OPTIMIZATION OF QUANTUM UNIVERSAL DETECTORS

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The expectation value $\langle O \rangle$ of an arbitrary operator O can be obtained via a universal measuring apparatus that is independent of O, by changing only the data-processing of the outcomes. Such a "universal detector" performs a joint measurement on the system and on a suitable ancilla prepared in a fixed state, and is equivalent to a positive operator valued measure (POVM) for the system that is "informationally complete". The data processing functions generally are not unique, and we pose the problem of their optimization, providing some examples for covariant POVM's, in particular for $SU(d)$ covariance group.

Universality and programmability are crucial features in quantum technology, for communication, processing, and storage of information. Different tasks should be achieved by a basic set of devices, that would allow to perform different kinds of quantum information processing, such as in quantum computation[1,2], teleportation[3,4], entanglement detection[5], and entanglement distillation[6]. In particular, a universal detector[7] achieves the estimation of the ensemble average of an arbitrary operator by changing only the data processing of the outcomes. In some way it is analogous to a quantum tomographic apparatus [8]: however, the latter would typically require a *quorum* of observables—corresponding to a set of devices or to a single tunable device—whereas a universal detector would measure only a single fixed observable on an extended Hilbert space that includes a suitable ancilla.

Universal detectors can be characterized via a necessary and sufficient condition given in terms of "frames of operators" (i. e. spanning sets of operators), and can be achieved via Bell measurements, which are described by projectors on maximally entangled states [7]. Entanglement, however, is not an essential ingredient, and there are universal detectors which are described by separable POVM's as well[9].

When attention is restricted to the system Hilbert space only, universal detectors are equivalent to informationally complete (shortly "info-complete") POVM's[10], which are frames made of positive operators. Info-complete POVM's are necessarily not-orthogonal, whence universal detectors have a more physical counterpart, in terms of an observable and an apparatus ancilla.

When using a universal detector the ensemble average of an arbitrary operator is by choosing the appropriate data processing function of the measurement outcomes. As we will see, the data processing functions are generally not unique, and are related to the concept of *dual operator frame*. In the following, after reviewing the main results on universal detectors, we pose the problem of optimization of data-processing functions, with particular focus on the case of covariant POVM's, and in particular for the $SU(d)$ covariance group.

Let us introduce the concept of universal detector, or, more abstractly, of universal POVM. We consider a quantum system in a Hilbert space $\mathcal{H}$, coupled to an ancilla with Hilbert space $\mathcal{K}$. A POVM $\{\Pi_i\}$, $\Pi_i \geq 0$ and $\sum_i \Pi_i = I_{\mathcal{H}} \otimes I_{\mathcal{K}}$ on the Hilbert space $\mathcal{H} \otimes \mathcal{K}$ is *universal* for the system iff there exists a state of the ancilla ν such that for any operator O one has

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O) \mathrm{Tr}[(\rho \otimes \nu)\Pi_i] \,, \tag{1}$$

where $f_i(\nu, O)$—parametrized by ν and O—is a suitable function of the outcome i of the measurement, and we will refer to it as the *data processing* function. The detector will be called *universal* when it is described by a universal POVM. In order to give a necessary and sufficient condition for universality, we need to introduce some notation, and the concept of *frame of operators*. We will use the following symbols for bipartite pure states in $\mathcal{H} \otimes \mathcal{K}$

$$|A\rangle\!\rangle = \sum_{n=1}^{\dim\mathcal{H}} \sum_{m=1}^{\dim\mathcal{K}} A_{nm} |n\rangle \otimes |m\rangle \ , \tag{2}$$

where $|n\rangle$ and $|m\rangle$ are fixed orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively. Equation (2) exploits the isomorphism [11] between the Hilbert space of the Hilbert-Schmidt operators A, B from $\mathcal{K}$ to $\mathcal{H}$, with scalar product $\langle A, B\rangle = \mathrm{Tr}[A^\dagger B]$, and the Hilbert space of bipartite vectors $|A\rangle\!\rangle, |B\rangle\!\rangle \in \mathcal{H} \otimes \mathcal{K}$, with $\langle\!\langle A|B\rangle\!\rangle \equiv \langle A, B\rangle$. It is easy to show the following identities

$$\begin{aligned} A \otimes B|\, C\rangle\!\rangle &= |\, ACB^\tau\rangle\!\rangle \,, \\ \mathrm{Tr}_{\mathcal{K}}[|\, A\rangle\!\rangle\langle\!\langle B\,|] &= AB^\dagger \,, \\ \mathrm{Tr}_{\mathcal{H}}[|\, A\rangle\!\rangle\langle\!\langle B\,|] &= A^\tau B^* \,, \end{aligned} \tag{3}$$

where τ and $*$ denote transposition and complex conjugation with respect to the fixed bases.

