Distributed Quantum Dense Coding

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We introduce the notion of distributed quantum dense coding, i.e., the generalization of quantum dense coding to more than one sender and more than one receiver. We show that global operations (as compared to local operations) of the senders do not increase the information transfer capacity, in the case of a single receiver. For the case of two receivers, using local operations and classical communication, a nontrivial upper bound for the capacity is derived. We propose a general classification scheme of quantum states according to their usefulness for dense coding. In the bipartite case (for any dimensions), bound entanglement is not useful for this task.

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Entanglement is considered to be the most important resource for quantum information [1], as it allows for new quantum protocols such as superdense coding, quantum teleportation, and quantum cryptography. It is therefore of great importance to classify quantum states according to their entanglement properties, in particular, with respect to their usefulness for a given quantum information task. An important example of such classification concerns the distillability of quantum states, i.e., the question of whether entanglement can be concentrated by local operations [2]. Recently, the question of usefulness of states for quantum teleportation [3] and quantum cryptography [4] has been addressed.

In this Letter, we introduce the general concept of distributed dense coding (see also [5]) and present a classification of mixed states according to their *dense codeability* (DC). The idea of dense coding is to use previously shared entanglement between a sender and a receiver to send more information than is possible without the resource of entanglement. We establish a full DC classification for two-party systems, generalizing Refs. [6–8]. In particular, we show that bipartite bound entangled states, in any dimensions, cannot be used for dense coding. Furthermore, we consider the case of *several* senders and receivers, in three different scenarios: (i) the senders or receivers are distant and not allowed to communicate among themselves; (ii) they can use local operations and classical communication (LOCC); (iii) they can perform global operations. We present the classification structure for these scenarios. For the case of a single receiver, we obtain the exact DC capacity. Surprisingly, this capacity cannot be increased by communication between the senders or their joint operations. Moreover, states which are bound entangled in the senders to receiver cut are not useful in this scenario. For the case of more than one receiver, we obtain upper bounds for the corresponding DC capacities.

Let us first consider the bipartite scenario. The amount of classical information that can be sent via a d-dimensional quantum system is at most $\log_2 d$ bits (binary digits). This is due to the Holevo bound [9]. In quantum dense coding, entanglement between the sender and receiver allows one to go beyond this bound [10]. If the sender and receive—hereafter called Alice (A) and Bob (B)—share an entangled bipartite state in $d_A \otimes d_B$, Alice is sometimes able to send more than $\log_2 d_A$ bits to Bob, i.e., more than the maximal information content of her subsystem without any shared entanglement. However, she certainly cannot send more than $\log_2 d_A + \log_2 d_B$ bits to Bob, as required by the Holevo bound.

Given a previously shared state ρ^{AB} in dimension $d_A \otimes d_B$, a general dense coding protocol consists of two steps.

(i) Alice performs a local unitary transformation U_i with probability p_i on her part of ρ^{AB}. This means that she transforms the state ρ^{AB} to the ensemble $\{p_i, \rho_i^{AB}\}$, where $\rho_i^{AB} = U_i \otimes I_{d_B} \rho^{AB} U_i^\dagger \otimes I_{d_B}$. Here I_{d_B} is the identity operator on Bob's Hilbert space. Alice then sends her part of the ensemble state to Bob.

(ii) Bob extracts the maximal information about the index i from the ensemble $\{p_i, \rho_i^{AB}\}$, where now the total state is at his side, by performing suitable measurements.

The maximum amount of information that Bob can gather from his measurement is bounded from above by the Holevo quantity [9]

$$S(\overline{\rho}) - \sum_i p_i S(\rho_i^{AB}) = \sum_i p_i S(\rho_i^{AB} \parallel \overline{\rho}). \tag{1}$$

Here $S(\varsigma) = -\mathrm{tr}(\varsigma \log_2 \varsigma)$ denotes the von Neumann entropy, $S(\varrho \parallel \varsigma) = \mathrm{tr}(\varrho \log_2 \varrho - \varrho \log_2 \varsigma)$ is the relative entropy, and $\overline{\rho} = \sum_i p_i \rho_i^{AB}$. This bound can be attained asymptotically [11], so that the capacity of dense coding is defined as $\chi = \max \sum_i p_i S(\rho_i^{AB} \parallel \overline{\rho})$, where the maximization is over all sets $\{U_i\}$ of unitaries performed by Alice, and all choices of probabilities $\{p_i\}$.

