

## Superbroadcasting and classical information

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We address the problem of broadcasting  $N$  copies of a generic qubit state to  $M > N$  copies by estimating its direction and preparing a suitable output state according to the outcome of the estimate. This semiclassical broadcasting protocol is more restrictive than a general one, since it requires an intermediate step where classical information is extracted and processed. However, we prove that a suboptimal superbroadcasting, namely broadcasting with simultaneous purification of the local output states with respect to the input ones, is possible. We show that in the asymptotic limit of  $M \rightarrow \infty$  the purification rate converges to the optimal one, proving the conjecture that optimal broadcasting and state estimation are asymptotically equivalent. We also show that it is possible to achieve superbroadcasting with simultaneous inversion of the Bloch vector direction (universal NOT). We prove that in this case the semiclassical procedure of state estimation and preparation turns out to be optimal. We finally analyze semiclassical superbroadcasting in the phase-covariant case.

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### I. INTRODUCTION

A basic feature of classical information is that it can be copied and distributed to an unlimited number of users. However, as one considers quantum information the fundamental task of broadcasting for pure states is impossible [1], and this implies severe limitations to very useful purposes such as parallel computation, networked communications, and secret sharing.

A perfect distribution of the information encoded into  $N$  input systems equally prepared in a pure state to  $M > N$  users would correspond to the so-called *quantum cloning*, which is forbidden by the laws of quantum mechanics. Nevertheless, the case of mixed input states is different, since one needs only the local state of each final user to be equal to the input state, whereas the global output state is allowed to be correlated. This fact opens the possibility to generalize the idea of cloning to quantum maps that output correlated states such as their local reduced states are copies of the input. This generalized version of quantum cloning was named *quantum broadcasting* in Ref. [2]. In this work the impossibility of perfect broadcasting was proved in the case of a single input copy whenever the set of states to be broadcast contains a pair of noncommuting density matrices. This proof was later often considered as the mixed states-scenario extension of the no-cloning theorem. However, it was recently shown that even noncommuting quantum states can be perfectly broadcast provided a suitable number of input copies is available [3]. Moreover, a new phenomenon can occur, which was named *superbroadcasting*: for  $N$  two-level systems (qubits), equally prepared in an unknown mixed input state, the information contained in the direction of the Bloch vector can be distributed to  $M > N$  users and the local state of each final user can be more pure than the initial copies.

An intuitive explanation of the superbroadcasting effect is provided by the statement that superbroadcasting shifts the noise from local purities to global correlations [3,4]. One of

the issues of superbroadcasting is then a deeper understanding of the role of correlations of different nature. While there are correlations which improve the accessibility of information encoded in multiple systems [5], the case of superbroadcasting points out that other kind of correlations are in fact detrimental in this respect. This leads to an amount of information in the global output state that is lower than the sum of informations contained in the local reduced states, i.e., the total information in absence of correlations. Natural questions then arise at this stage. Are the correlations among the final users solely quantum, or is it possible to purify the local states by introducing just classical correlations? Moreover, in the optimal broadcasting protocol, the distribution of information is achieved by coherently manipulating input systems, and the true direction of the Bloch vector remains unknown during the whole procedure. What happens if one first uses the  $N$  input copies to estimate the direction of the Bloch vector, and then distributes  $M$  pure states pointing in the estimated direction? Is it still possible, on average, to increase the purity of local states?

A preliminary extensive analysis of bipartite correlations at the output of superbroadcasting maps suggests that no bipartite entanglement is present [4], whereas the analysis of multipartite entanglement is still an open problem. The fact that the practical protocol for achieving superbroadcasting involves pure state cloning [6,7] suggests on the other hand that the output state contains quantum correlations coming from the structure of the tensor product Hilbert space and its symmetric subspace.

In this paper, we will consider a semiclassical procedure for broadcasting, which consists of measurement and subsequent reparation of the quantum states, usually referred to as the *measure-and-prepare* scheme. We call this scheme semiclassical because broadcasting occurs via extraction of classical information—though the information is retrieved by a collective measurement which might be strictly quantum, being generally a nonlocal measurement—and the output

states are separable, namely they contain only classical correlations. Determining the optimal semiclassical superbroadcasting provides then a boundary between the classical regime, and the quantum regime where entanglement and coherent processing are exploited.

We show that the phenomenon of superbroadcasting can still be observed in the semiclassical case, even though the scaling factors obtained by this scheme are suboptimal. The possibility of superbroadcasting even in the semiclassical scenario can be explained as a change of encoding of the classical information about the direction of the Bloch vector. In fact, the tensor product of  $N$  identical qubit states provides an encoding of direction, in which the information is spread in a nonlocal way over the whole  $N$ -qubits system. In order to extract such an information, one needs either a collective measurement or a statistical processing of single-qubit measurements. However, after the information has been extracted, it can be redistributed exploiting an encoding, which is more favorable to single users. This result allows one to determine a fidelity threshold which separates the classical superbroadcasting—essentially due to the capability of extracting classical information from the input copies—and the strictly quantum effect—which is due to a dynamical processing that optimally distributes information without accessing it. Moreover, we will show that the fidelity of the optimal estimation of direction coincides with the fidelity of the optimal superbroadcasting protocol in the limit  $M \rightarrow \infty$ . This provides the first example of generalization to arbitrary mixed states of the relation between cloning and state estimation, which was known in the literature for pure states [8–10].

In this paper we will also address the optimal approximation of a *universal NOT broadcasting*, namely the impossible transformation which corresponds to a combination of ideal purification, quantum cloning, and spin flip (universal NOT). We will derive the optimal physical map, observing how in this case the semiclassical procedure achieves the optimal fidelity. In other words, the optimal universal NOT broadcasting can be viewed as a purely classical processing of information, as it happens in the case of pure input states [11].

