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Informationally complete measurements and group representation

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Abstract
Informationally complete measurements on a quantum system allow one to estimate the expectation value of any arbitrary operator by just averaging functions of the experimental outcomes. We show that such kinds of measurement can be achieved through positive-operator valued measures (POVMs) related to unitary irreducible representations of a group on the Hilbert space of the system. With the help of frame theory we provide a constructive way to evaluate the data-processing function for arbitrary operators.

Keywords: quantum measurements, operator spaces, frames, groups, covariant POVMs

1. Introduction

The aim of any measurement is to retrieve information on the state of a physical system. In classical mechanics, measuring the location on the phase space provides complete information on the system. On the other hand, in quantum mechanics there are infinitely many elementary measurements—corresponding to different observables—that provide only partial information, whereas 'complementary' information can be achieved only with mutually exclusive experiments where non-commuting observables should be perfectly measured.

The problem then arises as to how to achieve a kind of quantum measurement that is 'complete' [1, 2], in the sense that it can be used to infer information on all possible (also exclusive) observables. The main idea is to perform a generalized 'unsharp' measurement, described by a so-called POVM (positive-operator valued measure), from which a specific type of information—i.e., a particular ensemble average of a given operator—is retrieved by just changing the data-processing of its experimental outcomes.

Informationally complete measurements are relevant for foundations of quantum mechanics as a kind of 'standard' for a purely probabilistic description [3]. Moreover, the existence of such measurements with a minimal number of outcomes is crucial for the quantum version of the de Finetti theorem [4].

The most popular example of informationally complete measurement is given by the coherent-state POVM for a single mode of the radiation field, whose probability distribution is the so-called Q-function (or Husimi function)[2] [5, 6]. Another example, though of a completely different kind, is the case of quantum tomography [7], in which one measures an observable randomly selected from an informationally complete set—a 'quorum'.

Investigations on informationally complete measurements have been extensively carried out in the framework of 'phase-space observables': various monographs [8–14] review different aspects of these developments. Phase-space observables are very useful in various fields of quantum physics, including quantum communication and information theory. They also lead to substantial advancement on some relevant conceptual issues, such as the problem of jointly measuring non-commuting observables, or the problem of the classical limit for quantum measurements. However, the problem of classifying all possible informationally complete measurements, also in view of feasibility, was never investigated in generality, and only elementary physical systems have been considered: the harmonic oscillator and the spin.

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[2] However, because of convergence problems this function gives only expectation values of operators that admit anti-normal ordered expansion [6].

In this paper, we present a more general treatment of the problem based on group-theoretic techniques. We will see that informationally complete measurements can be achieved through POVMs derived from unitary irreducible representations of a group on the Hilbert space of the system. With the help of frame theory, we will also provide a constructive way to evaluate the data-processing function for estimating ensemble averages of arbitrary operators.

The paper is organized as follows. In section 2 we prove the equivalence of the informational completeness of a measurement and the invertibility of an operator constructed with the POVM. This proof makes use of frame theory [15, 16], and also shows how to obtain the data-processing function for arbitrary operator. In section 3 we derive the conditions of informational completeness for POVMs that are covariant with respect to a (compact) group that has unitary irreducible representation on the Hilbert space of the system. We devote section 4 to explicit examples of informationally complete POVMs with different covariance group. In the example of the Weyl–Heisenberg group, we recover the results of [17]. Some concluding remarks are given in section 5.

2. Info-complete POVMs and frame of operators

In the following, we will make extensive use of the isomorphism [18] between the Hilbert space of the Hilbert–Schmidt operators A, B on $\mathcal{H}$, with scalar product $\langle A, B\rangle = \mathrm{Tr}[A^\dagger B]$, and the Hilbert space of bipartite vectors $|A\rangle\rangle, |B\rangle\rangle \in \mathcal{H}\otimes\mathcal{H}$, with $\langle\langle A|B\rangle\rangle \equiv \langle A, B\rangle$, and

$$|A\rangle\rangle = \sum_{n=1}^{d}\sum_{m=1}^{d} A_{nm}|n\rangle\otimes|m\rangle, \tag{1}$$

where $|n\rangle$ and $|m\rangle$ are fixed orthonormal bases for $\mathcal{H}$, $d = \dim\mathcal{H}$, and $A_{nm} = \langle n|A|m\rangle$. Notice the identities

$$A\otimes B|C\rangle\rangle = |ACB^\tau\rangle\rangle, \qquad A\otimes B^\dagger|C\rangle\rangle = |ACB^*\rangle\rangle, \tag{2}$$

where τ and $*$ denote transposition and complex conjugation with respect to the fixed bases in equations (1).

