

The Dirac quantum automaton: a short review

Giacomo Mauro D'Ariano and Paolo Perinotti

QUIT Group, Dipartimento di Fisica, Università di Pavia, INFN, Gruppo IV, sezione di Pavia, via Bassi 6, I-27100 Pavia, Italy

E-mail: dariano@unipv.it

Received 11 July 2014, revised 10 September 2014

Accepted for publication 22 September 2014

Published 19 December 2014

Abstract

We briefly review the derivation from first principles of the quantum cellular automata, from which the free quantum field theory (Weyl, Dirac and Maxwell) emerges in the relativistic limit. We illustrate the deviations from the relativistic dynamics occurring at the Planck scale in terms of the main vectors that rule the evolution of wave packets.

Keywords: quantum cellular automata, quantum walks, Dirac equation, Weyl equation

1. Introduction

The revolution in physics that started at the beginning of the 20th century with the invention of quantum mechanics led to one of the most astonishing paradigm shifts in the history of science. The amount and relevance of the commonly held beliefs that quantum mechanics forces us to abandon is so significant that even after a century of experimental success, many physicists still feel uncomfortable with the theory.

In the last two decades, the advent of quantum information theory provided an unprecedented advance in the understanding of the conceptual structure of quantum theory [1]. This happened by thinking of quantum systems as abstract, hardware-independent information carriers, rather than just mathematical descriptions of particles, in the very same way as bits in classical information theory. Looking at quantum systems in this way provides us with a new, lucid, nonparadoxical perspective for the abstract and elusive theory of Hilbert spaces, which can now be derived from down-to-earth principles [2]—a result that was unsuccessfully sought for several decades after the fertile Birkhoff's and Von Neumann's quantum logic program [3].

The above research experience has demonstrated the astonishing power of regarding information as more fundamental than matter—the informational paradigm by Wheeler [4] and Feynman [5]. Quantum theory is indeed a theory of information, as it can be axiomatically derived from six axioms of pure information-theoretical nature [2, 6]. The axioms describe six information processing tasks, and postulate their achievability or impossibility.

The first five of the postulates are shared by classical information theory. The one that singles out quantum theory is the principle of purification [13]. Quantum theory is a special operational probabilistic theory, which means that it describes processes in terms of circuits of input–output relations between events (i.e., transformations—) and associates probabilities to closed circuits between preparations and observations. The postulates that pin down the full quantum theory of abstract systems do not have any effect on the mechanical part of the theory, which is thus undetermined at this stage. The gap left in the informational formulation of quantum mechanics calls for new principles that lead to quantum field theory.

In a series of papers, some of which were presented at previous editions of the Vaxjo conference of the present volume [7–12], one of the authors of this paper proposed the extension of the information-theoretic axiomatization program to quantum field theory, and, therefore, to the whole of physics. Some additional axioms have been introduced to the informational axioms of quantum theory; these axioms, concern the network of the information-processing that describes the physical law. The general principle is the minimization of the algorithmic complexity of the information processing that describes the physical law. This implies minimizing the dimension of the denumerable quantum systems in interaction. A huge reduction of the algorithm's complexity is achieved by requiring linearity, unitarity, locality, homogeneity, and isotropy of the interactions. This reduces the the physical law to a quantum cellular automaton (QCA). The lattice of the

automaton represents the topological structure of the computation, which has no ‘physical’ location; space and time —along with the physical law and its relativistic covariance —emerge from the computation. In [13], a clear-cut derivation of Weyl’s and Dirac’s equations have been given, following the new additional information-theoretic principles.

