Quantum Convex Structures and their Physical Interrelations

Giacomo Mauro D'Ariano
Università di Pavia
www.qubit.it

quantum information theory group

Mathematisches Forschungsinstitut Oberwolfach
Meeting: *Entanglement and Decoherence: Mathematics and Physics of Quantum Information and Computation*
26 January 2005

quantum information theory group

Essential literature

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Convex structures of POVM's and Channels

- G. M. D'Ariano, P. Lo Presti, P. Perinotti, *Classical randomness in quantum measurements,* Phys. Rev. A (submitted), (quant-ph0408115)
- G. Chiribella, G. M. D'Ariano, P. Perinotti (unpublished)

Quantum calibration

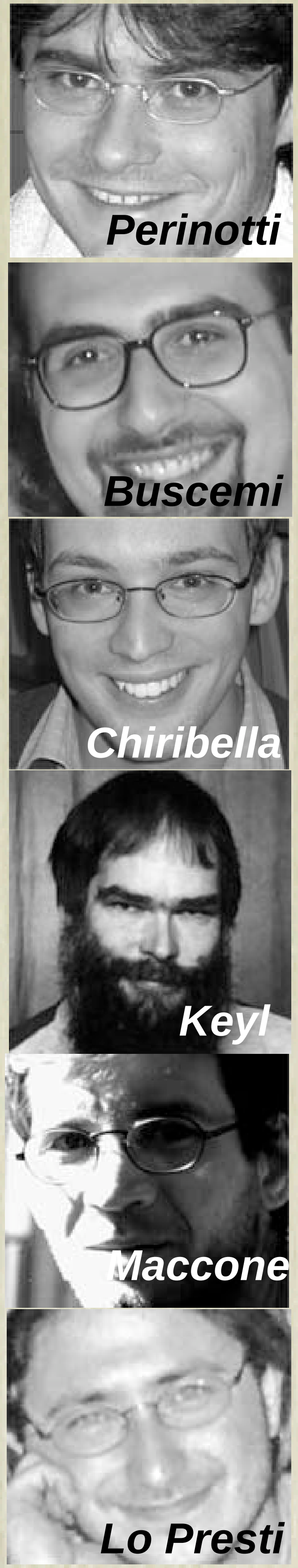
- G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **91** 047902 (2003)
- G. M. D'Ariano, P. Lo Presti, and L. Maccone, *Quantum Calibration of Measuring Apparatuses,* Phys. Rev. Lett. **93** 250407 (2004)

Programmability of channels and measurements

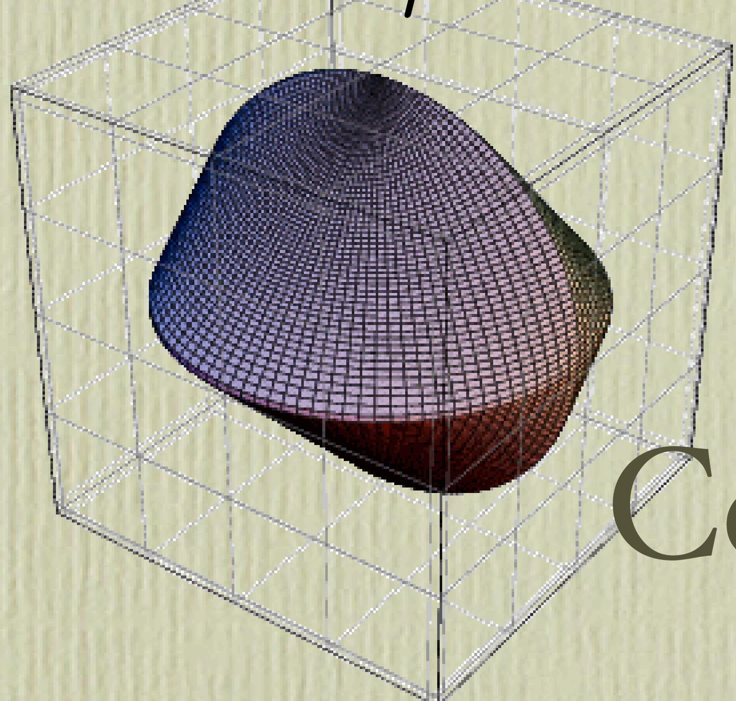
- G. M. D'Ariano, P. Perinotti, *Efficient universal programmable quantum measurements,* Phys. Rev. Lett. (in press) (quant-ph-0410169)
- G. M. D'Ariano and P. Perinotti, *On the realization of Bell observables,* Phys. Lett A **329** 188 -192 (2004)

Clean POVM's

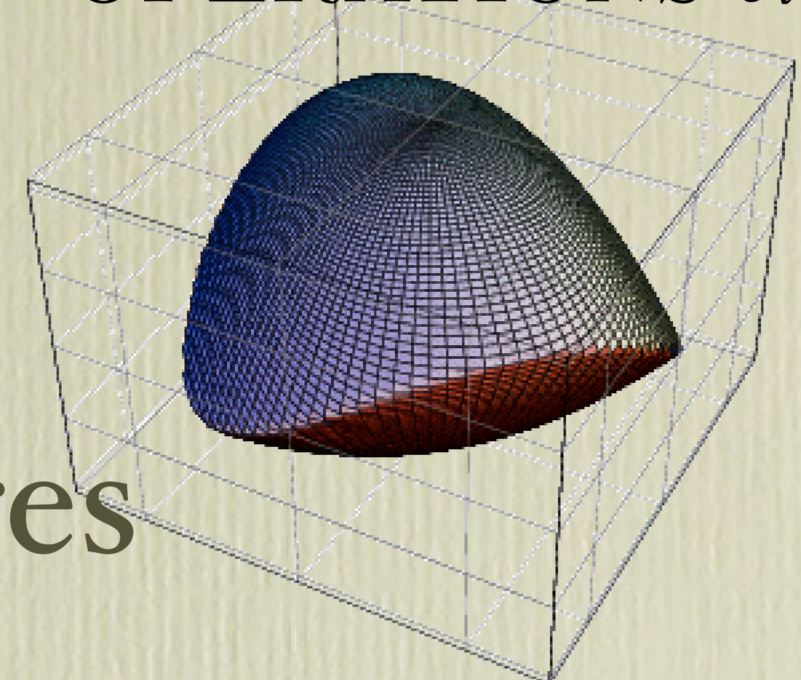
- F. Buscemi, P. Perinotti, G. M. D'Ariano, M. Keyl, R. Werner, (unpublished)