An informationally complete measurement for a quantum system with Hilbert space $\mathcal{H}$ is described by a POVM $\{\Pi_i\}$, $\Pi_i \geqslant 0$ and $\sum_i \Pi_i = I$, that allows one to obtain the expectation value of any operator O of the system in the state ρ as follows:

$$\langle O\rangle \equiv \mathrm{Tr}[\rho O] = \sum_i f_i(O)\,\mathrm{Tr}[\rho\Pi_i], \tag{3}$$

where $f_i(O)$ is the *data processing* function of the outcome i which depends on the operator O. Such a POVM will be referred to in short as an 'info-complete' POVM. Since equation (3) holds for any state ρ, it holds generally at the operator level without the ensemble average; namely, one has the expansion for operators

$$O = \sum_i f_i(O)\Pi_i. \tag{4}$$

Equation (4) states that the set of positive operators Π_i spans the linear space of operators of the system. Spanning sets of operators have been already used in quantum tomography [19]: their characterization as spanning sets of operators is naturally accomplished in the context of *frame theory* [15, 16].

An operator *frame* $\{\Xi_i\}$ is simply a set of operators Ξ_i that spans a normed linear space of operators; namely, there are two constants a and b, with $0 < a \leqslant b < \infty$, such that for all operators A one has $a\|A\|^2 \leqslant \sum_i |c_i(A)|^2 \leqslant b\|A\|^2$, where $c_i(A)$ are the coefficients of the expansion of A over the set. Here, for simplicity, we will consider the (Hilbert) space of Hilbert–Schmidt operators on $\mathcal{H}$, whence the norm will be the Frobenius norm $\|A\|_2 = \sqrt{\mathrm{Tr}[A^\dagger A]}$.

For an operator frame $\{\Xi_i\}$ there exists another frame $\{\Theta_i\}$—called the *dual frame*—giving the operator expansion in the form

$$A = \sum_i \mathrm{Tr}[\Theta_i^\dagger A]\Xi_i. \tag{5}$$

The completeness relation of the frame and its dual reads

$$\sum_i \langle\psi|\Xi_i|\phi\rangle\langle\varphi|\Theta_i^\dagger|\eta\rangle = \langle\psi|\eta\rangle\langle\varphi|\phi\rangle, \tag{6}$$

for any $\phi, \varphi, \psi, \eta \in \mathcal{H}$. For continuous sets, the sums in equations (5) and (6) are replaced by integrals. Given a frame $\{\Xi_i\}$, generally the dual set is not unique. However, all duals $\{\Theta_i\}$ of a given frame can be obtained via the linear relation [20]

$$|\Theta_i\rangle\rangle = F^{-1}|\Xi_i\rangle\rangle + |Y_i\rangle\rangle - \sum_j \langle\langle\Xi_j|F^{-1}|\Xi_i\rangle\rangle|Y_j\rangle\rangle, \tag{7}$$

where Y_i are arbitrary, and the positive and invertible operator F can be written as

$$F = \sum_i |\Xi_i\rangle\rangle\langle\langle\Xi_i|. \tag{8}$$

The operator F is called the 'frame operator' in frame theory, whereas the set of operators corresponding to the vectors $F^{-1}|\Xi_i\rangle\rangle$ through the above isomorphism is the so-called 'canonical dual' frame. On the other hand, given an arbitrary set of operators $\{\Xi_i\}$, the invertibility of F in equation (8) implies that the set is a frame. Notice that if the frame is bi-orthogonal, namely $\langle\langle\Xi_i|F^{-1}|\Xi_j\rangle\rangle = \delta_{ij}$, then the canonical one is the unique dual frame. One can also prove the converse statement [16], whence bi-orthogonality is equivalent to uniqueness of the canonical dual frame.

From the above considerations it follows that a POVM $\{\Pi_i\}$ is info-complete if and only if the corresponding operator $F = \sum_i |\Pi_i\rangle\rangle\langle\langle\Pi_i|$ is invertible. From linearity, by comparing equations (4) and (5), one can see that a dual frame of an info-complete POVM provides a data processing function as $f_i(O) = \mathrm{Tr}[\Theta_i^\dagger O]$, whence equation (7) allows a useful flexibility in the data processing, with the possibility of minimizing the statistical error of the estimation by varying the free operators Y_i. In fact, one can generally look for the dual set that minimizes the rms error in the expectations of arbitrary Hermitian operators, as we have shown in [22].