A *frame*[12] for operators—say A from $\mathcal{K}$ to $\mathcal{H}$—is just a set of operators spanning a normed linear space of operators, i. e. there are constants $0 < a \leq b < \infty$ such that for all operators A one has $a\|A\|^2 \leq \sum_i |\langle A, \Xi_i\rangle|^2 \leq b\|A\|^2$. Here, for simplicity, we will consider the (Hilbert) space of Hilbert-Schmidt operators from $\mathcal{K}$ to $\mathcal{H}$, and use the equivalent vector notation introduced in Eq. (2). Frames of operators have been already used disguised as *spanning sets of operators*[13] in the context of quantum tomography. For $\{\Xi_i\}$ an operator frame there exists another frame $\{\Theta_i\}$—called *dual frame*—providing operator expansions in the form

$$A = \sum_i \mathrm{Tr}[\Theta_i^\dagger A]\Xi_i \ . \tag{4}$$

The completeness relation of the frame and its dual reads

$$\sum_i \langle\psi|\Xi_i|\phi\rangle\langle\varphi|\Theta_i^\dagger|\eta\rangle = \langle\psi|\eta\rangle\langle\varphi|\phi\rangle \ , \tag{5}$$

for any $\phi, \varphi \in \mathcal{H}$ and $\psi, \eta \in \mathcal{K}$. For continuous sets, the sums in Eqs. (4) and (5) are replaced by integrals. Given a frame $\{\Xi_i\}$, generally the dual set is not unique. However, all duals $\{\Theta_i\}$ of a given frame can be obtained via the linear relation[14]

$$|\Theta_i\rangle\!\rangle = F^{-1}|\Xi_i\rangle\!\rangle + |Y_i\rangle\!\rangle - \sum_j \langle\!\langle \Xi_j|F^{-1}|\Xi_i\rangle\!\rangle |Y_j\rangle\!\rangle \, , \tag{6}$$

where Y_i are arbitrary, and the positive and invertible operator F writes

$$F = \sum_i |\Xi_i\rangle\!\rangle\langle\!\langle\Xi_i| \, . \tag{7}$$

The operator F is called *frame operator* in frame theory [12], whereas the set of operators corresponding to the vectors $F^{-1}|\Xi_i\rangle\!\rangle$ through the above isomorphism is called *canonical dual frame*. As we will show immediately, the dual frame provides the data processing function, whence Eq. (6) allows a useful flexibility in the data-processing, with the possibility of optimizing the statistical error in the estimation by minimization over the free operators Y_i.

Let us now consider a universal POVM on $\mathcal{H}\otimes\mathcal{K}$. The elements $\{\Pi_i\}$ can be diagonalized as follows

$$\Pi_i = \sum_{j=1}^{r_i} |\,\Psi_j^{(i)}\rangle\!\rangle\langle\!\langle\Psi_j^{(i)}\,| \, , \tag{8}$$

where the vectors $|\,\Psi_j^{(i)}\rangle\!\rangle$ have norm equal to the j-th eigenvalue of Π_i, and r_i is the rank of Π_i. From the normalization condition $\sum_i \Pi_i = I_\mathcal{H}\otimes I_\mathcal{K}$, it follows that the set of operators $\{\Psi_j^{(i)}\}$ from $\mathcal{K}$ to $\mathcal{H}$ must be an operator frame itself. The characterization of universal POVM's is then given by the condition that there exists a state $\nu \in \mathcal{K}$ such that the following operators

$$\Xi_i[\nu] \equiv \sum_{j=1}^{r_i} \Psi_j^{(i)}\nu^\tau\Psi_j^{(i)\dagger} \tag{9}$$

are a frame for operators on $\mathcal{H}$. In fact, using Eq. (8), Eq. (1) rewrites

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\mathrm{Tr}\left[\rho\sum_{j=1}^{r_i}\Psi_j^{(i)}\nu^\tau\Psi_j^{(i)\dagger}\right] \ , \tag{10}$$

and this is true independently of ρ iff

$$O = \sum_i f_i(\nu, O)\Xi_i[\nu] \, . \tag{11}$$

From linearity one has

$$f_i(\nu, O) = \mathrm{Tr}[\Theta_i^\dagger[\nu] O] \;, \tag{12}$$

where $\Theta_i[\nu]$ is a dual frame of $\Xi_i[\nu]$. Hence, after finding a dual frame for $\Xi_i[\nu]$, the data processing function is easily evaluated via Eq. (12).