For $d_A \otimes d_B$ systems, with $d_A = d_B = d$, it was shown in Ref. [7] that the maximum is reached for a complete set of orthogonal unitary operators $\{W_j\}$, sampled with equal probabilities, and obeying the *trace rule* $\frac{1}{d^2}\sum_j W_j^\dagger \Xi W_j = \mathrm{tr}[\Xi] I$, for any operator Ξ. A typical example of such a set is provided by the group of shift-and-multiply operators $W_{(p,q)}|j\rangle = \exp(\frac{2\pi p j}{d})|j + q(\text{mod}d)\rangle$, where $\{|j\rangle\}$ denotes an orthonormal basis and $p, q, j = 0, \ldots, d-1$.

In a similar way, one can show that the same sets of unitary operators with equal probabilities are also optimal for bipartite systems with $d_A \neq d_B$. Let us give a brief outline of the proof. As in Ref. [7], the optimization of the dense coding capacity proceeds in three steps.

Step 1. The average state of the ensemble $\{\frac{1}{d_A^2}, \rho_j\}$, which is obtained after Alice performs the unitary transformations W_j on her subsystem, is $\overline{\rho}' = \frac{1}{d_A} I_{d_A} \otimes \rho^B$, where I_{d_A} is the identity operator on Alice's Hilbert space, and $\rho^B = \text{tr}_A \rho^{AB}$. Let χ' be the capacity for this particular choice of unitaries, so that $\chi' = \frac{1}{d_A^2}\sum_j S(\rho_j \parallel \overline{\rho}')$.

Step 2. The capacity χ' is equal to the relative entropy $S(\sigma_{AB} \parallel \overline{\rho}')$, for $\sigma_{AB} = U \otimes I_{d_B} \rho^{AB} U^\dagger \otimes I_{d_B}$, and an arbitrary unitary transformation U on Alice's part.

Step 3. Consider now an arbitrary ensemble $\mathcal{E} = \{p_i, \rho_i = U_i \otimes I_{d_B} \rho^{AB} U_i^\dagger \otimes I_{d_B}\}$ produced by unitary operators U_i applied (with probability p_i) by Alice. Let $\chi_\mathcal{E}$ be the corresponding capacity, so that $\chi_\mathcal{E} = \sum_i p_i S(\rho_i \parallel \overline{\rho})$. Since $\chi' = S(\rho_i \parallel \overline{\rho}')$ for all i (see step 2), we have $\chi' = \sum_i p_i S(\rho_i \parallel \overline{\rho}')$. By Donald's identity [12], $\chi' = \sum_i p_i S(\rho_i \parallel \overline{\rho}) + S(\overline{\rho} \parallel \overline{\rho}') = \chi_\mathcal{E} + S(\overline{\rho} \parallel \overline{\rho}')$, which is $\geq \chi_\mathcal{E}$, as relative entropy is a positive quantity. So this implies that the complete orthogonal set of unitaries W_j, chosen with equal probabilities, is an optimal choice for achieving the capacity for dense coding in $d_A \otimes d_B$ systems. And consequently the capacity of dense coding for a given shared state ρ^{AB} is given by

$$\chi = \log_2 d_A + S(\rho^B) - S(\rho^{AB}). \tag{2}$$

The quantity χ could be increased when Alice and Bob were allowed to locally operate on the shared state. However, an increase of χ (e.g., via filtering) would require classical communication between them. As classical information (which is sent from the sender Alice to the receiver Bob) is the result of the dense coding protocol, we cannot allow them to perform classical communication to effect a change of the shared state.

A classical protocol (i.e., a protocol that does not require a shared quantum state) can be used by Alice to send at most $\log_2 d_A$ bits of classical information. A shared quantum state is thus said to be useful for dense coding or *dense codeable*, if the corresponding capacity is more than $\log_2 d_A$. From Eq. (2), it is clear that such states are precisely those for which $S(\rho^B) > S(\rho^{AB})$, i.e., states that are more mixed locally than globally. For separable states, this inequality is never satisfied [13]. We show that even bound entangled states [14] in $d_A \otimes d_B$, i.e., states that are entangled, and yet they are not distillable, i.e., it is not possible to obtain maximally entangled states from them by LOCC, cannot be used for dense coding. For $d \otimes d$ systems, this was pointed out in Ref. [15].

Let us first state the reduction criterion [16] for detecting distillable states: If a state ρ^{AB} is separable or bound entangled, then $\rho^A \otimes I_{d_B} \geq \rho^{AB}$ and $I_{d_A} \otimes \rho^B \geq \rho^{AB}$. There exist distillable states that violate this criterion. Any state ρ^{AB} for which $S(\rho^B) > S(\rho^{AB})$ violates the reduction criterion [17] (see also [18]), and hence is distillable. Thus, $S(\rho^B) > S(\rho^{AB})$ is not satisfied by *any* bound entangled state: Bipartite bound entanglement is not useful for dense coding. Note also that one cannot use a bound entangled state either to obtain a higher fidelity than classically, in a teleportation protocol [3,19].

