a quantum measurement, and the postulates of quantum theory severely limit what a measurement can achieve. The problem of evaluating how much a measurement is informative has obvious practical relevance in several contexts, such as the communication of classical information over noisy quantum channels and the storage and retrieval of information from quantum memories. When addressing such problem, one can consider two different figures of merit: the probability of correct detection [1] (in a discrimination scenario) and the mutual information [2] (in a communication scenario). The latter case is the object of our discussion.

The *informational power*  $W(\Pi)$  of a POVM  $\Pi$  was introduced in Ref. [2] as the maximum over all possible ensembles of states  $R$  of the mutual information between  $\Pi$  and  $R$ , namely

$$W(\Pi) = \max_R I(R, \Pi).$$

Informational power is an additive quantity. Given a channel  $\Phi$  from an Hilbert space  $\mathcal{H}$  to an Hilbert space  $\mathcal{K}$ , the *single-use channel capacity* is given by  $C_1(\Phi) := \sup_R \sup_\Lambda I(\Phi(R), \Lambda)$ , where the suprema are taken over all ensembles  $R$  in  $\mathcal{H}$  and over all POVMs  $\Lambda$  on  $\mathcal{K}$ . A *quantum-classical channel* [3] (q-c channel)  $\Phi_\Pi$  is defined as  $\Phi_\Pi(\rho) := \sum_j \text{Tr}[\rho \Pi_j] |j\rangle \langle j|$ , where  $\Pi = \{\Pi_j\}$  is a POVM and  $|j\rangle$  is an orthonormal basis. In Ref. [2] it was proved that the informational power of a POVM  $\Pi$  is equal to

the single-use capacity  $C_1(\Phi_\Pi)$  of the q-c channel  $\Phi_\Pi$ , i. e.

$$W(\Pi) = C_1(\Phi_\Pi).$$

The additivity of the informational power follows from the additivity of the single-use capacity of q-c channels.

The informational power of a quantum measurement is equal to the accessible information of a quantum ensemble that depends on the measurement. The *accessible information* [4]  $A(R)$  of an ensemble  $R = \{p_i, \rho_i\}$  is the maximum over all possible POVMs  $\Pi$  of the mutual information between  $R$  and  $\Pi$ , namely  $A(R) = \max_\Pi I(R, \Pi)$ . Given an ensemble  $S = \{q_i, \sigma_i\}$ , define [5] the POVM  $\Pi(S)$  as

$$\Pi(S) := \left\{ q_i \sigma_S^{-1/2} \sigma_i \sigma_S^{-1/2} \right\}. \quad (1)$$

Given a POVM  $\Lambda = \{\Lambda_j\}$  and a density matrix  $\sigma$ , define [5] the ensemble  $R(\Lambda, \sigma)$  as

$$R(\Lambda, \sigma) := \left\{ \text{Tr}[\sigma \Lambda_j], \frac{\sigma^{1/2} \Lambda_j \sigma^{1/2}}{\text{Tr}[\sigma \Lambda_j]} \right\}. \quad (2)$$

In Ref. [2] it was proved that the informational power of a POVM  $\Lambda = \{\Lambda_j\}$  is given by

$$W(\Lambda) = \max_\sigma A(R(\Lambda, \sigma)).$$

Moreover, the ensemble  $S^* = \{q_i^*, \sigma_i^*\}$  is maximally informative for the POVM  $\Lambda$  if and only if  $\sigma_{S^*} = \arg \max_\sigma A(R(\Lambda, \sigma))$  and the POVM  $\Pi(S^*)$  is maximally informative for the ensemble  $R(\Lambda, \sigma_{S^*})$ , as illustrated in the commuting diagram in Fig. 1. From

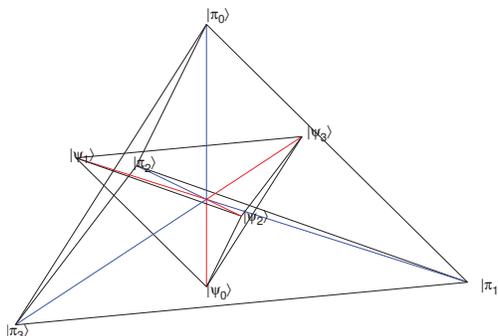
$$\begin{array}{ccc} \Lambda & \xrightarrow{\sigma_{S^*}} & R(\Lambda, \sigma_{S^*}) \\ \downarrow & & \downarrow \\ S^* & \xleftarrow{\sigma_{S^*}} & \Pi(S^*) \end{array}$$