The paper is organized as follows. In Sec. II we introduce the main tools that will be employed to describe symmetric and covariant broadcasting maps. In Sec. III we derive the covariant superbroadcasting map achieved by optimal estimation of the direction of the Bloch vector and conditional reparation of the  $M$  output states. In Sec. IV we derive the optimal covariant NOT broadcasting map and show that it can be achieved by semiclassical means. In Sec. V we study the phase covariant case, we derive the phase covariant semiclassical map, and compare it with the universal case. In Sec. VI we establish the relation between asymptotic superbroadcasting and state estimation. Finally, in Sec. VII we summarize the main results of this paper and discuss their perspectives.

## II. PRELIMINARY TOOLS

### A. Schur-Weyl duality and permutation invariant operators

Symmetry considerations play a fundamental role in the analysis of broadcasting maps, where the input states are  $N$

identically prepared states, and the output states are required to be permutationally invariant, in order to equally distribute information among many users. A very convenient tool to deal with permutation invariance is the so-called Schur-Weyl duality, which relates the irreducible representations of the permutation group to the irreducible representations of the group  $SU(d)$ .

For a system of  $N$  qubits, it is possible to decompose the Hilbert space  $\mathcal{H}=(\mathbb{C}^2)^{\otimes N}$  as a Clebsch-Gordan direct sum

$$\mathcal{H} \simeq \bigoplus_{j=j_0}^{N/2} \mathcal{H}_j \otimes \mathbb{C}^{m_j}, \quad (1)$$

where  $j_0$  is 0 (1/2) for  $N$  even (odd),  $\mathcal{H}_j = \mathbb{C}^{2j+1}$ , and

$$m_j = \frac{2j+1}{M/2+j+1} \binom{M}{M/2-j}. \quad (2)$$

Here  $j$  is the quantum number associated to the total angular momentum, and the spaces  $\mathcal{H}_j$  carry the irreducible representations of  $SU(2)$ . In other words, for any  $U_g \in SU(2)$ , we have

$$U_g^{\otimes N} = \bigoplus_{j=j_0}^{N/2} U_g^{(j)} \otimes \mathbb{1}_{m_j}, \quad (3)$$

where  $\{U_g^{(j)}\}$  is the irreducible representation labeled by the quantum number  $j$ , and  $\mathbb{1}_{m_j}$  is the identity in  $\mathbb{C}^{m_j}$ .

According to the Schur-Weyl duality, the action of a permutation of the Hilbert spaces in the tensor product  $\mathcal{H}^{\otimes N}$  can be represented in the same way as in Eq. (3), with the only difference that the roles of  $\mathcal{H}_j$  and  $\mathbb{C}^{m_j}$  are exchanged, namely the action of permutation is irreducible in  $\mathbb{C}^{m_j}$  and is trivial in  $\mathcal{H}_j$ . In this decomposition, a permutation invariant operator  $X$  has the form

$$X = \bigoplus_{j=j_0}^{N/2} X_j \otimes \mathbb{1}_{m_j}. \quad (4)$$

In particular, the state  $\rho^{\otimes N}$  of  $N$  identically prepared qubits can be written as

$$\rho^{\otimes N} = \bigoplus_{j=j_0}^{N/2} \rho_j \otimes \frac{\mathbb{1}_{m_j}}{m_j}, \quad (5)$$

where  $\rho_j$  is a positive operator on the Hilbert space  $\mathcal{H}_j$  with  $\sum_{j=j_0}^{N/2} \text{Tr}[\rho_j] = 1$ . This decomposition, and the explicit expression for the  $\rho_j$ 's was first given in [6]. If the single-qubit state is  $\rho = (1 + r\mathbf{n} \cdot \boldsymbol{\sigma})/2$ , where  $r$  and  $\mathbf{n}$  are the length and the direction of the Bloch vector, respectively, then  $\rho_j$  is given by

$$\begin{aligned} \rho_j &= m_j (r_+ r_-)^{N/2} \begin{pmatrix} r_+ \\ r_- \end{pmatrix}^{\mathbf{n} \cdot \mathbf{J}^{(j)}} \\ &= m_j (r_+ r_-)^{N/2} \sum_{m=-j}^j \begin{pmatrix} r_+ \\ r_- \end{pmatrix}^m |j, m; \mathbf{n}\rangle \langle j, m; \mathbf{n}|, \end{aligned} \quad (6)$$

where  $|j, m; \mathbf{n}\rangle$  is the eigenstate of the operator  $\mathbf{n} \cdot \mathbf{J}^{(j)}$  for eigenvalue  $m$ , and  $r_{\pm} = (1 \pm r)/2$ .

**B. Symmetric broadcasting maps**

In order to derive the optimal universal NOT broadcasting we will make use of the formalism of the Choi isomorphism between CP maps  $\mathcal{E}$  from states on the Hilbert space  $\mathcal{H}$  to states on the Hilbert space  $\mathcal{K}$ , and positive operators  $R$  on  $\mathcal{H} \otimes \mathcal{K}$ . The isomorphism is given by

$$R_{\mathcal{E}} = \mathcal{E} \otimes \mathcal{I}(|\Omega\rangle\langle\Omega|) \leftrightarrow \mathcal{E}(\rho) = \text{Tr}[1 \otimes \rho^T R_{\mathcal{E}}], \quad (7)$$

where  $|\Omega\rangle \in \mathcal{H} \otimes \mathcal{H}$  is the non-normalized maximally entangled vector  $|\Omega\rangle = \sum_m |m\rangle |m\rangle$ , and  $X^T$  denotes the transpose of  $X$  with respect to the fixed basis  $\{|m\rangle\}$ . For a broadcasting map from  $N$  input qubits to  $M$  output qubits, we have  $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$  and  $\mathcal{K} = (\mathbb{C}^2)^{\otimes M}$ .