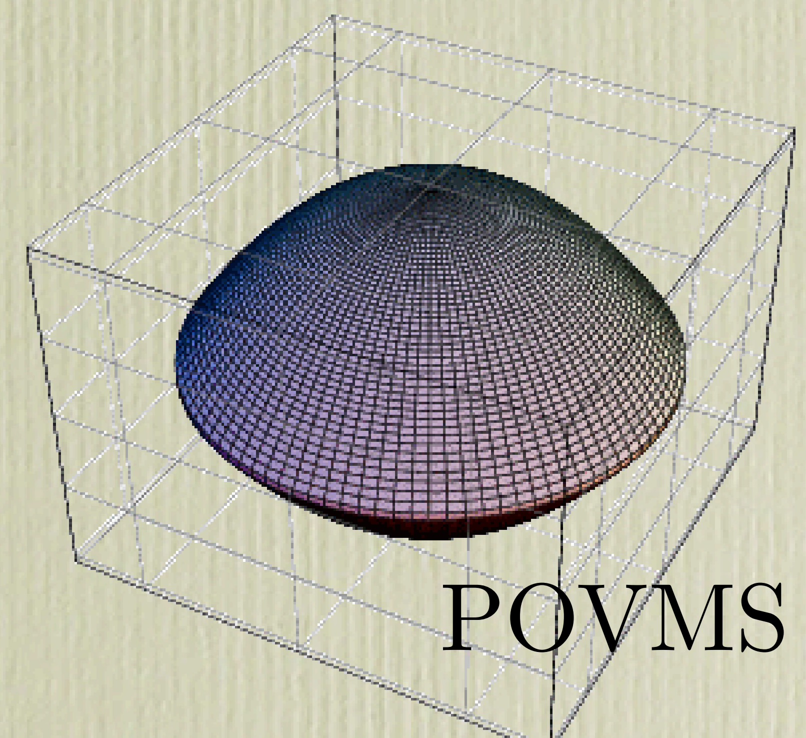
Convex structures



STATES ρ



OPERATIONS $\mathscr{M}$



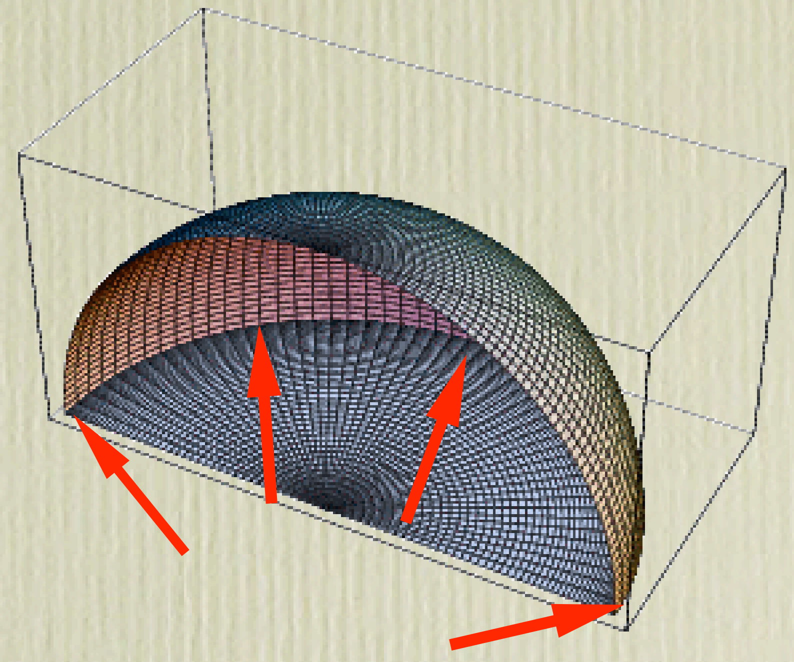
POVMS $\mathbf{P}$

Why to study convex structures?

Optimization problems:

