Quantum Convex Structures and their Physical Interrelations

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This talk reviews some results contained in Refs.[1]-[7], and the present extended abstract mainly recalls the open problems posed during the talk.

The quantum convex structures that will be considered are those of Quantum States, Quantum Operations (in particular trace-preserving, i. e. channels) and POVM's (Positive Operator Valued Measures). More than focusing only on the convex structures themselves, I will analyze some physically meaningful interrelations that link them each other: 1) one-to-one maps between States and Quantum Operations, and between States and POVM's, corresponding to Quantum Calibration; 2) dilation maps from the convex set of States to those of Quantum Operations and of POVM's, corresponding to Quantum Programmability; 3) mapping POVM's to POVM's via channels, corresponding to pre-processing of POVM's.

Quantum Calibration. The convex Quantum Operations and that of bipartite states are connected each-other (apart from a normalization) by the Choi-Jamiolkowski isomorphism between CP-maps and positive bipartite operators. Such correspondence can be extended to the following one: $R = \mathcal{M} \otimes \mathcal{I}(F)$, describing the output state R of the local action of the map $\mathcal{M}$ on the input state F ($\mathcal{I}$ denotes the identity map). One calls the state F *tomographically faithful*[3] when the correspondence $\mathcal{M} \leftrightarrow R$ is one-to-one. Using such correspondence, one can perform the quantum tomography of the operation/channel $\mathcal{M}$ via a a joint tomography on the bipartite system at the output. The inversion formula from the output state to the map is $\mathcal{M}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^\intercal)\mathcal{I} \otimes \mathcal{F}^{-1}(R)]$ where $\mathcal{F}$ is the map $\mathcal{F} = \mathrm{Tr}_2[(I \otimes \rho^\intercal)F]$ associated to the tomographically faithful state F.

The faithful state F establishes also the one-to-one correspondence between POVM's and ensembles of states: $p_n\rho_n = \mathcal{F}'(P_n)$ and $P_n = \mathcal{F}'^{-1}(p_n\rho_n)$,, where $\mathcal{F}'$ is the associated map $\mathcal{F}'(X) = \mathrm{Tr}_2[(I \otimes X)F]$, p_n being the probability of the outcome n and ρ_n the corresponding conditioned state (to be determined tomographically). As an example of application, Ref.[4] presents a Monte Carlo simulation of an experiment of quantum calibration of a typical photodetector using a realistic homodyne tomography setup, and a twin beam from down-conversion of vacuum for the state F.

Quantum Programmability. The Choi one-to-one correspondence between channels and bipartite states is not only the basis of tomography of channels, but carries also a physical interpretation in terms of *probabilistic programmability* of channels. Here, however, we are interested in *deterministic programmability* of channels. We want to program the channel by a fixed device as follows $\mathcal{M}_{U,\sigma}(\rho) \doteq \mathrm{Tr}_2[U(\rho\otimes\sigma)U^\dagger]$, with the system in the state ρ interacting with an ancilla in the state σ via the unitary operator U of the programmable device (the state of the ancilla is the *program*). For fixed U the above map can be regarded as a linear map from the convex set of the ancilla states $\mathscr{A}$ to the convex set of channels for the system. We will denote by $\mathscr{M}_{U,\mathscr{A}}$ the image of the ancilla states $\mathscr{A}$ under such linear map. According to the well known no-go theorem by Nielsen and Chuang it is impossible to program all unitary channels on the system with a single U and a finite-dimensional ancilla, namely the image convex $\mathscr{M}_{U,\mathscr{A}}$ is a proper subset of the whole convex of channels. This opens the following problem:

Problem 1 (The big U)**.** *For given dimension of the ancilla, find the unitary operators U that are the most efficient in programming channels, namely which minimize the largest distance of each channel $\mathcal{C} \in \mathscr{C}$ from the programmable set $\mathscr{M}_{\mathbf{Z},\mathscr{A}}$:* $\varepsilon(U) \doteq \max_{\mathcal{C}\in\mathscr{C}} \min_{\mathcal{P}\in\mathscr{M}_{U,\mathscr{A}}} \delta(\mathcal{C}, \mathcal{E})$.