This concludes our studies of bipartite dense coding, where the capacity for any given composite state is described by Eq. (2). Note that any *pure* entangled bipartite state is useful for dense coding, whereas there exist mixed entangled states, even in dimension $2 \otimes 2$, which are not, e.g., a Werner state with singlet fraction less than ≈ 0.7476. By contrast, all entangled states in $2 \otimes 2$ and $2 \otimes 3$ are useful for teleportation [3]. This shows that teleportation and dense coding are *inequivalent* tasks. In higher dimensions, at least the distillable states that violate the reduction criterion [16] are useful for teleportation. This is because states that violate the reduction criterion either already have nonclassical teleportation fidelity or can be transformed into such a state by single-side single-copy filtering operations [3]. Moreover, DC states violate the reduction criterion [17], and hence are useful for teleportation.

We will now consider a scheme of dense coding for multipartite states, starting with the case of a single receiver. Suppose that there are $N-1$ Alices, say, $A_1, A_2, \ldots, A_{N-1}$, and a single Bob ($B$). The Alices want to send (classical) information to Bob. The information of one Alice will in general be different from another Alice. To do this, they use a previously shared N-party state $\rho^{A_1,\ldots,A_{N-1}B}$. To start, the jth Alice A_j chooses the unitary tranformation $U_{i_j}^{A_j}$ with probability $p_{i_j}^{A_j}$ and applies it on her part of the state ρ. After performing the unitary transformations, the Alices send their respective parts to Bob. Then Bob makes a global measurement on the total system, to gather maximal information about Alices' ensemble. Here, Bob has no restriction in optimizing over the global measurement, and the Holevo quantity is defined by Alices' action. Note that the Holevo bound can be achieved asymptotically for *product* encodings of the signal states [11]. Therefore it can be reached asymptotically also in the present case of many Alices at distant locations. From the complete orthogonal set $\{W_{j_l}^{A_l}\}$ for A_l, we can construct the set of local operators $\otimes_l W_{j_l}^{A_l}$ which is a complete and orthogonal set for the composite system of all Alices, whence the trace rule holds for their global Hilbert space. Then, the situation is equivalent to the previous case of a single Alice. Using steps 2 and 3, discussed for $d_A \otimes d_B$ systems, it follows that the capacity of distributed dense coding with a single receiver is

$$\chi^{A_1,\ldots,A_{N-1}B} = \log_2 d_{A_1} + \cdots + \log_2 d_{A_{N-1}} + S(\rho^B) - S(\rho^{A_1,\ldots,A_{N-1}B}). \quad (3)$$

Notice that the right-hand side of Eq. (3) is equal to the capacity of dense coding when the Alices are together [see Eq. (2)]. We have thus shown the surprising fact that the Alices do *not* need to perform global unitaries to attain the maximal capacity in a dense coding protocol. We conclude that, also in the present scenario, a state which is bound entangled in the bipartite cut $A_1, \ldots, A_{N-1}$:B, cannot be used for dense coding, since analogous considerations as before show that one cannot have $S(\rho^B) > S(\rho^{A_1,\ldots,A_{N-1}B})$ for such a state.

We now consider the situation of several senders (called Alices, $A_1, \ldots, A_{N-1}$) and two receivers (called Bobs, B_1 and B_2). If the receivers are distant and do not communicate, the corresponding DC capacities are simply additive. This case is denoted in Fig. 1 as LO-DC. Let us therefore study the case where the Bobs are far apart, but are allowed to use LOCC between them, denoted as LOCC-DC in Fig. 1. Here, some of the Alices, say, $A_1, \ldots, A_k$, send their parts of the shared state $\rho^{A_1,\ldots,A_{N-1}B_1B_2}$ to B_1, while the rest of the Alices, $A_{k+1}, \ldots, A_{N-1}$, send their states to B_2. Finally, B_1 and B_2 share the ensemble $\{r_i, \zeta_i\}$, given by $r_i = p_{i_1}^{A_1}, \ldots, p_{i_{N-1}}^{A_{N-1}}$, $\zeta_i = U_{i_1}^{A_1} \otimes \cdots \otimes U_{i_{N-1}}^{A_{N-1}} \otimes I_{d_{B_1}} \otimes I_{d_{B_2}} \rho^{A_1,\ldots,A_{N-1}B_1B_2} U_{i_1}^{A_1\dagger} \otimes \cdots \otimes U_{i_{N-1}}^{A_{N-1}\dagger} \otimes I_{d_{B_1}} \otimes I_{d_{B_2}}$, where the unitary operator $U_{i_j}^{A_j}$ is applied by A_j with probability $p_{i_j}^{A_j}$. Note that B_1 and B_2 are allowed to apply LOCC in the bipartite cut $A_1, \ldots, A_k B_1$:$A_{k+1}, \ldots, A_{N-1} B_2$.