In the study of universal broadcasting maps, one requires the universal covariance property, which ensures that the output Bloch vectors point at the same direction as the input ones, and is defined as follows

$$\mathcal{E}(U_g^{\otimes N} \rho U_g^{\otimes N \dagger}) = U_g^{\otimes M} \mathcal{E}(\rho) U_g^{\otimes M \dagger}, \quad (8)$$

$\rho$  being any state on  $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ , and  $U_g$  being any element of  $SU(2)$ . The universal covariance of the map  $\mathcal{E}$  translates into the commutation relation

$$[R, U_g^{\otimes M} \otimes U_g^{* \otimes N}] = 0 \quad \forall U_g \in SU(2). \quad (9)$$

Using the property  $U_g^* = \sigma_y U_g \sigma_y$ , this relation can be rewritten as

$$[S, U_g^{\otimes M+N}] = 0, \quad \forall U_g \in SU(2), \quad (10)$$

where

$$S = (1^{\otimes M} \otimes \sigma_y^{\otimes N}) R (1^{\otimes M} \otimes \sigma_y^{\otimes N}). \quad (11)$$

Moreover, since the figures of merit for broadcasting maps are usually averaged over the output states, without loss of generality we can consider maps that are invariant under permutations of the output systems. Similarly, since we consider only permutation invariant input states, we can restrict attention to maps which are invariant under permutation of the input systems. Consequently  $R$  can be required to satisfy

$$[R, \Pi_{\sigma}^M \otimes \Pi_{\tau}^N] = 0, \quad \forall \sigma \in S_M, \forall \tau \in S_N, \quad (12)$$

where  $\sigma$  ( $\tau$ ) are permutations of the  $N$  input ( $M$  output) qubits, and  $\Pi_{\sigma}^N$  ( $\Pi_{\tau}^M$ ) are the unitary operators representing them. Clearly, this relation is equivalent to

$$[S, \Pi_{\sigma}^M \otimes \Pi_{\tau}^N] = 0. \quad (13)$$

Exploiting the decomposition (1), we can write

$$\mathcal{H} \otimes \mathcal{K} = \left( \bigoplus_{j=j_0}^{M/2} \mathcal{H}_j \otimes \mathbb{C}^{m_j} \right) \otimes \left( \bigoplus_{l=l_0}^{N/2} \mathcal{H}_l \otimes \mathbb{C}^{m_l} \right), \quad (14)$$

and, rearranging the factors in the tensor product, we have

$$\mathcal{H} \otimes \mathcal{K} = \bigoplus_{j=j_0}^{M/2} \bigoplus_{l=l_0}^{N/2} (\mathcal{H}_j \otimes \mathcal{H}_l) \otimes (\mathbb{C}^{m_j} \otimes \mathbb{C}^{m_l}). \quad (15)$$

According to Eq. (13), the operator  $S$  must be invariant under permutations of both the input and the output qubits, whence it must have the form (4)

$$S = \bigoplus_{j=j_0}^{M/2} \bigoplus_{l=l_0}^{N/2} S_{jl} \otimes (\mathbb{1}_{m_j} \otimes \mathbb{1}_{m_l}), \quad (16)$$

where  $S_{jl}$  is a positive operator on  $\mathcal{H}_j \otimes \mathcal{H}_l$ . Moreover, the product  $\mathcal{H}_j \otimes \mathcal{H}_l$  can be further decomposed as

$$\mathcal{H}_j \otimes \mathcal{H}_l = \bigoplus_{J=|j-l|}^{j+l} \mathcal{H}_J^{j,l}, \quad (17)$$

where  $\mathcal{H}_J^{j,l}$  are the  $(2J+1)$ -dimensional subspaces that carry the irreducible representations of the Clebsch-Gordan series of  $U_g^{(j)} \otimes U_g^{(l)}$ . According to Eq. (10),  $S$  must be invariant under  $U_g^{\otimes (N+M)}$ , therefore

$$S = \bigoplus_{j=j_0}^{M/2} \bigoplus_{l=l_0}^{N/2} \bigoplus_{J=|j-l|}^{j+l} s_{j,l}^J P_{j,l}^J \otimes \mathbb{1}_{m_j} \otimes \mathbb{1}_{m_l}, \quad (18)$$

where  $P_{j,l}^J$  is the projection from  $\mathcal{H}_j \otimes \mathcal{H}_l$  onto  $\mathcal{H}_J^{j,l}$ , and  $s_{j,l}^J$  are positive reals.

To find the optimal broadcasting maps, it is useful to know the extremal points of the convex set of the corresponding operators. According to the classification given in Ref. [4], a map is extremal if and only if

$$S = \bigoplus_{l=l_0}^{N/2} \frac{2l+1}{2J_l+1} \frac{1}{m_{j_l}} P_{j_l,l}^{J_l} \otimes \mathbb{1}_{m_{j_l}} \otimes \mathbb{1}_{m_l}, \quad (19)$$

where  $J_l$  and  $j_l$  are two vectors of quantum numbers functions of  $l$ . (Of course, the entries of  $j_l$  can range from  $j_0$  to  $M/2$ , and while the entries of  $J_l$  range from  $|j_l-l|$  to  $j_l+l$ . For universally covariant superbroadcasting one has  $J_l = |l-M/2|$  and  $j_l = M/2$  [3].)

**III. SUPERBROADCASTING VIA OPTIMAL ESTIMATION OF DIRECTION**

Let us consider a broadcasting map that distributes to  $M$  users the information contained into  $N$  qubits, each of them prepared in the same unknown state

$$\rho(\mathbf{n}, r) = \frac{1}{2}(1 + r\mathbf{n} \cdot \boldsymbol{\sigma}), \quad (20)$$

$r$  and  $\mathbf{n}$  being the length and the direction of the Bloch vector, respectively. Precisely, with the term ‘‘information’’ we mean the information about the direction  $\mathbf{n}$ , while the degree of mixedness of the input state is regarded only as an effect of noise. Accordingly, the aim of the broadcasting procedure is to distribute to each user a local state with a Bloch vector pointing in a direction as close as possible to the direction  $\mathbf{n}$ , and possibly with higher purity.