From the above assumptions, only two QCA follow, and Lorentz covariance is broken. Both automata converge to the Weyl equation in the relativistic limit of small wave vectors. In the opposite regime of ultrarelativistic wave vectors, Lorentz covariance is distorted, and one has additional invariants in terms of energy and length scales [14]. This feature is the characteristic trait of the deformed space–time of doubly special relativity by Amelino-Camelia [15] or the deformed Lorentz symmetry by Smolin and Magueijo [16]. It was proved in the literature that these deformed Lorentz symmetries exhibit the phenomenon of relative locality [17, 18], which consists of the observer-dependence of coincidence in space, generalizing the relativity of the simultaneity of special relativity.

The QCA thus provides an extended quantum field theory that unifies the Planck scale with the Fermi scale. Starting from the Weyl automaton pair as the building block, one can build other QCA pairs by direct sum and tensor products. In this way, one achieves automata that give the Dirac’s and the Maxwell field equations in the relativistic limit. The free electromagnetic field is recovered in the form of the neutrino theory of light [19] of De Broglie [20], considered also by Jordan [21], Born and Nagendra Nath [22, 23], and Kronig [24], and rediscovered in more recent times by Perkins [25]. With the principle of bounded information density, bosonic fields also become an emergent notion, and they are ultimately based on Fermionic systems evolved by the automaton.

The discreteness of the lattice avoids all problems that plague quantum field theory arising from the continuum, especially the problem of localization. Most relevantly, the theory is quantum *ab initio*, with no need of quantization rules. This is the bonus of regarding information as more fundamental than matter.

2. Quantum systems in interaction

The starting point for the construction of space–time and the physical laws therein is an unstructured, countably infinite set, G , of local Fermionic modes. These must be thought of as elementary quantum information carriers, as in the Fermionic quantum computation of [28–29], with labels that are logical addresses and do not represent space coordinates. Indeed, space will emerge from the evolution of the Fermionic systems, rather than being their container. It is natural to wonder why we introduce Fermionic quantum systems instead of qubits at this stage. The reason is that a register of qubits is incompatible with the requirements that

we will impose in the following, and in particular in linearity. The fermionic nature of field follows from the isotropy requirement along with the minimality of the field dimension. The derivation of Fermionic quantum theory from information theoretic axioms, namely as an operational probabilistic theory, can be achieved following methods very similar to those in [29].

Fermionic modes are conveniently described by the field operators, $\{\psi_{g,s}\}_{g \in G}$, satisfying the usual anticommutation relations

$$\{\psi_{g,l}, \psi_{g',l'}\} = 0, \quad \{\psi_{g,l}, \psi_{g',l'}^\dagger\} = \delta_{g,g'} \delta_{l,l'} I. \quad (1)$$

Note that we introduce the label $1 \leq l \leq s$ because in general we allow the site, g , to host a finite number $s \geq 1$ of local modes. In the following, we will denote the formal column vector by the symbol ψ_g

$$\psi_g := \begin{pmatrix} \psi_{g,1} \\ \psi_{g,2} \\ \vdots \\ \psi_{g,s} \end{pmatrix}. \quad (2)$$

The requirements of unitarity, linearity, locality, homogeneity, and isotropy on the QCA expressing the dynamics of the Fermionic modes provide the set, G , with a finitely presented group structure [13], with generator set $S = S_+ \cup S_-$, with $S_- := \{h^{-1}, h \in S_+\}$. Moreover, if we draw a colored arrow from g to g' iff $g' = gh$ for $h \in S_+$, with colors corresponding to generators $h \in S_+$, we build up a Cayley graph of group G .

We express linearity of the cellular automaton as follows

$$\psi_g(t+1) = \sum_{h \in S} A_h \psi_{gh}(t). \quad (3)$$

The isotropy requirement consists of the existence of a group, L , of permutations, acting transitively on S_+ , that is a graph automorphism of the Cayley graph of G , with a (possibly projective) unitary representation, V , such that

$$\forall l \in L: \quad A_{l(h)} = V_l A_h V_l^\dagger, \quad \forall h \in S_+. \quad (4)$$