Minimize a cost-function that is concave over the convex set

Minimum on the set of extremal points



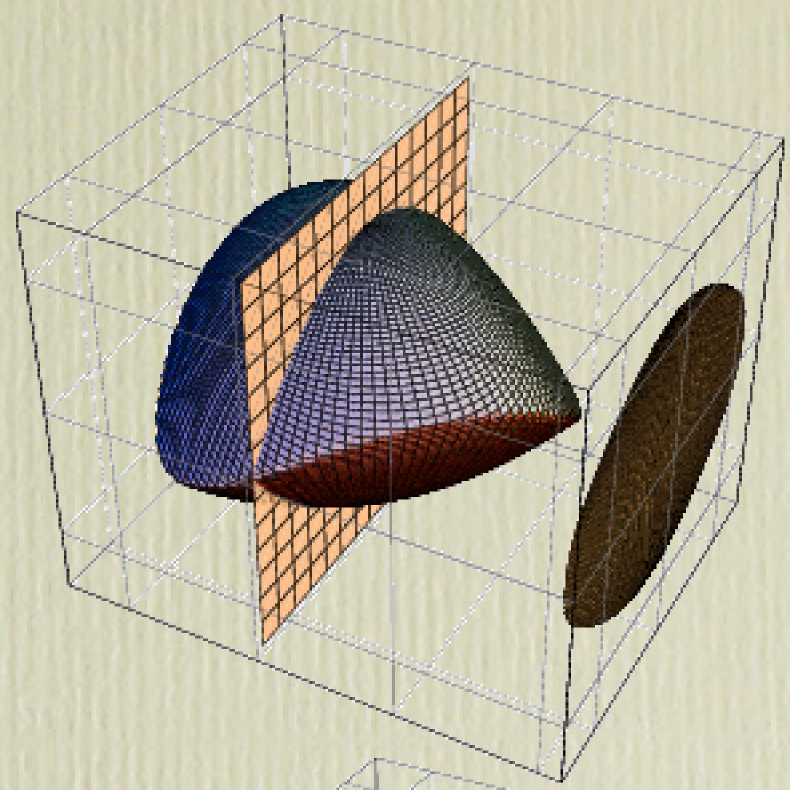
Why to study convex structures?

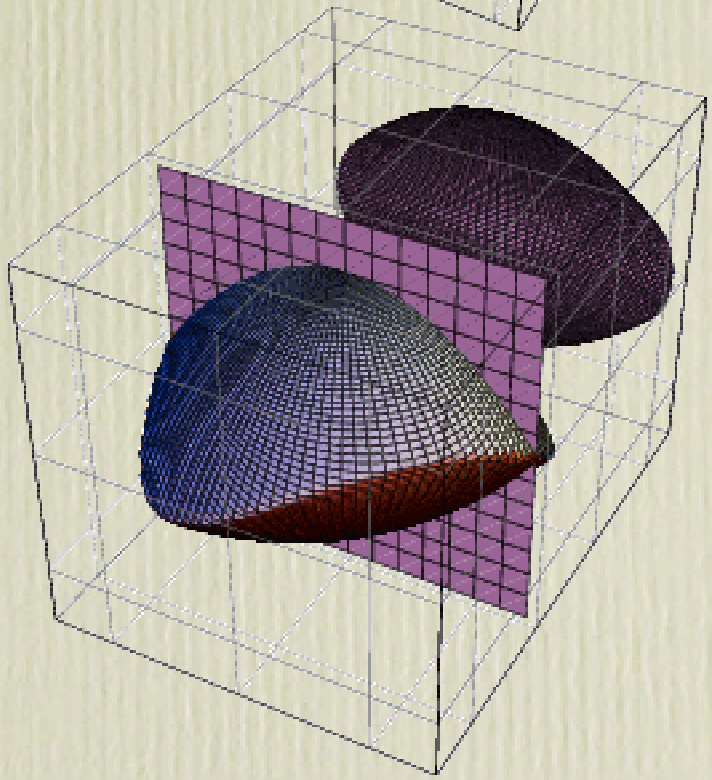
Linear Constraints

corresponds to plane sections of the convex

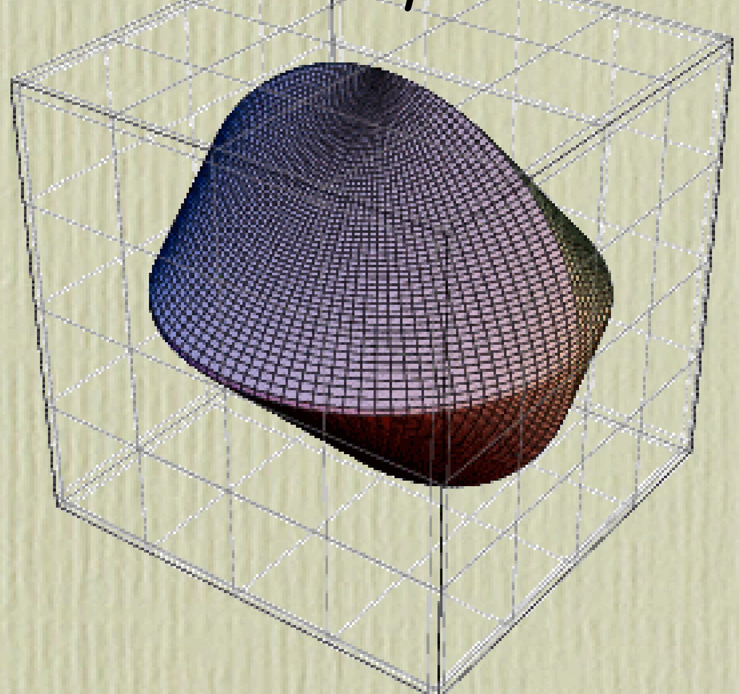
The border of the section is the section of the border