Let us denote the classical information that can be obtained by the Bobs in this setting as I_{acc}^{LOCC}. Its asymptotic version, maximized over all choices of unitaries and probabilities by the Alices, is the DC capacity in this case, denoted as χ^{LOCC}. A Holevo-like universal upper bound for I_{acc}^{LOCC}, valid also for its asymptotic version, is known [20]. In the present case, it reads $\chi^{\mathrm{LOCC}} \leq \max[S(\overline{\zeta}^{(1)}) + S(\overline{\zeta}^{(2)}) - \max_{x=1,2} \sum_i p_i S(\zeta_i^{(x)})]$, where $\overline{\zeta}^{(1)} = \mathrm{tr}_{A_{k+1},\ldots,A_{N-1}B_2} \overline{\zeta}$, $\overline{\zeta}^{(2)} = \mathrm{tr}_{A_1,\ldots,A_{k+1}B_1} \overline{\zeta}$, with $\overline{\zeta} = \sum_i r_i \zeta_i$, and $\zeta_i^{(1)} = \mathrm{tr}_{A_{k+1},\ldots,A_{N-1}B_2} \zeta_i$, $\zeta_i^{(2)} = \mathrm{tr}_{A_1,\ldots,A_{k+1}B_1} \zeta_i$. The unspecified maximization is over all choices of unitaries and probabilities by the Alices.

To obtain a more useful bound, note that for any bipartite state ϱ^{AB} local unitaries cannot change the spectrum of the global as well as the local density matrices. In particular, for arbitrary unitaries U^A and U^B acting on ϱ^{AB} to obtain $\varrho'^{AB} = U^A \otimes U^B \varrho^{AB} U^{A\dagger} \otimes U^{B\dagger}$, we have $S(\mathrm{tr}_B \varrho^{AB}) = S(\mathrm{tr}_B \varrho'^{AB})$ and $S(\mathrm{tr}_A \varrho^{AB}) = S(\mathrm{tr}_A \varrho'^{AB})$. Using this fact, the bound on χ^{LOCC} can be simplified to obtain $\chi^{\mathrm{LOCC}} \leq \max[S(\overline{\zeta}^{(1)}) + S(\overline{\zeta}^{(2)})] - \max_{x=1,2} S(\rho^{(x)})$, where $\rho^{(1)} = \mathrm{tr}_{A_{k+1},\ldots,A_{N-1}B_2} \rho$, $\rho^{(2)} = \mathrm{tr}_{A_1,\ldots,A_{k+1}B_1} \rho$, and the unspecified maximization is as before. This maximization can be performed as follows. First, note that the maximizations for the two subsets of Alices are independent, as they concern disjoint subspaces of the Hilbert space. Thus, we have to find the maximum of the concave function $S(\overline{\zeta}^{(x)})$ $(x = 1, 2)$. Moreover, the $\overline{\zeta}^{(x)}$ form a convex set, for all choices of unitaries $U_{i_j}^{A_j}$ and probabilities $p_{i_j}^{A_j}$. Thus to achieve this maximum, it is sufficient to show that the first derivative of S vanishes, because here a local maximum is the global one. Perturbation of the solution from the previous maximization tasks, namely, W_j with equal probabilities, shows in a straightforward way that this solution is again the optimal one. Thus, we arrive at

$$\chi^{\mathrm{LOCC}} \leq \log_2 d_{A_1} + \cdots + \log_2 d_{A_{N-1}} + S(\rho^{B_1}) + S(\rho^{B_2}) - \max_{x=1,2} S(\rho^x) \equiv \mathcal{B}^{\mathrm{LOCC}}, \quad (4)$$

where $\rho^{B_1} = \mathrm{tr}_{A_1,\ldots,A_{N-1}B_2} \rho$ and $\rho^{B_2} = \mathrm{tr}_{A_1,\ldots,A_{N-1}B_1} \rho$. Analogous arguments as in steps 2 and 3 also give Eq. (4).