Here we want to obtain the broadcasting map in two steps, namely by first performing a measurement on the ini-

tial qubits, in order to optimally extract the classical information about their direction, and then by preparing  $M$  identical pure states pointing in the estimated direction. This approach can also be used for the NOT broadcasting, with the only difference that after estimation we have to prepare pure states pointing in the opposite direction.

In the following we denote with  $\hat{\mathbf{n}}$  the estimated direction of the Bloch vector, and the measurement statistics will be described by a positive operator valued measure (POVM)  $M(\hat{\mathbf{n}})$ , namely by a set of positive semidefinite operators satisfying the normalization condition

$$\int_{S^2} d^2\mathbf{n} M(\mathbf{n}) = \mathbb{1}, \quad (21)$$

where  $d^2\mathbf{n}$  is the normalized Haar measure on the unit sphere  $S^2$ . The probability density of estimating  $\hat{\mathbf{n}}$  when the true direction is  $\mathbf{n}$  is given by the Born rule  $p(\hat{\mathbf{n}}|\mathbf{n}) = \text{Tr}[M(\hat{\mathbf{n}})\rho(\mathbf{n}, r)]$ . Once the estimation is performed the output state of the broadcasting procedure is

$$\rho_{out}^M(\mathbf{n}, r) = \int_{S^2} d\hat{\mathbf{n}} p(\hat{\mathbf{n}}|\mathbf{n}) |\hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}}|^{\otimes M}, \quad (22)$$

where  $|\mathbf{n}\rangle$  denotes the eigenvector of  $\mathbf{n} \cdot \boldsymbol{\sigma}$  for the eigenvalue +1 (a NOT broadcasting can be obtained replacing  $|\mathbf{n}\rangle$  with its orthogonal complement  $|\!-\mathbf{n}\rangle$  in the above formula). Accordingly, the local state of each user is

$$\rho_{out}^1(\mathbf{n}, r) = \int d\hat{\mathbf{n}} p(\hat{\mathbf{n}}|\mathbf{n}) |\hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}}|, \quad (23)$$

and it is independent of the number of users  $M$ .

In the following we will require the broadcasting map to be covariant under rotations. This corresponds to require the property

$$\rho_{out}^M(g\mathbf{n}, r) = U_g^{\otimes M} \rho_{out}^M(\mathbf{n}, r) U_g^{\otimes M\dagger}, \quad (24)$$

where  $g \in \text{SO}(3)$  denotes a rotation in the three-dimensional space, and  $U_g \in \text{SU}(2)$  is a two by two matrix representing the rotation  $g$  in the single-qubit Hilbert space. In other words, we require that, if the Bloch vector of the input copies is rotated by  $g$ , then also the output state is rotated by the same rotation. In order to have a covariant broadcasting map the POVM density  $M(\mathbf{n})$  must be itself covariant, namely it must satisfy the property [12]

$$M(g\mathbf{n}) = U_g^{\otimes N} M(\mathbf{n}) U_g^{\otimes N\dagger}, \quad (25)$$

for any rotation  $g$ . In this way, the probability distribution has the property

$$p(g\hat{\mathbf{n}}|g\mathbf{n}) = p(\hat{\mathbf{n}}|\mathbf{n}), \quad \forall g \in \text{SO}(3), \quad (26)$$

and, therefore, the output state (22) satisfies the covariance property (24).

In this framework, we want the local state  $\rho_{out}^1(\mathbf{n}, r)$  to be as close as possible to the pure state  $|\mathbf{n}\rangle\langle\mathbf{n}|$ . For this purpose, the estimation strategy will be optimized in order to maximize the single-site fidelity

$$F(\rho_{out}^1(\mathbf{n}, r), |\mathbf{n}\rangle\langle\mathbf{n}|) = \int_{S^2} d\hat{\mathbf{n}} p(\hat{\mathbf{n}}|\mathbf{n}) |\langle\hat{\mathbf{n}}|\mathbf{n}\rangle|^2 \quad (27)$$

$$= \int_{S^2} d\hat{\mathbf{n}} p(\hat{\mathbf{n}}|\mathbf{n}) \frac{1 + \hat{\mathbf{n}} \cdot \mathbf{n}}{2}. \quad (28)$$

In the case of the universal NOT broadcasting, the single-user output state is

$$\tilde{\rho}_{out}^1(\mathbf{n}, r) = \int_{S^2} d\hat{\mathbf{n}} p(\hat{\mathbf{n}}|\mathbf{n}) |-\hat{\mathbf{n}}\rangle\langle-\hat{\mathbf{n}}|, \quad (29)$$

and one considers its fidelity with the pure state  $|\!-\mathbf{n}\rangle\langle\!-\mathbf{n}|$ . Clearly, in the classical procedure both broadcasting and NOT broadcasting have the *same fidelity*. Due to the invariance property (26), the fidelity does not depend on the actual value of the direction  $\mathbf{n}$ , and it is enough to maximize it for a fixed direction, for example the positive direction  $\mathbf{k}$  of the  $z$  axis. For this reason, from now on we will denote the fidelity simply with  $F$ .