As a definition of distance one could consider any of those given in Ref.[8]. For POVM's we have a similar situation. In the following we will consider discrete spectrum and denote the POVM with the vector notation $\mathbf{P} \doteq (P_1, P_2, \ldots)$, P_i denoting the POVM elements. Here the deterministic programmability is represented by the extension map $\mathcal{M}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}] = \mathbf{P}$

from states to POVM's. A no-go theorem analogous to that of channels holds for POVM's[5], and this opens the following problem (in the following $\mathscr{P}_N$ denotes the convex of POVM's with N outcomes)

Problem 2 (The big $\mathbf{Z}$)**.** *For given dimension of the ancilla Hilbert space and cardinality of the POVM $N = |\mathbf{Z}| = |\mathbf{P}|$, find the joint observables $\mathbf{Z}$ that are the most efficient in programming POVM's, namely which minimize the largest distance of each POVM from the programmable set $\mathcal{M}_{\mathbf{Z},\mathscr{A}}$: $\varepsilon(\mathbf{Z}) \doteq \max_{\mathbf{P}\in\mathscr{P}_N} \min_{\mathbf{Q}\in\mathscr{M}_{\mathbf{Z},\mathscr{A}}} \delta(\mathbf{P},\mathbf{Q})$.*

As a definition of distance we can use the physical distance $\delta(\mathbf{P},\mathbf{Q}) = \max_\rho \sum_i |\operatorname{Tr}[\rho(P_i - Q_i)]|$. The solution of Problems 1 and 2 are unknown even for dimension $d = 2$ of the system. In Ref.[5] it is shown that using a joint observable $\mathbf{Z}$ of the form of a fixed local system observable evolved with a controlled-unitary interaction, one can program observables with polynomial growth of the dimension of the ancilla versus the accuracy ε^{-1}. For qubits one can even achieve linear growth.

Notice that if we pose restrictions on the set of programmable POVM's, then it maybe possible to program the full convex set exactly. This is the case of covariance under a unitary irreducible representation of a group, where the POVM density can be programmed by means of a fixed covariant Bell POVM density[5, 6] [the "seed" of the POVM is just the state of the ancilla, apart from a simple antilinear transformation]. This suggests that for ancilla having the same dimension of the system the observable $\mathbf{Z}$ should be Bell. Notice that the controlled-unitary form also occurs in connecting local to Bell observables[6]. Here another problem arises:

Problem 3 (The "Bellizing" U)**.** *Classify all unitary operators U that connect a fixed separable orthonormal basis to a Bell orthonormal basis.*

This problem needs the solution of another problem, namely that of the classification of Bell basis:

Problem 4 (Bell basis classification)**.** *Classify all orthonormal Bell basis, or, equivalently, classify all orthonormal basis of unitary operators.*

Regarding the last problem more material can be found on Ref.[9].

Processing of POVM's and the problem of Clean POVM's. If we precede a measuring apparatus by a quantum channel $\mathcal{E}$, the series of channel-measurement is equivalent to a new measurement, whose POVM is given by $\mathbf{Q} = \mathcal{E}(\mathbf{P})$. We call this pre-processing of the POVM (this is the case, e.g. of optical pre-amplification of photodetection or homodyning). The pre-processing scheme should be contrasted with that of post-processing, in which the output outcomes of the measurement are processed numerically, corresponding to an endomorphism of the probability space of the POVM (for discrete probability space this is just the composition of the POVM with a Markov matrix). Such post-processing is completely classical, whereas the pre-processing is quantum.