Since the number of elements of an operator frame for $\mathcal{H}$ cannot be smaller than d^2, an info-complete POVM is necessarily not orthogonal, whence it is overcomplete. Hence, it is simple to prove that an arbitrary frame for operators in $\mathcal{H}$ made of positive operators $\{K_i\}$ allows one to construct an info-complete POVM. In fact, since the operator $S \equiv \sum_i K_i$ is invertible, the set $\{\tilde{K}_i = S^{-1/2}K_iS^{-1/2}\}$ satisfies the completeness relation $\sum_i \tilde{K}_i = I$.

3. Group-theoretic techniques

The representation theory of groups provides the easiest way to construct frames made of unitary operators. Consider for example a unitary irreducible representation (UIR) $\{U_g, g \in G\}$ of a compact group G on the Hilbert space $\mathcal{H}$. From Shur's lemma, one has

$$\int_G \mathrm{d}\mu\,(g) U_g O U_g^\dagger = \mathrm{Tr}[O] I, \tag{9}$$

where $\mathrm{d}\mu(g)$ denotes the left-invariant measure normalized as $\int_G \mathrm{d}\mu\,(g) = d$. As one can see, equation (9) is equivalent to equation (6) with $\{U_g\}$ self-dual operator frame. On the other hand, the direct construction of info-complete POVMs is not as simple, since it involves the searching of frames of *positive* operators. A way to construct info-complete POVMs is suggested by equation (9). For any density matrix ν the set of positive operators

$$\Pi_g = U_g \nu U_g^\dagger \tag{10}$$

provides a resolution of the identity, whence $\{\Pi_g\}$ is a POVM. Moreover, the POVM is info-complete iff the operator

$$\begin{aligned} F &= \int_G \mathrm{d}\mu\,(g) |\Pi_g\rangle\!\rangle\langle\!\langle \Pi_g| \\ &= \int_G \mathrm{d}\mu\,(g) U_g \otimes U_g^* |\nu\rangle\!\rangle\langle\!\langle \nu| U_g^\dagger \otimes (U_g^*)^\dagger, \end{aligned} \tag{11}$$

is invertible, where we have used equation (2). Representation theory allows one to evaluate the integral in equation (11). When $U_g \otimes U_g^*$ has only inequivalent irreducible representations on $\mathcal{H} \otimes \mathcal{H}$, upon denoting by P_σ the projectors over the invariant subspaces, one has

$$F = d \sum_\sigma \frac{\mathrm{Tr}[P_\sigma |\nu\rangle\!\rangle\langle\!\langle \nu|]}{\mathrm{Tr}[P_\sigma]} P_\sigma. \tag{12}$$

As a consequence, the POVM $\{\Pi_g\}$ is info-complete iff the state ν satisfies the condition

$$\mathrm{Tr}[P_\sigma |\nu\rangle\!\rangle\langle\!\langle \nu|] \neq 0 \qquad \forall \sigma. \tag{13}$$

In this case the inverse of F is easily calculated as follows:

$$F^{-1} = d^{-1} \sum_\sigma \frac{\mathrm{Tr}[P_\sigma]}{\mathrm{Tr}[P_\sigma |\nu\rangle\!\rangle\langle\!\langle \nu|]} P_\sigma, \tag{14}$$

and the canonical dual Θ_g is obtained by the identity $|\Theta_g\rangle\!\rangle = F^{-1} |U_g \nu U_g^\dagger\rangle\!\rangle$, namely

$$|\Theta_g\rangle\!\rangle = d^{-1} U_g \otimes U_g^* \sum_\sigma \frac{\mathrm{Tr}[P_\sigma]}{\mathrm{Tr}[P_\sigma |\nu\rangle\!\rangle\langle\!\langle \nu|]} P_\sigma |\nu\rangle\!\rangle, \tag{15}$$

where we have used the property $[U_g \otimes U_g^*, P_\sigma] = 0$. By equation (15) one can notice that the canonical dual is covariant itself, namely $\Theta_g = U_g \xi U_g^\dagger$, where ξ is given by

$$|\xi\rangle\!\rangle = d^{-1} \sum_\sigma \frac{\mathrm{Tr}[P_\sigma]}{\mathrm{Tr}[P_\sigma |\nu\rangle\!\rangle\langle\!\langle \nu|]} P_\sigma |\nu\rangle\!\rangle. \tag{16}$$