The estimation strategy that maximizes the fidelity  $F$  can be found in a simple way by exploiting the decomposition (5) of the input state. First, due to the special form of the states, without loss of generality we can restrict our attention to POVMs of the form

$$M(\mathbf{n}) = \bigoplus_{j=j_0}^{N/2} M_j(\mathbf{n}) \otimes \mathbb{1}_{m_j}, \quad (30)$$

where each  $M_j(\mathbf{n})$  is a POVM in the representation space  $\mathcal{H}_j$ , namely  $M_j(\mathbf{n}) \geq 0$  and

$$\int d\mathbf{n} M_j(\mathbf{n}) = \mathbb{1}_{2j+1}. \quad (31)$$

In fact, if  $\tilde{M}(\mathbf{n})$  is any POVM, then the corresponding probability distribution is

$$\begin{aligned} p(\hat{\mathbf{n}}|\mathbf{n}) &= \sum_j \text{Tr} \left[ \tilde{M}(\hat{\mathbf{n}}) \left( \rho_j(\mathbf{n}, r) \otimes \frac{\mathbb{1}_{m_j}}{m_j} \right) \right] \\ &= \sum_j \text{Tr} [\tilde{M}_j(\hat{\mathbf{n}}) \rho_j(\mathbf{n}, r)], \end{aligned}$$

where  $\tilde{M}_j(\mathbf{n}) = 1/m_j \text{Tr}_{m_j}[\tilde{M}(\mathbf{n})]$ . The same probability distribution can be obtained by a POVM of the form (30), just by choosing  $M_j(\mathbf{n}) = \tilde{M}_j(\mathbf{n})$ .

The fidelity (27) then becomes a sum of independent contributions  $F = \sum_j f_j$  with

$$f_j = \int_{S^2} d\hat{\mathbf{n}} \text{Tr} [M_j(\hat{\mathbf{n}}) \rho_j(\mathbf{k}, r)] |\langle\hat{\mathbf{n}}|\mathbf{k}\rangle|^2, \quad (32)$$

where  $\mathbf{k}$  is the unit vector pointing in the positive  $z$  direction. Since all contributions are independent, each of them can be maximized separately. For this purpose, we can exploit the result by Holevo [12] about the optimal estimation of rotations for mixed states. For any value  $j$  the optimal POVM is given by

$$M_j(\mathbf{n}) = (2j+1)|j, j; \mathbf{n}\rangle\langle j, j; \mathbf{n}|, \quad (33)$$

where  $|j, j; \mathbf{n}\rangle$  is the eigenvector of  $\mathbf{n} \cdot \mathbf{J}^{(j)}$  corresponding to the eigenvalue  $j$ , and the contribution to the fidelity is

$$f_j(\mathbf{n}) = \frac{1}{2} \left( 1 + \frac{\text{Tr}[\rho_j(\mathbf{k}, r) J_z^{(j)}]}{j+1} \right). \quad (34)$$

Finally, by using the expression (6), we can calculate explicitly the fidelity as

$$F = \frac{1}{2} \left[ 1 + (r_+ r_-)^{N/2} \sum_{j=j_0}^{N/2} \frac{m_j}{(j+1)} \sum_{m=-j}^j m \left( \frac{r_+}{r_-} \right)^m \right]. \quad (35)$$

As already mentioned, this is also the value of the fidelity for the universal NOT broadcasting obtained via optimal estimation of the direction.

Now we want to investigate whether the phenomenon of superbroadcasting takes place in the classical broadcasting procedure. To do this we consider the Bloch vector of the local output state by writing

$$\rho_{out}^1(\mathbf{n}, r) = \frac{1}{2} [1 + r'(r) \mathbf{n}'(\mathbf{n}) \cdot \boldsymbol{\sigma}]. \quad (36)$$

The first observation is that the direction  $\mathbf{n}' = \mathbf{n}'(\mathbf{n})$  of the output state is the same as the direction  $\mathbf{n}$  of the input state  $\mathbf{n}$ . Due to covariance, it is enough to prove this fact for  $\mathbf{n}$  pointing in the  $z$  direction. In order to prove it, suppose that  $\mathbf{n}' \neq \mathbf{k}$ , then we would have

$$\langle \mathbf{n}' | \rho_{out}^1(\mathbf{k}, r) | \mathbf{n}' \rangle \geq \langle \mathbf{k} | \rho_{out}^1(\mathbf{k}, r) | \mathbf{k} \rangle = F, \quad (37)$$

namely the fidelity of the output state with  $|\mathbf{n}'\rangle$  would be higher than the fidelity with  $|\mathbf{k}\rangle$  (the equality holds only if the output state is maximally mixed). Since we can write  $\mathbf{n}' = \bar{g}\mathbf{k}$  for some suitable rotation  $\bar{g}$ , in that case we could replace the optimal POVM  $M(\mathbf{n})$  with a new POVM  $M'(\mathbf{n}) = M(\bar{g}^{-1}\mathbf{n})$ , where  $g^{-1}$  denotes the inverse rotation of  $g$ . In this way the fidelity associated to the new POVM would be  $F' = \langle \mathbf{n}' | \rho_{out}^1(\mathbf{k}, r) | \mathbf{n}' \rangle \geq F$ , in contradiction to the fact that  $M(\mathbf{n})$  is the optimal POVM. Therefore, for the optimal POVM the Bloch vector of the output state *must* point in the same direction as the Bloch vector of the input state.

Once we know that the Bloch vector of the output state points in the correct direction, we can simply calculate its length  $r'$  by the relation

$$F = \langle \mathbf{n} | \rho_{out}^1(\mathbf{n}, r) | \mathbf{n} \rangle = \frac{1 + r'}{2}, \quad (38)$$

which is straightforward from Eq. (36). Thus we obtain

$$r'(r) = (r_+ r_-)^{N/2} \sum_{j=j_0}^{N/2} \frac{m_j}{(j+1)} \sum_{m=-j}^j m \left( \frac{r_+}{r_-} \right)^m. \quad (39)$$

The significant parameter in order to assess the quality of broadcasting is the *scaling factor* given by the ratio of input and output single site Bloch vector length  $p(r) = r'/r$ , which is plotted in Fig. 1 for  $N=4, 6, 8$ .