A trivial lower bound on χ^{LOCC} is given by the case where the two Bobs do not use communication; thus their two channels are independent, and the capacities add. We denote the capacity without communication as $\chi^{B_1B_2}$, and thus have $\chi^{\mathrm{LOCC}} \geq \chi^{B_1} + \chi^{B_2} = \chi^{B_1B_2}$. A trivial upper bound on χ^{LOCC} is obtained by using the fact that the Bobs can obtain more (at least, not less) information, if they are together and are allowed to use global measurements, referred to as G-DC in Fig. 1. Let us call this bound the global DC capacity χ^{glob}: $\chi^{\mathrm{LOCC}} \leq \log_2 d_{A_1} + \cdots + \log_2 d_{A_{N-1}} + S(\rho^{B_1B_2}) - S(\rho) = \chi^{\mathrm{glob}}$. We summarize our results for the dense codeability of a given multipartite quantum state, for two receivers, in Fig. 1. We call a state dense codeable if its capacity is greater than $\log_2 d_{A_1} + \cdots + \log_2 d_{A_{N-1}}$, and locally dense codeable if $\chi^{B_1B_2} > \log_2 d_{A_1} + \cdots + \log_2 d_{A_{N-1}}$.

We now provide examples for the sets indicated in Fig. 1, and thus show that the sets are nonempty. Note

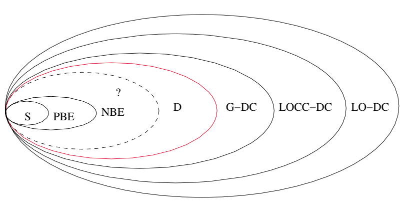

FIG. 1 (color online). Classification of *multipartite* quantum states, according to their usefulness for dense coding with more than one receiver. S, PBE, NBE, and D stand for separable, bound entangled states with positive partial transpose, bound entangled states with nonpositive partial transpose (if existing), and distillable non-G-DC states, respectively (each with respect to the bipartite split between the senders and receivers); see text for other notations. For a single receiver in the multiparty case, and for bipartite systems, there are shells for S, PBE, NBE, D, and G-DC only. The NBE to D boundary is not convex, provided a certain NBE state exists [26], while the convexity of G-DC to LOCC-DC boundary remains an open problem. Other boundaries are convex. In particular, the convexity of the D to G-DC boundary follows from [27].

also that for these examples of DC states one can add the identity to the corresponding state (up to a certain limit) and still keep the noisy state dense codeable. Therefore the sets are not of measure zero.

An example of a state that is G-DC but not LOCC-DC (i.e., $\mathcal{B}^{\rm LOCC} \leq \log_2 d_{A_1} + \cdots + \log_2 d_{A_{N-1}} < \chi^{\rm glob}$) is $\frac{1}{2} \times (|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle)$ from [21], where the first two parties are senders and the last two parties are receivers, with the first (second) party sending her subsystem to the third (fourth) one.

The four-qubit Greenberger-Horne-Zeilinger state [22], namely, $(|0000\rangle + |1111\rangle)/\sqrt{2}$, is not locally DC, as the two-party reduced density matrices are separable. However, it is useful for LOCC-DC: When the two senders choose the Pauli unitaries with equal probabilities, one can show that the two receivers can completely distinguish the resulting ensemble of eight orthogonal states by LOCC. This protocol and the upper bound in Eq. (4) give $\chi^{\rm LOCC} = 3$.

A trivial example for a state that is already locally DC is the tensor product of two singlets. A four-party W state [23] is not locally DC, but it is yet unknown whether it is LOCC-DC. The general problem in proving that a state is useful for LOCC dense coding is that the bound in (4) is sometimes not very tight, and can even be higher than the global bound $\chi^{\rm glob}$, as is the case for the bound entangled states of Ref. [24]. The question whether there exist multipartite bound entangled states that are DC remains open. We point out here that the ordering of states that is induced by the task "dense coding," as illustrated in Fig. 1, is different from the ordering induced by other entanglement criteria, e.g., as described in [21]. Each quantum information processing objective may even lead to its own structure of quantum states.

Finally, it is formally possible to generalize these considerations to the case where there are more than two receivers. However, the main obstacle is that there is as of yet no good estimation of mutual information that is accessible locally, for the case of more than two parties. For an attempt in this direction, see Ref. [25].

In summary, we have introduced the notion of dense codeability, i.e., the usefulness of a given quantum state for dense coding. We have generalized bipartite dense coding to the multipartite case and investigated the classification of entangled states according to their dense codeability. We have presented a full classification for the bipartite case and have shown that here bound entangled states in any dimensions are not dense codeable. In the multipartite case, the capacity of dense coding depends on the possibility of interactions between the receivers. Here, we proposed a classification scheme and showed examples for the various identified classes.

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