Extremals of the section belong to the original border

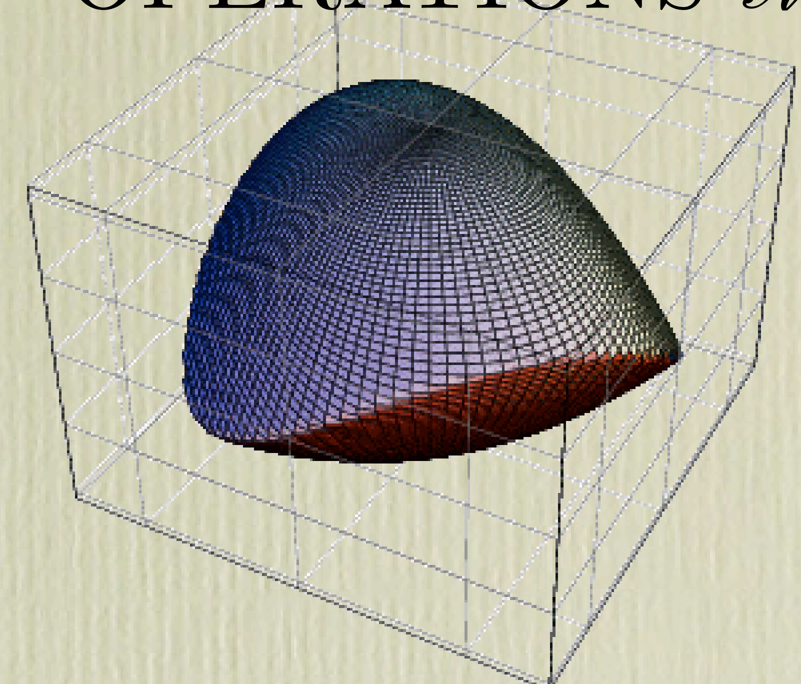




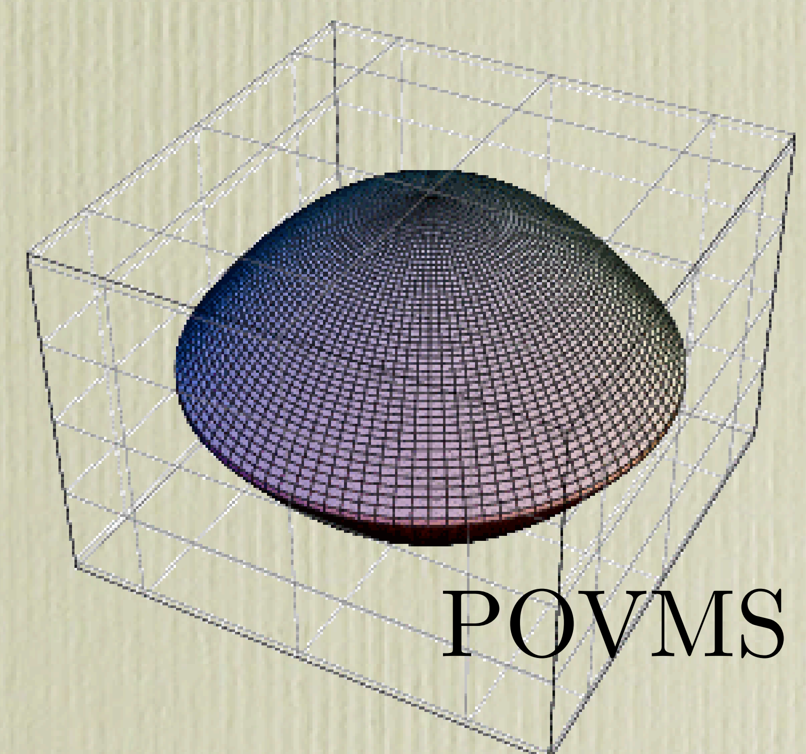
STATES ρ



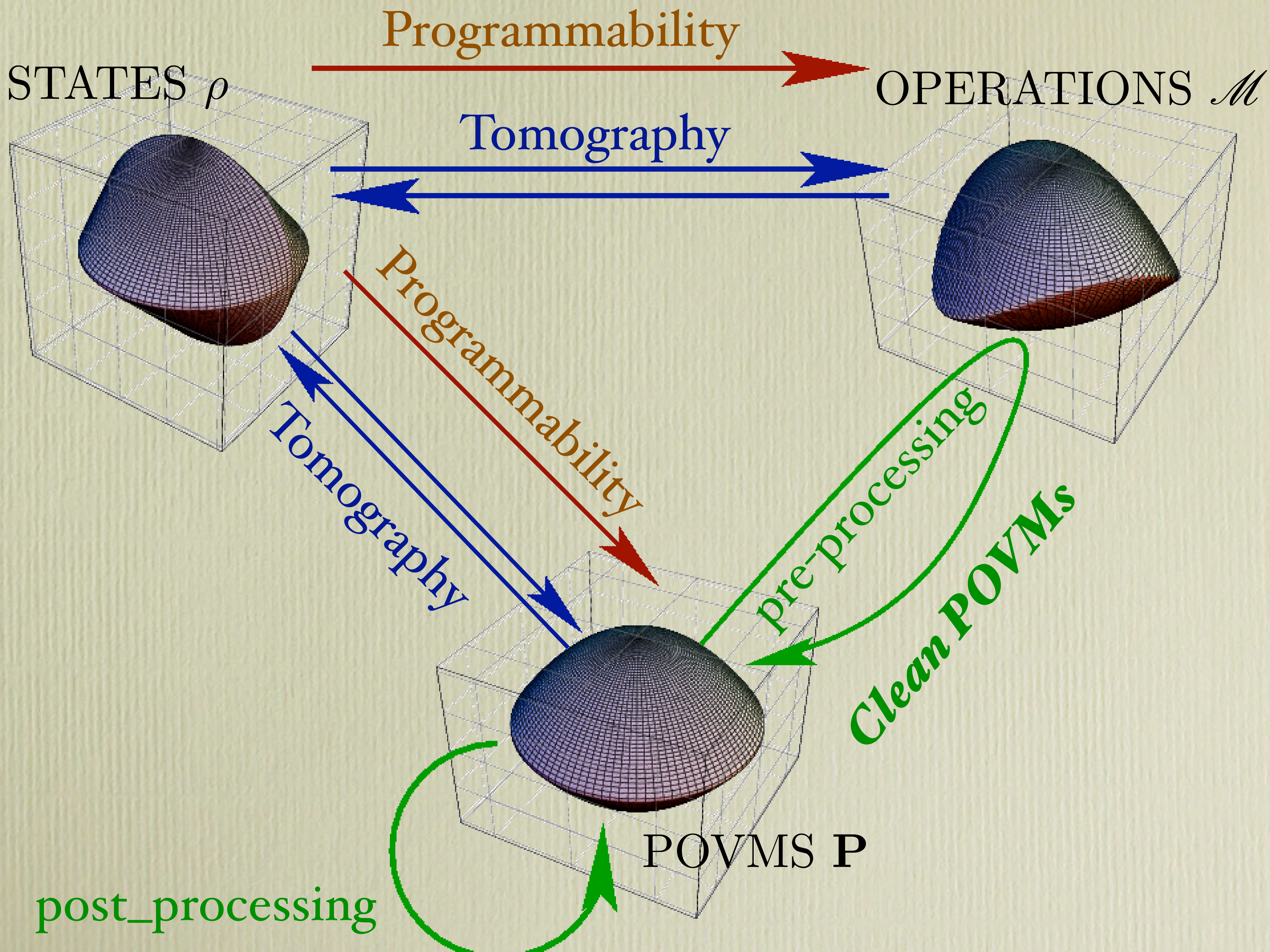
OPERATIONS $\mathscr{M}$



Interrelations



POVMS $\mathbf{P}$


Programmability
STATES ρ
OPERATIONS $\mathscr{M}$
Tomography
Programmability
Tomography
pre-processing
Clean POVMs
POVMS $\mathbf{P}$
post_processing

Notation

- Bipartite states $|\Psi\rangle\!\rangle \in \mathsf{H} \otimes \mathsf{K} \Longleftrightarrow$ operators $\Psi \in \mathsf{HS}(\mathsf{K}, \mathsf{H})$

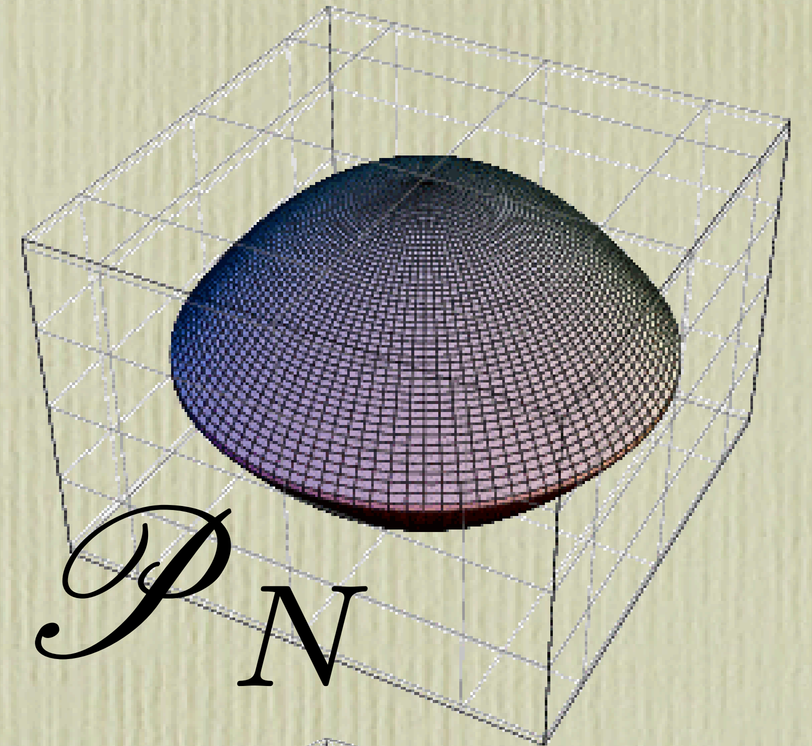