The unitarity requirement corresponds the unitarity of the following operator on $\ell^2(G) \otimes \mathbb{C}^s$

$$A := \sum_{h \in S} T_h \otimes A_h, \quad (5)$$

where the unitary representation, T , of G on $\ell^2(G)$ is conveniently defined by the right multiplication $T_h |g\rangle := |gh\rangle$, with $\{|g\rangle\}_{g \in G}$ denoting the canonical orthonormal basis of $\ell^2(G)$. This operator acts on a single copy of the field, ψ , which is considered to be an element of $\ell^2(G) \otimes \mathbb{C}^s$. By linearity, the action over N copies of the field is just the N th tensor power of A . The operator, A , is what is usually called the *quantum walk*. However, here it also defines the full QCA, due to linearity.

We now focus on the case where the group, G , can be quasi-isometrically embedded in \mathbb{R}^d . In such a case, group

G is a virtually Abelian group, having \mathbb{Z}^d as the Abelian subgroup of finite index, i_G (namely with a finite number of cosets)¹. It is possible to prove that an automaton with such a G can be recast as an automaton on \mathbb{Z}^d with the field-array dimension, $s' = si_G$. Among the infinitely many Cayley graphs of G , we will restrict our attention to those for which the embedding is also isotropic, namely the action of L does not change the length in the embedding space.

For the above reasons, we will restrict our attention to the group $G = \mathbb{Z}^d$, and, in particular, $d = 3^2$. We will therefore adopt the additive composition notation. The isotropy requirement restricts Cayley graphs of \mathbb{Z}^3 to the three-dimensional Bravais lattices.

Being \mathbb{Z}^3 Abelian, the operators $T_{\mathbf{h}_i}$, $\{\mathbf{h}_i\} = S_+$ can be jointly diagonalized, and the automaton can be expressed in terms of the $s \times s$ unitary matrices, $\tilde{A}_{\mathbf{k}} := \sum_{\mathbf{h}_i \in S} e^{-i\mathbf{h}_i \cdot \mathbf{k}} A_{\mathbf{h}_i}$, 1 acting on the eigenspaces of $T_{\mathbf{h}_i} \otimes I$. In this case, one can prove that a nontrivial unitary automaton can exist only for $s > 1$ [13]. In particular, if we take the minimal dimension $s = 2$, the unitarity condition is satisfied only by the body-centered cubic (BCC) lattice Cayley graph, whose generators are $S_+ = \{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4\}$, related by the following constraint

$$\mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3 + \mathbf{h}_4 = 0. \quad (7)$$

There are only two unitary automata, A^\pm , for $s = 2$ on the BCC lattice, corresponding to the matrices

$$\tilde{A}_{\mathbf{k}}^\pm := d_{\mathbf{k}}^\pm I - i\tilde{\mathbf{n}}_{\mathbf{k}}^\pm \cdot \boldsymbol{\sigma}^{(\pm)}, \quad (8)$$

$$d_{\mathbf{k}}^\pm := c_x c_y c_z \mp s_x s_y s_z, \quad (9)$$

$$(\tilde{\mathbf{n}}_{\mathbf{k}}^\pm)_x := s_x c_y c_z \pm c_x s_y s_z, \quad (10)$$

$$(\tilde{\mathbf{n}}_{\mathbf{k}}^\pm)_y := c_x s_y c_z \mp s_x c_y s_z, \quad (11)$$

$$(\tilde{\mathbf{n}}_{\mathbf{k}}^\pm)_z := c_x c_y s_z \pm s_x s_y c_z, \quad (12)$$

where $\sigma_i^{(+)} := \sigma_i$ and $\sigma_i^{(-)} := \sigma_i^*$ are conjugated representations of the Pauli matrices, $s_i := \sin \frac{k_i}{\sqrt{3}}$ and $c_i := \cos \frac{k_i}{\sqrt{3}}$. The eigenvalues of $\tilde{A}_{\mathbf{k}}^\pm$ are $e^{-i\omega_{\mathbf{k}}^\pm}$, $e^{i\omega_{\mathbf{k}}^\pm}$, with dispersion relations $\omega_{\mathbf{k}}^\pm := \arccos d_{\mathbf{k}}^\pm$. If we define parity reflection, P , by $\mathbf{k} \rightarrow -\mathbf{k}$, we can observe that the two Weyl automata are converted into each other by the P symmetry followed by a σ_y rotation.