When restricting our attention just on the system Hilbert space, notice that from Eqs. (9) and (11) a universal detector corresponds to a system POVM whose elements make a frame of positive operators. Then, from Eqs. (11) and (12) it follows that such POVM is "informationally complete"[10], namely it allows evaluation of the expectation of an arbitrary system operator. Since the number of elements of an operator frame for $\mathcal{H}$ cannot be smaller than $(\dim \mathcal{H})^2$, an info-complete POVM is necessarily not orthogonal. Viceversa, it is simple to prove that an arbitrary frame for operators in $\mathcal{H}$ made of positive operators $\{K_i\}$ allows to construct an info-complete POVM. In fact, since the operator $S \equiv \sum_i K_i$ is invertible, the set $\{\tilde{K}_i = S^{-1/2} K_i S^{-1/2}\}$ satisfies the completeness relation $\sum_i \tilde{K}_i = I_{\mathcal{H}}$. The direct construction of info-complete POVM's is not trivial, since it involves the searching of *positive* operator frames. A way to construct universal POVM's is suggested by group-theoretic techniques[7]. For example, one can consider projectors on maximally entangled states, namely a Bell POVM on $\mathcal{H} \otimes \mathcal{H}$. In the notation of Eq. (2), a Bell POVM has elements of the form

$$\Pi_i = \frac{\alpha_i}{d} |\, U_i \rangle\!\rangle \langle\!\langle U_i \,| \,, \tag{13}$$

where d is the dimension of $\mathcal{H}$, α_i are suitable positive constants and U_i are unitaries. When the POVM is orthogonal, one has $\alpha_i = 1$ and $\mathrm{Tr}\left[U_i^\dagger U_j\right] = d\,\delta_{ij}$. Particular cases of Bell POVM's are those in which U_i are a unitary irreducible representation (UIR) of some group $\mathbf{G}$. As an example, consider a projective UIR of an abelian group, which therefore satisfies the relation

$$U_\alpha U_\beta U_\alpha^\dagger = e^{ic(\alpha,\beta)} U_\beta \;. \tag{14}$$

In this case the Bell POVM is orthogonal, with number of elements equal to the cardinality of the group d^2. One can show[7] that for any ancilla state ν such that $\mathrm{Tr}[U_\alpha^\dagger \nu^\tau] \neq 0$ for all α, the set of $\Xi_\alpha[\nu] = \frac{1}{d} U_\alpha \nu^\tau U_\alpha^\dagger$ is an operator frame. By identifying $U_1 \equiv I$, a possible choice of the ancilla state is

$$\nu = \frac{1}{d} I + \frac{1}{d(d^2-1)} \sum_{\alpha > 1} U_\alpha \;. \tag{15}$$

The dual frame in this case is unique, and is given by

$$\Theta_\alpha[\nu] = \frac{1}{d} \sum_{\beta=1}^{d^2} \frac{U_\beta}{\mathrm{Tr}\,[U_\beta \nu^*]}\, e^{-ic(\beta,\alpha)} \;. \tag{16}$$

Correspondingly, according to Eq. (12), also the data processing function is unique.

There are universal Bell POVM's also from non-abelian groups. An interesting example is provided by the group $SU(d)$. In this case the universality of the corresponding Bell POVM is proved by showing that the set of $\Xi_\alpha[\nu] = U_\alpha\, \nu^\tau\, U_\alpha^\dagger$ is an operator frame. Let us start by evaluating the frame operator, which is given through Eq. (7) by

$$\begin{aligned} F &= \int \mathrm{d}\alpha\, (U_\alpha \otimes U_\alpha^*)|\nu^\tau\rangle\!\rangle\langle\!\langle\nu^\tau|(U_\alpha^\dagger \otimes U_\alpha^\tau) \\ &= \frac{1}{d}|I\rangle\!\rangle\langle\!\langle I| + \frac{d\mathrm{Tr}[(\nu^\tau)^2]-1}{d^2-1}\left(I - \frac{1}{d}|I\rangle\!\rangle\langle\!\langle I|\right), \end{aligned} \tag{17}$$

where we used Shur's lemma to compute the integral. It can be noticed that F is expressed in diagonal form with eigenvalues 1 and $\frac{d\mathrm{Tr}[(\nu^\tau)^2]-1}{d^2-1}$, thus it is invertible for any ν^τ unless $\mathrm{Tr}[(\nu^\tau)^2] = d^{-1}$, corresponding to the state $\nu = I/d$. The expression for the inverse of the frame operator is easily evaluated

$$F^{-1} = \frac{1}{d}|I\rangle\!\rangle\langle\!\langle I| + \frac{d^2-1}{d\mathrm{Tr}[(\nu^\tau)^2]-1}\left(I - \frac{1}{d}|I\rangle\!\rangle\langle\!\langle I|\right). \tag{18}$$