$$|\Psi\rangle\!\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle.$$

- Matrix notation (for fixed reference basis in the Hilbert spaces)

$$A \otimes B|C\rangle\!\rangle = |AC\, B^{\intercal}\rangle\!\rangle,$$

$$\langle\!\langle A|B\rangle\!\rangle \equiv \mathrm{Tr}[A^{\dagger} B].$$

$$|I\rangle\!\rangle = \sum_{n} |n\rangle \otimes |n\rangle$$



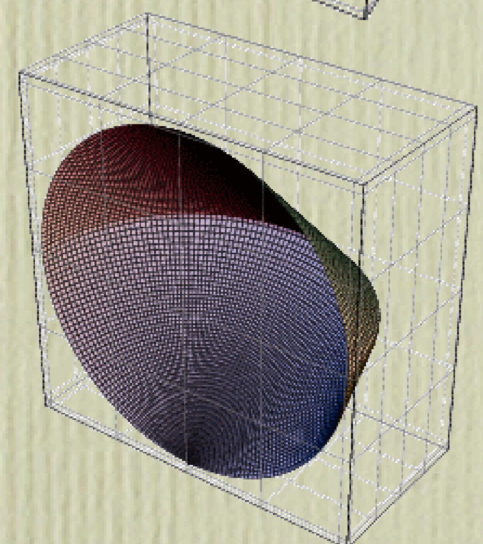
POVM's

Hilbert space H, $d = \dim(\mathsf{H})$

$\mathscr{P}_N$ convex set of POVM's on H with N outcomes

$\mathbf{P} \in \mathscr{P}_N$, $\mathbf{P} = \{P_1, \ldots, P_N\}$

$\{|v_n^{(e)}\rangle\}$: eigenvectors of P_e



Border of the convex

$b(\mathbf{P})$: dimension of the "face"