If we interpolate the discrete dynamics by a continuous time, we can express the evolution of the automaton through

¹ The embedding, \mathbb{T} , of a group, G , in a metric space, R , is quasi-isometric if the following bounds hold

$$\frac{1}{a}d(g, f)_G - b \leq d(\mathbb{T}(g), \mathbb{T}(f))_R \leq ad(g, f)_G + b, \quad (6)$$

for some constants $a, b \in \mathbb{R}$, where $d(x, y)_R$ is the distance in R while $d(x, y)_G$ is the distance induced by the word-metric in G . Moreover, the distance of any element of R from the image of G is uniformly bounded.

² It is still not proved that the automaton on the Abelian subgroup retains the isotropy property.

the following differential equation

$$i\partial_t \psi_{\mathbf{k},t} = \mathbf{n}_{\mathbf{k}}^\pm \cdot \boldsymbol{\sigma}^\pm \psi_{\mathbf{k},t}, \quad (13)$$

with $\mathbf{n}_{\mathbf{k}}^\pm := \frac{\omega_{\mathbf{k}}^\pm}{\sqrt{1-d_{\mathbf{k}}^{\pm 2}}} \tilde{\mathbf{n}}_{\mathbf{k}}^\pm$. If we now interpolate the lattice by the continuum \mathbb{R}^3 , allowing for real coordinates, the equation describes a spinorial plane wave whose phase velocity is $\omega_{\mathbf{k}}^\pm$, as in Weyl's equation, with polarizations described by the generalized helicity $\mathbf{n}_{\mathbf{k}}^\pm \cdot \boldsymbol{\sigma}^\pm$, representing the projection of spin along $\mathbf{n}_{\mathbf{k}}^\pm$ instead of the wave vector, \mathbf{k} . Moreover, in the limit of small wave vector $|\mathbf{k}| \ll 1$ one can easily verify that the differential equation becomes exactly Weyl's equation

$$i\partial_t \psi_{\mathbf{k},t} = \frac{1}{\sqrt{3}} \mathbf{k} \cdot \boldsymbol{\sigma}^\pm \psi_{\mathbf{k},t}. \quad (14)$$

It is worth stressing that if we take the time step of the order of the Planck time, t_P , and the lattice step of the order of the Planck length, l_P , the small-wave-vector limit encompasses all relativistic regimes tested so far in high-energy physics experiments (for a detailed estimation of the order of magnitude of first-order corrections, see [13]).

According to equation (14) the evolution of a wave packet that peaked around the wave vector, \mathbf{k} , is ruled by the group velocity

$$\mathbf{v}_{\mathbf{k}}^\pm := \nabla_{\mathbf{k}} \omega_{\mathbf{k}}^\pm, \quad (15)$$

which is the speed of translation of the wave packet in the position representation. In the relativistic limit $|\mathbf{k}| \ll 1$, this amounts to

$$\mathbf{v}_{\mathbf{k}}^\pm = \frac{1}{\sqrt{3}} \frac{\mathbf{k}}{|\mathbf{k}|}, \quad (16)$$

whose modulus is clearly constant. In order to have $|\mathbf{v}_{\mathbf{k}}^\pm| = c$ in the relativistic limit, it is then necessary to take the ratio between the lattice-step, a_0 , and the time-step, t_0 , as $a_0/t_0 = \sqrt{3}c$; if we keep $t_0 = t_P$, then $a_0 = \sqrt{3}l_P$; it is half of the BCC cell edge. This choice removes the factor $\frac{1}{\sqrt{3}}$ in equation (14) when converting to dimensionful units. The automaton then introduces corrections to physical predictions from the standard model. The main correction consists of the first-order correction to the relativistic group velocity of Weyl spinors of equation (16), which reads