**FIGURE 1.** The commuting diagram makes clear the duality between informational power and accessible information. Here  $S^* = \arg \max_S I(S, \Lambda)$  and  $\Pi(S^*) = \arg \max_\Pi I(R(\Lambda, \sigma_{S^*}), \Pi)$ . Horizontal arrows correspond to the duality operation of Eqs. (1) and (2). Moving in the sense of the arrow corresponds to apply Eq. (2), thus requiring  $\sigma_{S^*}$ . Moving in the opposite sense corresponds to apply Eq. (1). The vertical arrow from  $\Lambda$  to  $S^*$  indicates that  $S^*$  is maximally informative for the POVM  $\Lambda$ , whereas the vertical arrow from  $R(\Lambda, \sigma_{S^*})$  to  $\Pi(S^*)$  indicates that  $\Pi(S^*)$  is maximally informative for the ensemble  $R(\Lambda, \sigma_{S^*})$ .

this result it immediately follows that for any given  $D$ -dimensional POVM there exists a maximally informative ensemble of  $M$  pure states, with  $D \leq M \leq D^2$ , a result similar to Davies theorem for accessible information [6]. For POVMs with real matrix elements [7], the above bound can be strengthened to  $D \leq M \leq D(D+1)/2$ .

For any  $D$ -dimensional POVM  $\Pi = \{\Pi_j\}_{j=1}^N$  with commuting elements, there exists a maximally informative ensemble  $R = \{p_i^*, |i\rangle\langle i|\}_{i=1}^M$  of  $M \leq D$  states, where  $|i\rangle$  denotes the common orthonormal eigenvectors of  $\Pi$ , and the prior probabilities  $p_i^*$  maximize the mutual information. This result applies to the problem of the purification of noisy quantum measurements [8].

For some class of 2-dimensional and group-covariant POVMs (namely, SIC POVMs [6], real-symmetric POVMs [7], mirror-symmetric POVMs [9]), it is possible to provide an explicit form for a maximally informative ensemble which enjoys the same symmetry. For example (see Fig. 2), when  $\Pi = \{\frac{1}{2}|\pi_j\rangle\langle\pi_j|\}_{j=0}^3$  is the 2-dimensional SIC POVM (tetrahedral POVM), the ensemble  $R = \{\frac{1}{4}, |\psi_i\rangle\langle\psi_i|\}_{i=0}^3$  (anti-tetrahedral ensemble) with  $\langle\pi_i|\psi_i\rangle = 0$  for any  $i$  is maximally informative, and the informational power is  $W(\Pi) = \log \frac{4}{3}$ .



**FIGURE 2.** (Color online) Representation of the 2-dimensional SIC POVM (tetrahedral POVM, blue vectors)  $\Pi = \{\frac{1}{2}|\pi_j\rangle\langle\pi_j|\}_{j=0}^3$  and the corresponding maximally informative ensemble (anti-tetrahedral ensemble, red vectors)  $R = \{\frac{1}{4}, |\psi_i\rangle\langle\psi_i|\}_{i=0}^3$ . Orthogonal states form an angle of  $\pi$  as in the Bloch sphere representation, but their length is rescaled according to their norm.

The results we presented have obvious relevance in the theory of quantum communication and measurement, and interesting related works [10, 11] recently appeared. In particular, in Ref. [11] Holevo extends the results we presented to the relevant infinite dimensional case.

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## REFERENCES

1. N. Elron and Y. C. Eldar, IEEE Trans. Inf. Theory **53**, 1900 (2007).
2. M. Dall'Arno, G. M. D'Ariano, and M. F. Sacchi, Phys. Rev. A **83**, 062304 (2011).
3. A. S. Holevo, Russ. Math. Surv. **53**, 1295 (1998).
4. A. S. Holevo, J. Multivariate Anal. **3**, 337 (1973).
5. M. J. W. Hall, Phys. Rev. A **55**, 100 (1997); F. Buscemi, Phys. Rev. Lett. **99**, 180501 (2007); F. Buscemi and M. Horodecki, OSID **16**, 29 (2009).
6. E. B. Davies, IEEE Trans. Inf. Theory **24**, 596 (1978).
7. M. Sasaki, S. M. Barnett, R. Jozsa, M. Osaki, and O. Hirota, Phys. Rev. A **59**, 3325 (1999).
8. M. Dall'Arno, G. M. D'Ariano, and M. F. Sacchi, Phys. Rev. A **82**, 042315 (2010).
9. M. R. Frey, Phys. Rev. A **73**, 032309 (2006).
10. O. Oreshkov, J. Calsamiglia, R. Muñoz-Tapia, and E. Bagan, New J. Phys. **13**, 073032 (2011).
11. A. S. Holevo, Problems of Information Transmission **48**, 1 (2012).