border $\partial\mathscr{P}_N$ of $\mathscr{P}_N$:

$$b(\mathbf{P}) < d^2(N-1)$$

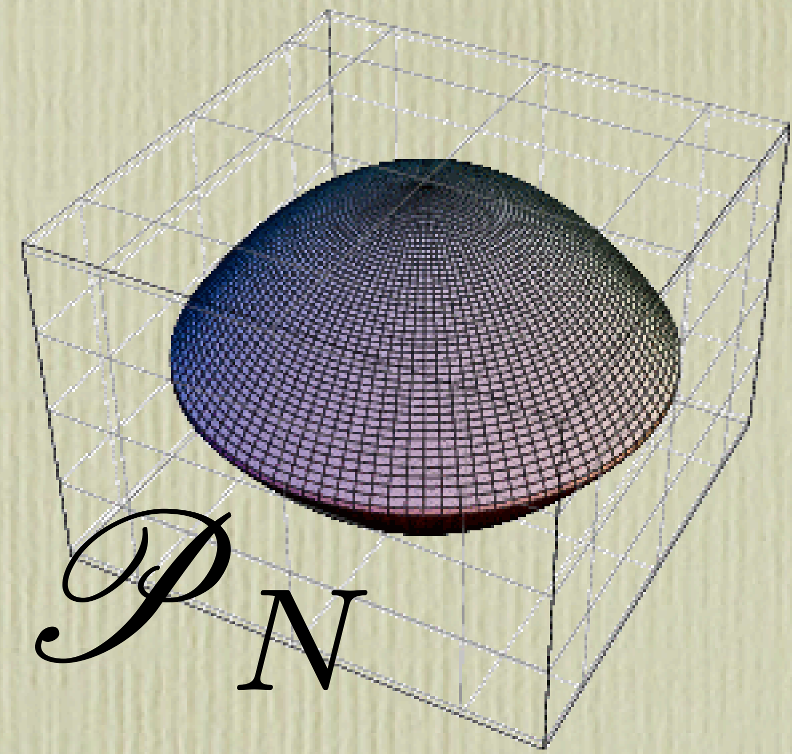
$$b(\mathbf{P}) = r(\mathbf{P}) - l(\mathbf{P})$$

where

$$r(\mathbf{P}) = \sum_e \mathrm{rank}(P_e)^2,$$

$$l(\mathbf{P}) = \dim[\mathsf{Span}\{|v_m^{(e)}\rangle\langle v_n^{(e)}|\}_{nme}]$$

G. M. D'Ariano, P. Lo Presti, and P. Perinotti, Phys. Rev. A (submitted) (quant-ph0408115)



POVM's

Extremal POVM's

A POVM $\mathbf{P} = \{P_e\}_{e \in \mathsf{E}}$ is extremal iff the supports $\mathsf{Supp}(P_e)$ are weakly independent for all $e \in \mathsf{E}$.

We call a generic set of orthogonal projections $\{Z_e\}_{e \in \mathsf{E}}$ weakly independent if for any set of operators $\{T_e\}_{e \in \mathsf{E}}$ on H one has

$$\sum_{e \in \mathsf{E}} Z_e T_e Z_e = 0 \quad \Rightarrow \quad Z_e T_e Z_e = 0, \ \forall e \in \mathsf{E}.$$

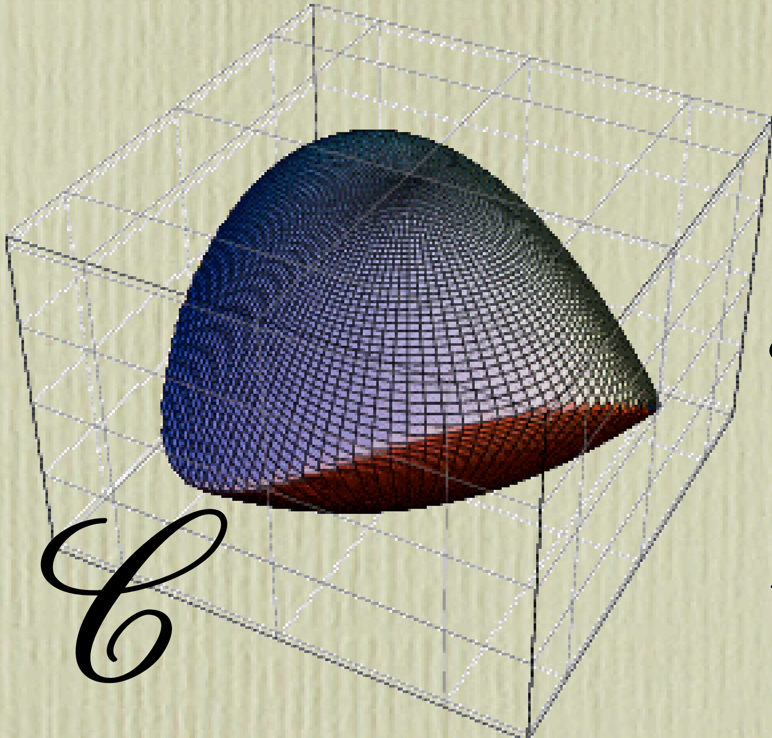
Extremal POVM's are not necessarily rank-one!

There are extremal POVM's only for $N \leq d^2$ outcomes.