A quantum channel transforms POVM's into POVM's, generally irreversibly, thus loosing some of the information retrieved from the measurement. This poses the following problem:

Problem 5 (Clean POVM's)**.** *Which POVM's are "undisturbed", namely they are not irreversibly connected to another POVM via a channel?*

We will call such POVM *clean*. To define more precisely the problem, we introduce a pre-ordering relation, which we call *cleanness*, defined as follows: For two POVM's $\mathbf{P}$ and $\mathbf{Q}$ we define $\mathbf{P} \succ \mathbf{Q}$ iff there exists a channel $\mathcal{E}$ such that $\mathbf{Q} = \mathcal{E}(\mathbf{P})$. We will say that the POVM $\mathbf{P}$ is *cleaner* than the POVM $\mathbf{Q}$. We will say that $\mathbf{P} \simeq \mathbf{Q}$ if both $\mathbf{Q} \succ \mathbf{P}$ and $\mathbf{P} \succ \mathbf{Q}$ hold. We call a POVM $\mathbf{P}$ *clean* when for any POVM $\mathbf{Q}$ such that $\mathbf{Q} \succ \mathbf{P}$ one has $\mathbf{Q} \simeq \mathbf{P}$. Partial solutions to the problem are the following[7]: *1)* for $N < d$ outcomes there are no clean POVM's, and for $N = d$ the set of clean POVM's coincides with the set of observable; *2)* all rank-one POVM's are clean; *3)* for $d = 2$, $\mathbf{P} \simeq \mathbf{Q}$ iff $\mathbf{P}$ is unitarily equivalent to $\mathbf{Q}$; *4)* for $\mathbf{A}$ and $\mathbf{B}$ effects, $\mathbf{A} \succ \mathbf{B}$ iff $[\lambda_m(A), \lambda_M(A)] \supseteq [\lambda_m(B), \lambda_M(B)]$; *5)* if the POVM $\mathbf{Q}$ is infocomplete then every

$\mathbf{P}$ such that $\mathbf{P} \succeq \mathbf{Q}$ is infocomplete; *6)* for infocomplete POVM's cleanness-equivalence is the same as unitary equivalence. One can easily see that generally *cleanness equivalence is different from unitary equivalence.* In fact it is possible to connect each other two unitarily inequivalent POVM's via two different channels (consider two effects with different spectrum and the same spectral interval). Moreover *cleanness is different from extremality in the POVM convex.* In fact, there are extremal POVM's that are not clean (e. g. any extremal POVM with $N < d$ outcomes, such as for $d = 3$, $\mathbf{P} = \{Z_0, Z_1\}$ with $Z_0 = |0\rangle\langle 0|, \qquad Z_1 = |1\rangle\langle 1| + |2\rangle\langle 2|$), and viceversa there are clean POVM's that are not extremal (e.g. any rank one POVM with $N > d^2$).

What does it mean that there are extremal POVM that are not clean? At a first sight this looks quite strange, since an extremal POVM is already perfect, in some sense. The answer is simply that sometimes we need to give-up some amount of information for the quality of the information. This is because maximizing the information is not necessarily compatible with the achievement of the minimal cost function in an optimization problem. Therefore, even though the channel decrease the information, this is the only way to achieve the minimal cost. On the other hand, once the measurement is performed, there is no classical post-processing that can achieve the same result of a quantum pre-processing, and achieving the full available amount of information is then useless. If we want to decide a posteriori the purpose of the measurement, then we need to use an informationally complete measurement, and the same amount of information is then available for each purpose.

Clearly, we can also define *cleanness for post-processing*, i.e. a POVM is cleaner than another when the latter can be obtained from the former via an irreversible classical processing. This classical case is very simple, since here cleanness is just equivalent to be rank-one. Therefore we conclude that rank-one POVM's are clean under both pre-processing and post-processing, On the other hand, both observables and rank-one informationally complete POVM's have all the following properties: they are extremal, clean under post-processing, and clean under pre-processing.

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