$$(\mathbf{v}_{\mathbf{k}}^\pm)_i = \mp \frac{k_x k_y k_z}{3|\mathbf{k}|^2} \times \frac{1 + \sqrt{3}(k_i/|\mathbf{k}|)^2}{k_i/|\mathbf{k}|}. \quad (17)$$

This term is clearly anisotropic, as expected from the very beginning due to the breaking of rotational symmetry by the lattice structure. The appearance of this kind of anisotropy was predicted in Feynman's proposal of a QCA model for the simulation of quantum physics in [5].

We now consider the Dirac automata E^\pm , defined by the matrices, $\tilde{E}_{\mathbf{k}}^\pm$, obtained as a direct sum of two Weyl automata, with a term independent of \mathbf{k} representing a local coupling, as

follows

$$\tilde{E}_{\mathbf{k}}^{\pm} = \begin{pmatrix} n\tilde{A}_{\mathbf{k}}^{\pm} & imI \\ imI & n\tilde{A}_{\mathbf{k}}^{\pm\dagger} \end{pmatrix}, \quad (18)$$

with $n^2 + m^2 = 1$. If we now define charge conjugation by conjugation with $\gamma^0\gamma^2$, parity reflection by $\mathbf{k} \rightarrow -\mathbf{k}$, and time reversal by the adjoint of the unitary operators $\tilde{A}_{\mathbf{k}}^{\pm}$, the two automata are interchanged by charge conjugation, parity and time reversal (CPT).

Also, in this case the dynamics is determined by the eigenvalues of $E_{\mathbf{k}}^{\pm}$ which can be expressed as $e^{-i\omega_{\mathbf{k}}^{\pm}}$, $e^{i\omega_{\mathbf{k}}^{\pm}}$ with

$$\omega_{\mathbf{k}}^{\pm} := \arccos\left(\sqrt{1 - m^2 d_{\mathbf{k}}^{\pm}}\right). \quad (19)$$

For the Dirac automata, the interpolating Hamiltonian gives the following differential equation

$$i\partial_t\varphi_{\mathbf{k},t} = \left(n\mathbf{n}_{\mathbf{k}}^{\pm} \cdot \boldsymbol{\alpha}^{\pm} + m\beta\right)\varphi_{\mathbf{k},t}, \quad (20)$$

where φ is now a Fermionic field with $s = 4$, and the matrices $\boldsymbol{\alpha}$ and β are given by $\alpha_i^{\pm} := \sigma_z \otimes \sigma_i^{\pm}$, and $\beta = i\sigma_x \otimes I$. Upon defining $\gamma^0 := \beta$ and $\gamma^i := \beta\alpha_i$, we obtain the Weyl representation of Dirac's matrices, γ^{μ} , satisfying $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}I$. In the relativistic limit $|\mathbf{k}| \ll 1$, and for small values of m , one has $n = 1$ at the first-order expansion in m , and the differential equation becomes

$$i\partial_t\varphi_{\mathbf{k},t} = \left(\frac{1}{\sqrt{3}}\mathbf{k} \cdot \boldsymbol{\alpha}^{\pm} + im\beta\right)\varphi_{\mathbf{k},t}, \quad (21)$$

which is Dirac's equation in the wave-vector representation.