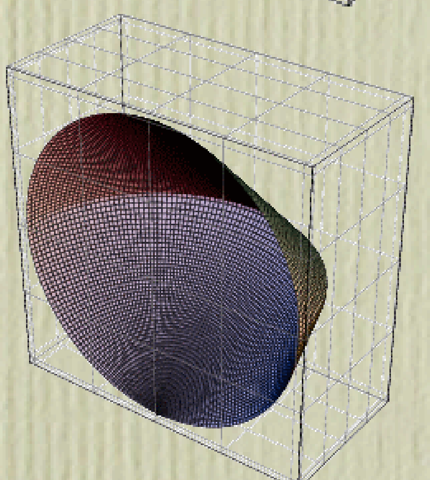
For $N = d^2$ outcomes there exists always an extremal POVM, which is rank-one and informationally complete.

G. M. D'Ariano, P. Lo Presti, and P. Perinotti, Phys. Rev. A (submitted) (quant-ph0408115)

Channels



$\mathscr{C}$



$\mathscr{C}$ convex set of channels $\mathscr{E}$ from $\mathcal{S}(\mathsf{H})$ to $\mathcal{S}(\mathsf{K})$

$R_{\mathscr{E}} = \mathscr{E} \otimes \mathscr{I}(|I\rangle\rangle\langle\langle I|)$ *Choi isomorphism*

$\mathscr{E} = \sum_n E_n \rho E_n^\dagger$ canonical Kraus

Border of the convex

$$b(\mathscr{E}) = r(\mathscr{E}) - l(\mathscr{E})$$

where

$r(\mathscr{E}) = \mathrm{rank}(R_{\mathscr{E}})^2$,

$l(\mathscr{E}) = \dim(\mathsf{Span}\{E_i^\dagger E_j\})$

$b(\mathscr{E})$: dimension of the "face"

border $\partial\mathscr{C}$ of $\mathscr{C}$:

$b(\mathscr{E}) < \dim(\mathsf{H})^2(\dim(\mathsf{K})^2 - 1)$

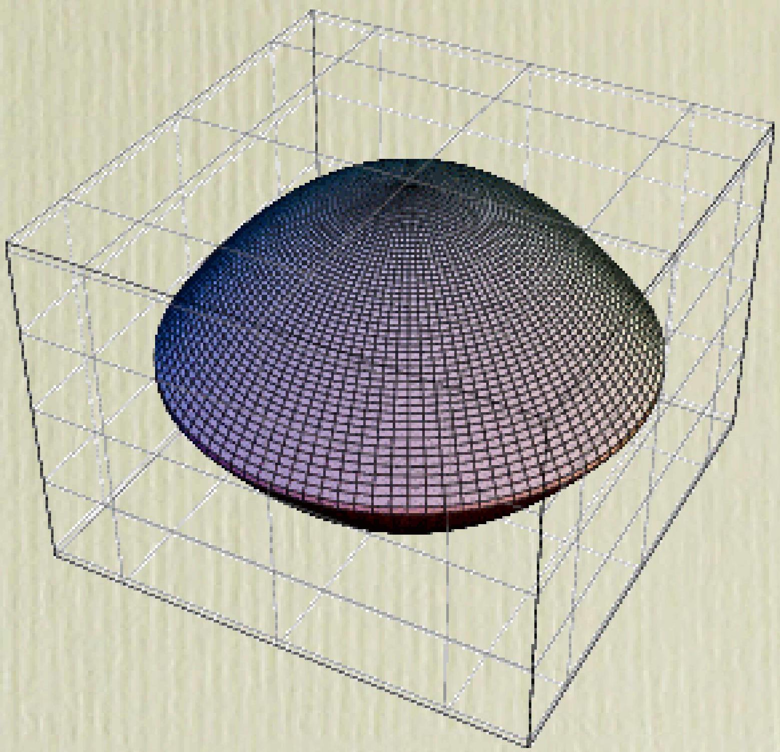
G. Chiribella, G. M. D'Ariano, and P. Perinotti, (unpublished)

Convex of covariant POVM's

Covariance is a linear constraint

G. M. D'Ariano, *Extremal covariant Quantum Operations and POVM's*, J. Math. Phys. **45** 3620-3635 (2004)

G. Chiribella and G. M. D'Ariano, *Extremal covariant positive operator measures*, J. Math. Phys. **45** 4435-4447 (2004)

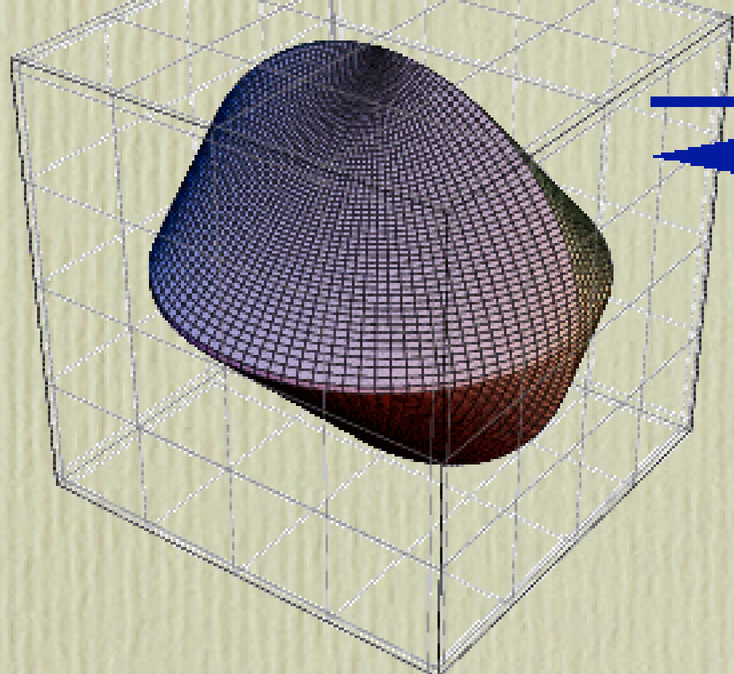


Extremal POVM's are not necessarily rank-one!