The plot of this expression demonstrates the presence of superbroadcasting for  $N \geq 6$ : in this case the length of the

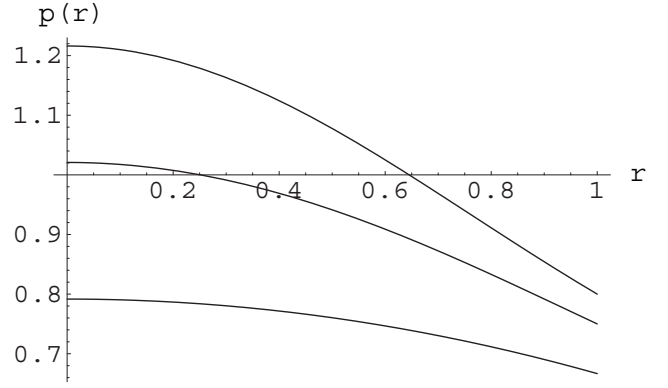


FIG. 1. Scaling factor  $p(r) = r'/r$ , with  $r'$  given in Eq. (39), for the classical universal broadcasting procedure. The three curves, from bottom to top, refer to  $N=4, 6, 8$  input copies, respectively. Notice that for  $N=4$  there is no superbroadcasting, namely we always have  $r' < r$ . The same plots also describe the optimal universal NOT broadcaster described in Sec. IV.

Bloch vector of each single-user state is increased after the broadcasting map. In other words, the classical broadcasting procedure allows to distribute to many users the information about the direction of the input Bloch vector, and, at the same time, to increase the purity of the final local states. This proves that superbroadcasting can be achieved with a classical procedure, by first extracting the classical information about the Bloch vector direction, and then distributing this information among the final users. The increase in the length of the Bloch vector at each site corresponds to an encoding of information which is more favorable to each single user.

Moreover, the expression (39) can be compared with the corresponding one for the optimal universal superbroadcasting [3,4], where the information about the direction is not extracted from the input states, but coherently manipulated and distributed. Remarkably, in the asymptotic limit of a large number  $M$  of output copies, the two expressions coincide, namely the optimal distribution of information is achieved asymptotically by the classical broadcasting procedure. This result provides the generalization to mixed states of the well known relation between cloning and state estimation [8–10].

#### IV. OPTIMAL UNIVERSAL NOT BROADCASTING

As mentioned above, a set of  $N$  qubits, equally prepared in the state  $\rho(\mathbf{n}, r) = (1 + r\mathbf{n} \cdot \boldsymbol{\sigma})/2$ , can be viewed as an encoding of the classical information about the direction  $\mathbf{n}$ . Suppose now that we want to distribute such an information to  $M > N$  users, and, at the same time, change the encoding by flipping the direction of the Bloch vector. In other words, we are interested in the best approximation of the impossible transformation

$$\rho(\mathbf{n}, r)^{\otimes N} \rightarrow |-\mathbf{n}\rangle\langle -\mathbf{n}|^{\otimes M}. \quad (40)$$

For pure states, and for  $N=M=1$ , this transformation coincides with the universal NOT gate, which flips the spin of a qubit for any possible direction [11]. In general, the transfor-

mation (40) corresponds to the combination of a perfect purification of the input states, followed by a perfect  $N \rightarrow M$  cloning, and by a perfect flipping of the output qubits. We will call the impossible transformation (40) *ideal universal NOT broadcasting*.

In order to derive the optimal CP map  $\mathcal{N}$  approximating the transformation (40), we define the single-site output state

$$\rho_{out}^1(\mathbf{n}, r) = \text{Tr}_{M-1}[\mathcal{N}(\rho(\mathbf{n}, r)^{\otimes N})], \quad (41)$$

where  $\text{Tr}_{M-1}$  denotes the partial trace over  $M-1$  output qubits. Notice that, since we consider symmetric broadcasting maps, the output state must be invariant under permutations, and the above definition does not depend on the choice of the  $M-1$  qubits that are traced out. The optimization of the map  $\mathcal{N}$  corresponds then to the maximization of the single-site fidelity

$$F(\rho_{out}^1(\mathbf{n}, r), |-\mathbf{n}\rangle\langle-\mathbf{n}|) = \langle-\mathbf{n}|\rho_{out}^1(\mathbf{n}, r)|-\mathbf{n}\rangle. \quad (42)$$

Here we consider universal broadcasting, which corresponds to require the fidelity to have the same value for any direction  $\mathbf{n}$ . Accordingly, the search for the optimal map can be restricted to the class of maps with the covariance property (8). In the following, we denote by  $F_{\text{NOT}}$  the value of the single-site fidelity (42). Moreover, since  $F_{\text{NOT}}$  is a linear function of the map  $\mathcal{N}$ , in order to maximize  $F_{\text{NOT}}$  we can restrict the attention to the set of extremal universal broadcasting maps. Such extremal maps are a finite number, and in the formalism of the Choi isomorphism (7) are characterized by Eq. (19).

To evaluate  $F_{\text{NOT}}$ , we choose the direction  $\mathbf{k}$  of the  $z$  axis in Eq. (42), and exploit the relation

$$r' \doteq \text{Tr}[\rho_{out}^1(\mathbf{k}, r)\sigma_z] = 1 - 2F_{\text{NOT}}. \quad (43)$$

Therefore the optimal map corresponds to the minimum value for  $r'$ . The value of  $r'$  for an extremal map has been calculated in Refs. [3,4]

$$r' = \frac{2}{M}(r_+r_-)^{N/2} \sum_{l=l_0}^{N/2} \beta(J_l, j_l, l) m_l \sum_{n=-l}^l n \left( \frac{r_-}{r_+} \right)^n, \quad (44)$$

where

$$\beta(J, j, l) = \frac{J(J+1) - j(j+1) - l(l+1)}{2l(l+1)}. \quad (45)$$

Since  $r_- \leq r_+$ , the sum  $\sum_{n=-l}^l n(r_-/r_+)^n$  is always negative, and the maximization of the fidelity corresponds to the maximization of  $\beta$  over  $J$  and  $j$ . Of course, for fixed values of  $j$  and  $l$ , to maximize  $\beta$  one has to take  $J$  maximum, i.e.,  $J=j+l \equiv J_l$ . Moreover, since  $\beta(j+l, j, l) = j/(l+1)$ , the maximum  $\beta$  is obtained by maximizing also  $j$ , i.e., by taking  $j=M/2 \equiv j_l$ . Therefore  $\beta_{\text{max}} = M/(2l+2)$ , corresponding to the fidelity

$$F_{\text{NOT}} = \frac{1}{2} \left[ 1 + (r_+r_-)^{N/2} \sum_{l=l_0}^{N/2} \frac{m_l}{l+1} \sum_{m=-l}^l m \left( \frac{r_+}{r_-} \right)^m \right]. \quad (46)$$

Remarkably, this expression coincides with the expression (35) of the fidelity of the optimal estimation of direction.