At this stage we can make some general remarks. Among the invariant subspaces there is always the span of $\frac{1}{\sqrt{d}} |I\rangle\!\rangle$, and thus equation (15) always has the term

$$d^{-1} U_g \otimes U_g^* |I\rangle\!\rangle = d^{-1} |I\rangle\!\rangle. \tag{17}$$

The other invariant subspaces depend on the representation U_g, but we can prove that for any UIR U_g such that $U_g \otimes U_g^*$ has inequivalent irreducible representations, there always exists a suitable ν such that the POVM $U_g \nu U_g^\dagger$ is info-complete. In fact, upon writing the projectors P_σ in terms of their eigenvectors

$$P_\sigma = \sum_j |\Psi_j^{(\sigma)}\rangle\!\rangle\langle\!\langle \Psi_j^{(\sigma)}|, \tag{18}$$

from $\sum_\sigma P_\sigma = I$, it follows that $\{\Psi_j^{(\sigma)}\}$ is an orthonormal basis for the Hilbert–Schmidt operators. By identifying $\Psi_0^{(0)} \equiv \frac{I}{\sqrt{d}}$, from the orthogonality one has $\mathrm{Tr}[\Psi_j^{(\sigma)}] = \sqrt{d} \delta_{\sigma 0}$. Then, one can find suitable phases $\{\theta_\mu\}$ and a real constant α such that the Hermitian operator

$$\nu = \frac{I}{d} + \alpha \sum_{\mu \neq 0} (\mathrm{e}^{\mathrm{i}\theta_\mu} \Psi_j^{(\mu)} + \mathrm{e}^{-\mathrm{i}\theta_\mu} \Psi_j^{(\mu)\dagger}) \tag{19}$$

is a density matrix satisfying condition (13). Notice that in equation (19) we just need a single label $j = j(\mu)$ for each μ.

In the last part of this section, we want to note that all POVMs of the form (10) are equivalent to a generalized Bell measurement [18] on a tensor-product space $\mathcal{H} \otimes \mathcal{H}$, where the second space describes an ancilla prepared in the state ν^τ. In fact, one has

$$U_g \nu U_g^\dagger = \mathrm{Tr}_{\mathcal{A}}[(I \otimes \nu^\tau) |U_g\rangle\!\rangle\langle\!\langle U_g|], \tag{20}$$

where $\mathrm{Tr}_{\mathcal{A}}$ denotes the partial trace over the ancilla space. In general, the projectors on the maximally entangled states $|U_g\rangle\!\rangle$ are not orthogonal. The above considerations allow one to understand the construction of the quantum universal detectors introduced in [21].

4. Examples

In this section we will provide some examples of info-complete POVMs, showing their underlying group structure.

4.1. $\mathbb{Z}_d \times \mathbb{Z}_d$

This first example involves a minimal info-complete POVM, namely a POVM having d^2 elements, and will give us some general insight in the case of projective representations of Abelian groups. Consider the group $\mathbb{Z}_d \times \mathbb{Z}_d$, and the following d-dimensional projective UIR:

$$U_{m,n} = \sum_{k=0}^{d-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} km} |k\rangle\langle k \oplus n|, \tag{21}$$

where $m, n \in [0, d-1]$, and $\oplus$ denotes sum modulo d. The composition and orthogonality relations of the set are given by

$$\begin{aligned} U_{m,n} U_{p,q} U_{m,n}^\dagger &= \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}(np - mq)} U_{p,q}, \\ \mathrm{Tr}\left[U_{p,q}^\dagger U_{m,n}\right] &= d \delta_{mp} \delta_{nq}. \end{aligned} \tag{22}$$