The canonical dual set $\Theta^0_\alpha[\nu]$ for $\Xi_\alpha[\nu]$ is is obtained by definition as follows

$$|\Theta^0_\alpha[\nu]\rangle\!\rangle = F^{-1}|U_\alpha\, \nu^\tau U_\alpha^\dagger\rangle\!\rangle\,, \tag{19}$$

and one has

$$\Theta^0_\alpha[\nu] = a\, U_\alpha\, \nu^\tau U_\alpha^\dagger + b\, I\,, \tag{20}$$

where $a = \frac{d^2-1}{d\mathrm{Tr}[(\nu^\tau)^2]-1}$ and $b = \frac{\mathrm{Tr}[(\nu^\tau)^2]-d}{d\mathrm{Tr}[(\nu^\tau)^2]-1}$. According to Eq. (12) the processing function corresponding to the canonical dual frame is then

$$f_\alpha(\nu, O) = a\,\mathrm{Tr}[U_\alpha\, \nu^\tau U_\alpha^\dagger O] + b\mathrm{Tr}[O]\,. \tag{21}$$

The knowledge of the canonical dual frame allows to parameterize all the alternate duals as in Eq. (6) by the arbitrary operators Y_α, greatly simplifying the task of optimizing the statistical error in the estimate of a given operator. Such "noise" can be generally defined in terms of the eigenvalues of the covariance matrix

$$C = \begin{pmatrix} \overline{(\mathrm{Re}f)^2} - \overline{\mathrm{Re}f}^2 & \overline{\mathrm{Re}f\mathrm{Im}f} - \overline{\mathrm{Re}f}\,\overline{\mathrm{Im}f} \\ \overline{\mathrm{Re}f\mathrm{Im}f} - \overline{\mathrm{Re}f}\,\overline{\mathrm{Im}f} & \overline{(\mathrm{Im}f)^2} - \overline{\mathrm{Im}f}^2 \end{pmatrix}, \tag{22}$$

where

$$\overline{g} = \int \mathrm{d}\alpha\, g_\alpha(\nu, O)\mathrm{Tr}[\rho\Xi_\alpha[\nu]]\,. \tag{23}$$

The noise clearly depends on the state on which the estimate is done. For a state-independent definition of noise one could use either the maximum noise or the average noise over all (pure or mixed) states. If one considers averages of Hermitian operators, the imaginary parts of the processing functions can be discarded, and this is equivalent to consider $\mathrm{Re}f$ only. The noise can thus be evaluated by the customary variance $\overline{(\mathrm{Re}f)^2} - \overline{\mathrm{Re}f}^2$. As an example, we now evaluate the optimal dual frame for the estimation of Hermitian operators, restricting our attention on covariant dual frames, i. e. of the form

$$\Theta_\alpha[\nu] = U_\alpha \xi U_\alpha^\dagger \,. \tag{24}$$

It can be proved that such a set is a dual frame of $\Xi_\alpha[\nu]$ iff

$$\mathrm{Tr}[\xi] = 1\,, \quad \mathrm{Tr}[\nu^\tau \xi] = d\,. \tag{25}$$

Since we are considering Hermitian operators, the processing function can be written

$$\mathrm{Re}f_\alpha(\nu, O) + i\mathrm{Im}f_\alpha(\nu, O) = \mathrm{Tr}[U_\alpha(\mathrm{Re}\xi)U_\alpha^\dagger O] + i\mathrm{Tr}[U_\alpha(\mathrm{Im}\xi)U_\alpha^\dagger O]\,, \tag{26}$$

where $\mathrm{Re}\xi = \frac{1}{2}(\xi + \xi^\dagger)$, and $\mathrm{Im}\xi = \frac{1}{2i}(\xi - \xi^\dagger)$. As stated before, we can restrict attention on $\mathrm{Re}f_\alpha(\nu, O)$, and thus we need to consider only the self-adjoint case $\xi \equiv \mathrm{Re}\xi$. Our optimization consists in minimizing the average variance over all pure states, namely