This proves that there is no better way of performing universal NOT broadcasting than first estimating the direction of the Bloch vector, and subsequently preparing the  $M$  output qubits in the opposite direction with respect to the estimated one, analogously to what happens in the case of pure input states [11,13].

## V. PHASE COVARIANT CASE

In this section we consider the *phase covariant* case, where in Eq. (8), instead of allowing  $U_g$  to move within the whole  $\text{SU}(2)$  group, we restrict it to belong to a fixed proper subgroup  $\text{U}(1) \subset \text{SU}(2)$  of rotations around a fixed axis, say around the  $z$  axis. Channels satisfying this covariance property act “equally well” not on the whole Bloch sphere, as in the universal case, but only on circles orthogonal to the rotation axis. Intuitively, since the phase covariance property is not as strict as in the universal case, we expect that phase covariant procedures generally achieve better performances compared to their universal counterparts.

In Ref. [4] the optimal phase covariant superbroadcasting was derived and was shown to act more efficiently than the optimal universal superbroadcasting. Analogously to the procedure of Sec. III for the universal case, we will now try to figure out whether there exists a classical procedure that achieves phase covariant superbroadcasting and reaches the fidelity of the optimal phase covariant superbroadcaster in the limit of infinite number of final users. In this Section we show that in fact such a measure-and-prepare scheme exists and consists of an optimal phase estimation over mixed qubit states [14,15] followed by the preparation of a suitable pure state.

Let us start considering input states lying on the  $xy$  equator of the Bloch sphere, namely

$$\rho(\phi, r) = \frac{1}{2}(1 + r \cos \phi \sigma_x + r \sin \phi \sigma_y). \quad (47)$$

We then require covariance under the one-phase rotations group around the  $z$  axis, namely

$$U_\phi = e^{i\phi\sigma_z/2}. \quad (48)$$

The action of a unitary operator  $U_\phi$  over a state of the form (47) is

$$U_\phi \rho(\phi_0, r) U_\phi^\dagger = \rho(\phi_0 + \phi, r), \quad (49)$$

hence it is clear that the action of  $\text{U}(1)$  rotates the Bloch vector around the  $z$  axis without affecting its length, namely without changing the purity of the state.

The semiclassical phase covariant broadcasting procedure we propose is the following. We optimally estimate the value of the phase  $\phi$  by a measurement over  $N$  copies of  $\rho(\phi, r)$  given by the POVM density  $P(\phi)$  derived in Ref. [14],

$$P(\phi) = U_\phi^{\otimes N} \xi U_\phi^{\dagger \otimes N}. \quad (50)$$

In the above expression the *seed*  $\xi$  of the optimal POVM is given by

$$\xi = \bigoplus_{j=j_0}^{N/2} (2j+1) \sum_{n=-j}^j |j,n;\mathbf{k}\rangle \langle j,n;\mathbf{k}| \otimes \mathbb{1}_{m_j}, \quad (51)$$

where  $\mathbf{k}$  is the rotation axis and  $|j,n;\mathbf{k}\rangle$  denotes eigenvectors of the angular momentum along  $\mathbf{k}$  with total angular momentum  $j$ . The POVM density  $P(\phi)$  obeys the normalization condition

$$\int_0^{2\pi} \frac{d\phi}{2\pi} P(\phi) = 1. \quad (52)$$

After performing the estimation, which gives the conditional probability density  $p(\hat{\phi}|\phi) = \text{Tr}[\rho(\phi,r)P(\hat{\phi})]$ , the output state for  $M$  final users is prepared as

$$\rho_{out}^M(\phi,r) = \int_0^{2\pi} d\hat{\phi} p(\hat{\phi}|\phi) |\hat{\phi}\rangle \langle \hat{\phi}|^{\otimes M}, \quad (53)$$

where  $|\phi\rangle$  denotes the eigenvector of  $\cos\phi\sigma_x + \sin\phi\sigma_y$  for the eigenvalue  $+1$ . As in the previous sections, we focus on the single-site reduced output, namely

$$\rho_{out}^1(\phi,r) = \int_0^{2\pi} d\hat{\phi} p(\hat{\phi}|\phi) |\hat{\phi}\rangle \langle \hat{\phi}|, \quad (54)$$

and the fidelity of this procedure is given by

$$F = \langle \phi | \rho_{out}^1(\phi,r) | \phi \rangle. \quad (55)$$

Following the same arguments presented in Sec. III, it can be proved that  $\rho_{out}^1(\phi,r)$  and  $\rho(\phi,r)$  have parallel Bloch vectors, that is,

$$\rho_{out}^1(\phi,r) = \frac{1}{2} (1 + r' \cos\phi\sigma_x + r' \sin\phi\sigma_y), \quad (56)$$

and the fidelity can be again calculated as  $F = (1+r')/2$ . By exploiting the results of Ref. [14], the single-site output Bloch vector length  $r'$  turns out to be

$$r' = 4(r_+ r_-)^{N/2} \sum_{j=j_0}^{N/2} m_j \text{Tr} \left[ E_+^{(j)} \left( \frac{r_+}{r_-} \right)^{j(j)} \right], \quad (57)$$

where  $E_+^{(j)} = \sum_{m=-j}^{j-1} |j,m+1;\mathbf{k}\rangle \langle j,m;\mathbf{k}|$ . In Fig. 2 we report the plot of the scaling factor  $p(r) = r'/r$  for the phase covariant classical broadcasting procedure.