We will now look for an info-complete POVM of the form $\Xi_{m,n} = \frac{1}{d} U_{m,n} \nu U_{m,n}^\dagger$. The properties of the projective UIR help us to find the conditions for informational completeness, and to evaluate the dual frame directly, as an alternative way to the general method developed in the previous section. First, let us expand the state ν on the basis of $\{U_{m,n}\}$,

$$\Xi_{m,n} = \frac{1}{d} \sum_{p,q} e^{\frac{2\pi i}{d}(np-mq)} \operatorname{Tr}[U_{p,q}^\dagger \nu] U_{p,q}, \qquad (23)$$

where we used equation (22). From the identity $\sum_n e^{\frac{2\pi i}{d}np} = d\delta_{p0}$, one easily checks that a dual frame for the POVM $\Xi_{m,n}$ is given by

$$\Theta_{m,n} = \frac{1}{d} \sum_{p,q} e^{\frac{2\pi i}{d}(np-mq)} \frac{U_{p,q}}{\operatorname{Tr}[U_{p,q}\nu]}. \qquad (24)$$

Then, the condition for informational completeness on ν is simply

$$\operatorname{Tr}[U_{p,q}^\dagger \nu] \neq 0, \qquad \forall (p,q) \in \mathbb{Z}_d \times \mathbb{Z}_d. \qquad (25)$$

A pure state $\nu = |\psi\rangle\langle\psi|$ that satisfies the above condition is given by

$$|\psi\rangle = \sqrt{\frac{1-|\alpha|^2}{1-|\alpha|^{2d}}} \sum_{n=0}^{d-1} \alpha^n |n\rangle, \qquad (26)$$

for any α with $0 < |\alpha| < 1$. Notice that the sets $\Theta_{m,n}$ and $\Xi_{m,n}$ are bi-orthogonal, hence $\Theta_{m,n}$ is the unique dual set. The results given in this example are consistent with the general treatment of section 2. In fact, the irreducible representations of $U_{m,n} \otimes U_{m,n}^*$ are all inequivalent and one-dimensional, and the invariant subspaces are just the spans of $U_{m,n}$'s.

4.2. $SU(d)$

In this second example we will examine the POVM $\frac{1}{d} U_g \nu U_g^\dagger$ with $U_g \in SU(d)$. Here the invariant subspaces of $U_g \otimes U_g^*$ are the linear span of $\frac{1}{\sqrt{d}} |I\rangle\rangle$ and its orthogonal complement. From equation (12), the frame operator can be expressed as

$$F = \frac{1}{d} |I\rangle\rangle\langle\langle I| + \frac{d\operatorname{Tr}[\nu^2]-1}{d^2-1} \left(I - \frac{1}{d} |I\rangle\rangle\langle\langle I| \right), \qquad (27)$$

and it is invertible iff $d\operatorname{Tr}[\nu^2] \neq 1$. Notice that $\operatorname{Tr}[\nu^2] = \frac{1}{d}$ only for $\nu = \frac{I}{d}$, which leads to a trivial POVM. Any other ν gives an info-complete POVM. The inverse of the frame operator can be written as

$$F^{-1} = \frac{1}{d} |I\rangle\rangle\langle\langle I| + \frac{d^2-1}{d\operatorname{Tr}[\nu^2]-1} \left(I - \frac{1}{d} |I\rangle\rangle\langle\langle I| \right). \qquad (28)$$

Finally, by equation (15), the canonical dual reads

$$\Theta_g = U_g \left(\frac{d^2-1}{d\operatorname{Tr}[\nu^2]-1} \nu - \frac{d-\operatorname{Tr}[\nu^2]}{d\operatorname{Tr}[\nu^2]-1} I \right) U_g^\dagger. \qquad (29)$$

In [22] we showed that this canonical dual is optimal among all covariant duals, for estimating expectations of arbitrary Hermitian operators with minimal rms error.

4.3. Weyl–Heisenberg group

The last example can be considered as the continuous counterpart of the first example, for infinite dimensional Hilbert spaces. We will consider the Weyl–Heisenberg group in the representation of displacements $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$, where a and $a^\dagger$ are the annihilation and creation operators of a boson field, i.e. $[a, a^\dagger] = 1$. Notice that the group is not compact, hence the general treatment of section 2 does not directly apply. However, the displacement representation is square integrable [23], and the main identity (9) still holds in the form

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} D(\alpha) O D^\dagger(\alpha) = \operatorname{Tr}[O] I. \qquad (30)$$

The group structure is revealed by the following identities:

$$\begin{aligned} D(\alpha)D(\beta)D^\dagger(\alpha) &= e^{\alpha\beta^* - \alpha^*\beta} D(\beta), \\ \operatorname{Tr}[D^\dagger(\alpha)D(\beta)] &= \pi\delta^{(2)}(\alpha-\beta), \end{aligned} \qquad (31)$$

where $\delta^{(2)}(\alpha) \equiv (1/\pi^2) \int_{\mathbb{C}} d^2\gamma\, e^{\alpha\gamma^* - \alpha^*\gamma}$ denotes the Dirac-delta on the complex plane. From equations (30) and (31), also follow both the completeness on $\mathcal{H} \otimes \mathcal{H}$