3. Conclusion

We reviewed the derivation of Weyl's and Dirac's equations as approximate laws ruling the propagation of information on two Fermionic QCAs, which can be derived as the only two possible automata satisfying unitarity, linearity, homogeneity, and isotropy on a three-dimensional Bravais lattice. The geometry of the lattice is determined by the requirement of quasi-isometric embedding in the Euclidean space, \mathbb{R}^3 . Remarkably, we obtain relativistically invariant equations without assuming relativity at any step. This implies that the Lorentz symmetry is recovered as an emergent feature of our dynamical laws, which is only approximately valid in the relativistic limit of small wave vectors. On the other hand, the Lorentz symmetry is generally violated both by Weyl's and Dirac's automata, as can be expected by observing the violation of charge conjugation and parity and CPT by the Weyl and Dirac automata, respectively. The way in which the

Lorentz invariance is distorted in the one-dimensional case is the subject of [14], where an instance of doubly special relativity is derived from the automaton dynamics. A similar analysis can be carried out also in the three-dimensional case.

Acknowledgments

This work has been supported in part by the Templeton Foundation under the project ID# 43796, *A Quantum-Digital Universe*.

References

- [1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [2] Chiribella G, D'Ariano G M and Perinotti P 2011 *Phys. Rev. A* **84** 012311
- [3] Birkhoff G and von Neumann J 1936 *Ann. Math.* **37** 743
- [4] Wheeler J A 1990 *Complexity, Complexity, Entropy, and the Physics of Information* ed W H Zurek (Redwood City, CA: Addison-Wesley)
- [5] Feynman R 1982 *J. Int. Theor. Phys.* **21** 467
- [6] D'Ariano G M 2011 *AIP Conf. Proc.* **1327** 7–18
- [7] D'Ariano G M 2010 *AIP Conf. Proc.* **1232** 5
- [8] D'Ariano G M 2010 *AIP Conf. Proc.* **1232** 3
- [9] D'Ariano G M 2012 *Phys. Lett. A* **376** 678
- [10] D'Ariano G M 2011 *AIP Conf. Proc.* **1327** 7
- [11] D'Ariano G M 2012 *AIP Conf. Proc.* **1508** 146
- [12] D'Ariano G M 2012 *Il Nuovo Saggiatore* **28** 13
- [13] Chiribella G, D'Ariano G M and Perinotti P 2010 *Phys. Rev. A* **81** 062348
- [14] D'Ariano G M and Perinotti P 2013 *Phys. Rev. A* at press
- [15] Bibeau-Delisle A, Bisio A, D'Ariano G M, Perinotti P and Tosini A 2013 arXiv:1310.6760
- [16] Amelino-Camelia G 2001 *Phys. Lett. B* **510** 255–63
- [17] Magueijo J and Smolin L 2002 *Phys. Rev. Lett.* **88** 190403
- [18] Schutzhold R and Unruh W G 2003 *JETP Lett.* **78** 431
- [19] Amelino-Camelia G, Freidel L, Kowalski-Glikman J and Smolin L 2011 *Phys. Rev. D* **84** 084010
- [20] Bisio A, D'Ariano G M and Perinotti P 2014 arXiv:1407.6928
- [21] de Broglie L 1932 *Comput. Rend.* **195** 536–862
- [22] Jordan P 1935 *Z. Phys.* **93** 464
- [23] Born M and Nagendra Nath N S 1936 *Proc. Indian Acad. Sci.* **3** 318
- [24] Born M and Nath N S Nagendra 1936 *Proc. Indian Acad. Sci.* **6** 611
- [25] de R and Kronig L 1936 *Physica* **3** 1120–32
- [26] Perkins W A 1965 *Phys. Rev.* **137** 1291
- [27] Bravyi S B and Kitaev A Y 2002 *Ann. Phys.* **298** 210–26
- [28] Bañuls M-C, Cirac J I and Wolf M M 2007 *Phys. Rev. A* **76** 022311
- [29] D'Ariano G M, Manessi F, Perinotti P and Tosini A 2014 *Europhys. Lett.* **107** 20009
- [30] D'Ariano G M, Manessi F, Perinotti P and Tosini A 2014 *Int. J. Mod. Phys. A* **29** 1430025