Notice that its performances are always better than in the universal case reported in Fig. 1. As expected, the single-site output Bloch vector length (57) coincides with the corresponding quantity calculated for the optimal phase covariant superbroadcaster in Ref. [4] in the limit of infinite output copies.

Finally, notice that in the phase covariant case for states of the form (47) the NOT gate can always be achieved unitarily by a  $\pi$  rotation around the  $z$  axis. Therefore the optimal phase covariant NOT broadcasting has the same fidelity as the optimal phase covariant superbroadcaster in Ref. [4].

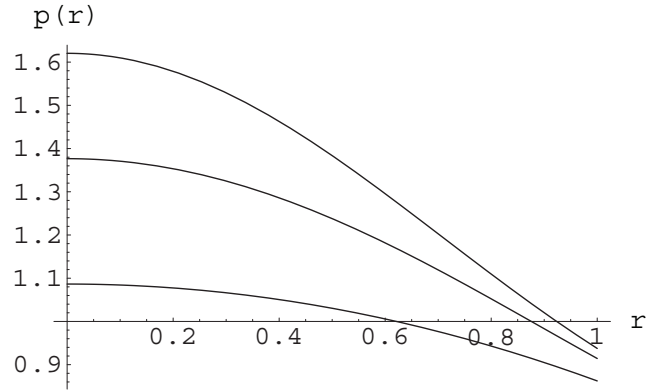


FIG. 2. Scaling factor  $p(r) = r'/r$ , with  $r'$  given in Eq. (57), for the classical phase covariant broadcasting procedure. The three curves, from bottom to top, refer to  $N=4, 6, 8$  input copies, respectively. Compared with the universal case, shown in Fig. 1, the phase covariant procedure always achieves better performances.

## VI. ASYMPTOTIC SUPERBROADCASTING AND STATE ESTIMATION

Recently, Bae and Acín gave an argument to prove that the asymptotic cloning of pure states is equivalent to state estimation [10]. The argument consists in noticing that, when restricted to a single output Hilbert space, a symmetric cloning from  $N$  to  $M=\infty$  copies is an entanglement breaking channel [16], and, therefore, it can be realized by the semiclassical measure-and-prepare scheme, namely the single user output states are given by

$$\rho_{out}^1 = \sum_i \text{Tr}[P_i \rho^{\otimes N}] \rho_i, \quad (58)$$

where the POVM  $\{P_i\}$  represents the quantum measurement performed on the input, and  $\rho_i$  is the (single user) output state prepared conditionally to the outcome  $i$ . As a consequence, if the input of the cloning machine is the pure state  $|\psi\rangle$ , then the single clone fidelity is  $F_{clon}[N, \infty] = \langle \psi | \rho_{out}^1 | \psi \rangle$ , and coincides with the estimation fidelity  $F_{est}[N] = \sum_i \text{Tr}[P_i \rho^{\otimes N}] \langle \psi | \rho_i | \psi \rangle$  of the POVM  $\{P_i\}$  with the guess states  $\{\rho_i\}$ . This proves that the problem of optimal symmetric  $N$ -to- $\infty$  cloning is equivalent to the problem of optimal state estimation with  $N$  input copies, and  $F_{clon}[N, \infty] = F_{est}[N]$ .

In the case of mixed states, a similar argument can be exploited to give a general explanation to the fact that in the ideal case of infinite users the fidelity of the optimal superbroadcasting is achieved by a semiclassical scheme. In fact, analogously to Ref. [10] since the output states of superbroadcasting are invariant under permutations, for  $M=\infty$  also the superbroadcasting transformation is an entanglement breaking channel, when restricted to a single user. Therefore, it can be realized by measurement and subsequent reparation, and the single user output states are written as in Eq. (58), with suitable  $\{P_i\}$  and  $\{\rho_i\}$ . Moreover, as for the case of cloning, also in the case of superbroadcasting the figure of merit is the fidelity of the output state with a *pure* state—the eigenvector of the input density operator corresponding to

the maximal eigenvalue [see Eq. (27) for the universal, and Eq. (55) for the phase covariant case]. It is then clear that asymptotically the fidelity of the optimal universal (phase covariant) superbroadcasting coincides with that of the optimal estimation of direction (phase). In general, the above reasoning shows that superbroadcasting with infinite users is equivalent to the estimation of the eigenstate corresponding to the largest eigenvalue of the input density matrix. This result generalizes the well known relation between cloning and state estimation to the case of mixed states.

## VII. CONCLUSIONS

In this paper we considered the problem of quantum broadcasting, and in particular we analysed the possibility of broadcasting  $N$  input qubit states to  $M$  output qubits with the same Bloch vector direction, just by estimating the direction by a collective measurement on the input qubits and then preparing  $M$  outputs correspondingly. The main result is that this strategy allows to achieve superbroadcasting, namely to have output copies which are even more pure than the input ones, at the expense of classical correlations in the global output state. This superbroadcasting is suboptimal, but asymptotically converges to the optimal one, confirming

also in the case of mixed states the fact that state estimation and cloning are asymptotically equivalent. We first considered the universal broadcasting, and then the broadcasting of the antipodal state, the so-called universal NOT. For this purpose, we proved that the semiclassical strategy is optimal. Finally, we considered the phase covariant version of the broadcasting problem, showing that superbroadcasting occurs with suboptimal purification rate, which is still better than the one for universal semiclassical superbroadcasting. The main interest of the summarized results is twofold. On one hand, our results prove that superbroadcasting can be achieved by a semiclassical procedure, and then coherent manipulation of quantum information is not necessary, even though optimal superbroadcasting requires it. On the other hand, the practical interest of our results is that the semiclassical rates exhibit a good approximation of the optimal rates, and can be more easily achieved experimentally.

## ACKNOWLEDGMENTS

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