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} |D(\alpha)\rangle\rangle\langle\langle D(\alpha)| = I \otimes I, \qquad (32)$$

and the orthogonality in the Dirac sense

$$\langle\langle D(\alpha)|D(\beta)\rangle\rangle = \pi\delta^{(2)}(\alpha-\beta). \qquad (33)$$

We consider the POVM $\Pi(\alpha) = \frac{1}{\pi} D(\alpha)\nu D^\dagger(\alpha)$, where ν is an arbitrary normalized state. By expanding ν as $\nu = \int_{\mathbb{C}} \frac{d^2\gamma}{\pi} \operatorname{Tr}[D^\dagger(\gamma)\nu] D(\gamma)$ and using equation (31), one has

$$\Pi(\alpha) = \int_{\mathbb{C}} \frac{d^2\gamma}{\pi} e^{\alpha\gamma^* - \alpha^*\gamma} \operatorname{Tr}[D^\dagger(\gamma)\nu] D(\gamma). \qquad (34)$$

The frame operator can then be written as follows:

$$F = \int_{\mathbb{C}} \frac{d^2\beta}{\pi} |\operatorname{Tr}[D^\dagger(\beta)\nu]|^2 |D(\beta)\rangle\rangle\langle\langle D(\beta)|. \qquad (35)$$

The POVM is then info-complete iff $\operatorname{Tr}[D(\beta)^\dagger \nu] \neq 0$ for all β. We note that such a condition was also found in [17] in the context of phase-space representation and covariant localization observables.

The inverse of the frame operator can be written as

$$F^{-1} = \int_{\mathbb{C}} \frac{d^2\beta}{\pi} \frac{1}{|\operatorname{Tr}[D^\dagger(\beta)\nu]|^2} |D(\beta)\rangle\rangle\langle\langle D(\beta)|. \qquad (36)$$

The canonical dual can be finally evaluated using equation (15), and is given by

$$\Theta(\alpha) = D(\alpha) \left(\int_{\mathbb{C}} \frac{d^2\beta}{\pi} \frac{D(\beta)}{\operatorname{Tr}[D(\beta)\nu]} \right) D^\dagger(\alpha). \qquad (37)$$

Notice that the dual is unique since it can be readily checked that the POVM and the canonical dual are bi-orthogonal.

The present example in infinite dimension needs some care in checking the convergence of the processing function $f(\alpha, O) = \operatorname{Tr}[\Theta^\dagger(\alpha) O]$. In fact, if we take the vacuum state $\nu = |0\rangle\langle 0|$, the POVM will be reduced to the customary projection on coherent states $|\alpha\rangle$, and the measurement will correspond to phase-space averaging with the Q-function $Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$. We know that this gives expectations only for operators admitting anti-normal ordered field expansion [6]. In particular, the matrix elements of the density operator cannot be recovered in this way [7]. Therefore, in infinite dimensions the universality can be limited by convergence.

5. Conclusions

We have presented a group-theoretical method to construct informationally complete quantum measurements. This method allows one to find the conditions for such completeness, and to construct a wide class of info-complete POVMs from unitary irreducible representation of groups. These POVMs can always be viewed as projectors on maximally entangled states—generally not orthogonal—of the system coupled with an ancilla, thus relating info-complete POVMs with the quantum universal detectors of [21]. The processing functions pertaining to any arbitrary operator have been obtained using the general method of frame theory. Such functions are generally not unique, and this allows optimizing the frame in order to minimize the statistical error of the estimation. We have finally provided some examples of info-complete POVMs corresponding to different kind of groups: discrete and Abelian ($\mathbb{Z}_d \times \mathbb{Z}_d$), continuous and compact ($SU(d)$), and non-compact and Abelian (Weyl–Heisenberg). We argue that a frame-theory approach could provide a general framework for quantum tomography where many subtle aspects could be clarified. In particular, in infinite-dimensional cases such as in homodyne tomography, a unifying picture where the non-trivial formulae for estimating unbounded operators [24] or the existence of null functions that allow adaptive tomography [25] could emerge.

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