$$\begin{aligned}(\Delta_\xi O^2) = \frac{1}{d}\int \mathrm{d}\beta \int \mathrm{d}\alpha\, \langle\psi_0|U_\beta^\dagger \Theta_\alpha[\nu] U_\beta|\psi_0\rangle \mathrm{Tr}[\Xi_\alpha^\dagger[\nu]O]^2 \\ -\frac{1}{d}\int \mathrm{d}\beta\, \langle\psi_0|U_\beta^\dagger O U_\beta|\psi_0\rangle^2\,,\end{aligned} \tag{27}$$

where the pure states have been parametrized as $U_\beta|\psi_0\rangle$, for a fixed arbitrary $\psi_0 \in \mathcal{H}$ and $U_\beta \in SU(d)$. We will compare Eq. (27) with the variance of the ideal measurement of O averaged over all pure states, namely

$$(\Delta_{\mathrm{obs}} O^2) = \frac{1}{d}\int \mathrm{d}\alpha\, \langle\psi_0|U_\alpha^\dagger O^2 U_\alpha|\psi_0\rangle - \frac{1}{d}\int \mathrm{d}\alpha\, \langle\psi_0|U_\alpha^\dagger O U_\alpha|\psi_0\rangle^2\,. \tag{28}$$

Equations (27) and (28) can be evaluated using the following identities

$$\begin{aligned}&\int \mathrm{d}\alpha\, U_\alpha A U_\alpha^\dagger = \mathrm{Tr}[A]I \\ &\int \mathrm{d}\alpha\, U_\alpha^{\otimes 2} A U_\alpha^{\otimes 2\dagger} = \frac{2}{d+1}\mathrm{Tr}[P_S A]P_S + \frac{2}{d-1}\mathrm{Tr}[P_A A]P_A \\ &\mathrm{Tr}[EB\otimes B] = \mathrm{Tr}[B^2]\,,\end{aligned} \tag{29}$$

where $P_S = \frac{1}{2}(I+E)$ and $P_A = \frac{1}{2}(I-E)$ are the projections on the totally symmetric and antisymmetric subspaces of $\mathcal{H}^{\otimes 2}$, and E denotes the swap operator $E|\phi\rangle\otimes|\psi\rangle = |\psi\rangle\otimes|\phi\rangle$. The results are

$$(\Delta_{\mathrm{obs}} O^2) = \frac{1}{d+1}\left(\mathrm{Tr}[O^2] - \frac{1}{d}\mathrm{Tr}[O]^2\right) \tag{30}$$

$$(\Delta_\xi O^2) = \frac{\mathrm{Tr}[\xi^2] - 1}{d-1}(\Delta_{\mathrm{obs}} O^2). \tag{31}$$

The optimization can be achieved by minimizing the coefficient $\frac{\mathrm{Tr}[\xi^2]-1}{d-1}$ with the constraints $\mathrm{Tr}[\xi] = 1$ and $\mathrm{Tr}[\nu^\tau \xi] = d$. By the method of Lagrange multipliers, one can write the variational equation

$$\frac{\delta}{\delta \langle\langle \xi |} \left(\frac{\langle\langle \xi | \xi \rangle\rangle - 1}{d-1} - \lambda \langle\langle \xi | \nu^\tau \rangle\rangle - \mu \langle\langle \xi | I \rangle\rangle \right) = 0 \,, \tag{32}$$

which leads to the following result

$$\xi_{\mathrm{opt}} = \frac{d^2-1}{d\mathrm{Tr}[(\nu^\tau)^2]-1} \nu^\tau - \frac{d - \mathrm{Tr}[(\nu^\tau)^2]}{d\mathrm{Tr}[(\nu^\tau)^2]-1} I \,, \tag{33}$$

namely, the optimal covariant dual frame is the canonical one. The optimization can be finally completed by looking for the least noisy ancilla state. By calculating $\mathrm{Tr}[\xi^2_{\mathrm{opt}}]$ and substituting in Eq. (31) one obtains

$$(\Delta_{\mathrm{opt}} O^2) = \frac{d^2+d-1-p}{dp-1} (\Delta_{\mathrm{obs}} O^2), \tag{34}$$

where $p \equiv \mathrm{Tr}[(\nu^\tau)^2]$. A simple differentiation of the expression in Eq. (34) with respect to p shows that the best choice corresponds to $p = 1$, namely the minimal added noise is achieved by an arbitrary pure ancilla state. In this case the expression is simplified and is equal to

$$(\Delta_{\mathrm{opt}} O^2) = (d+2)(\Delta_{\mathrm{obs}} O^2) \,. \tag{35}$$

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