For some (reducible) representations rank-one POVM are forbidden!

Tomography of operations

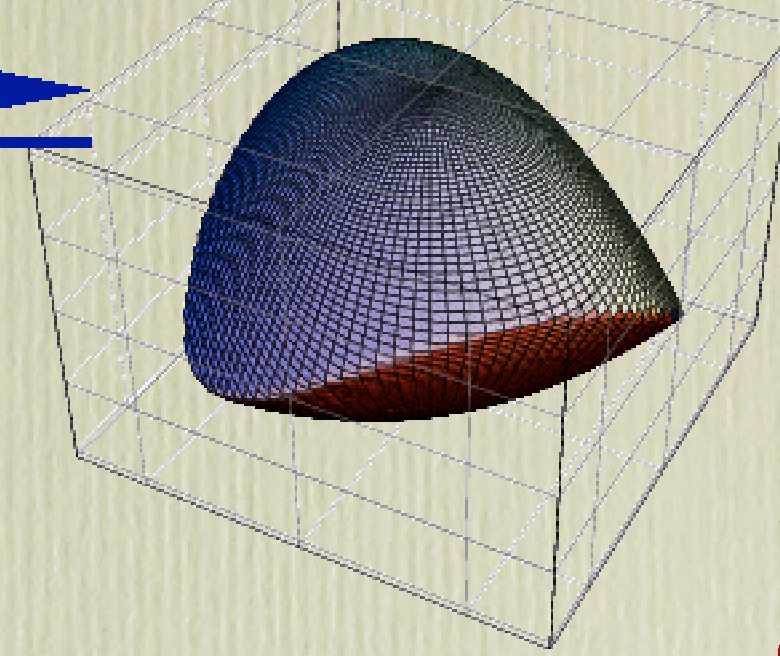
STATES ρ



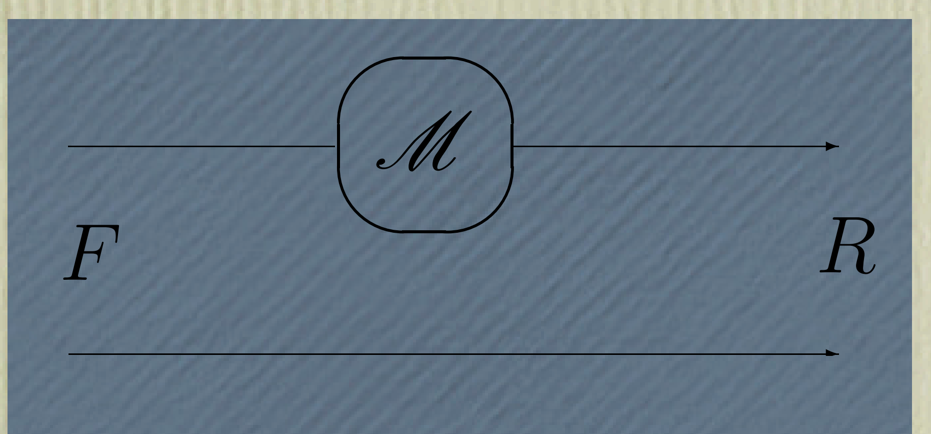
Tomography

$$R_{\mathscr{E}} = \mathscr{E} \otimes \mathscr{I}(|I\rangle\rangle\langle\langle I|)$$

OPERATIONS $\mathscr{M}$



$$R \Longleftrightarrow \mathscr{M}$$



F : faithful state

$$R = \mathscr{M} \otimes \mathscr{I}(F)$$

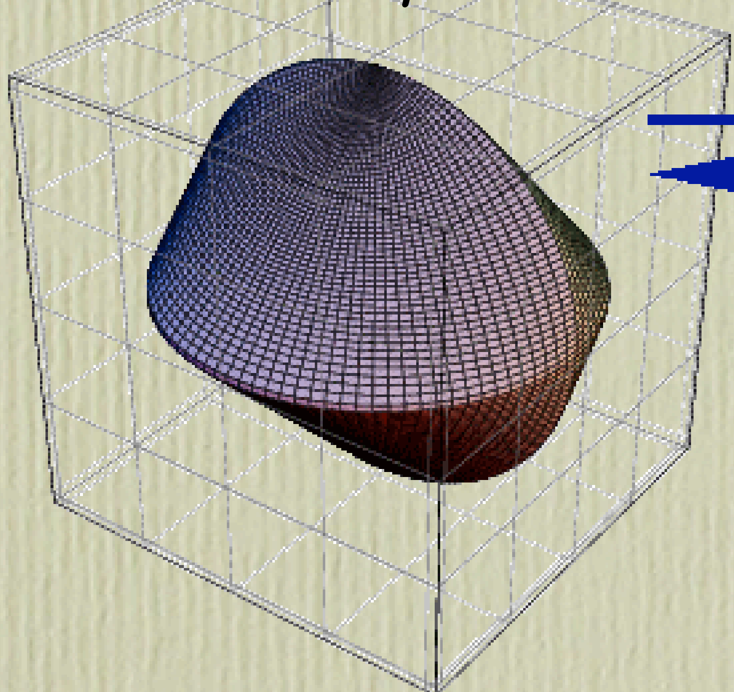
$$\mathscr{M}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^{\intercal})\mathscr{I} \otimes \mathscr{F}^{-1}(R)]$$

$$\mathscr{F}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^{\intercal})F]$$

G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **91** 047902-(2003)

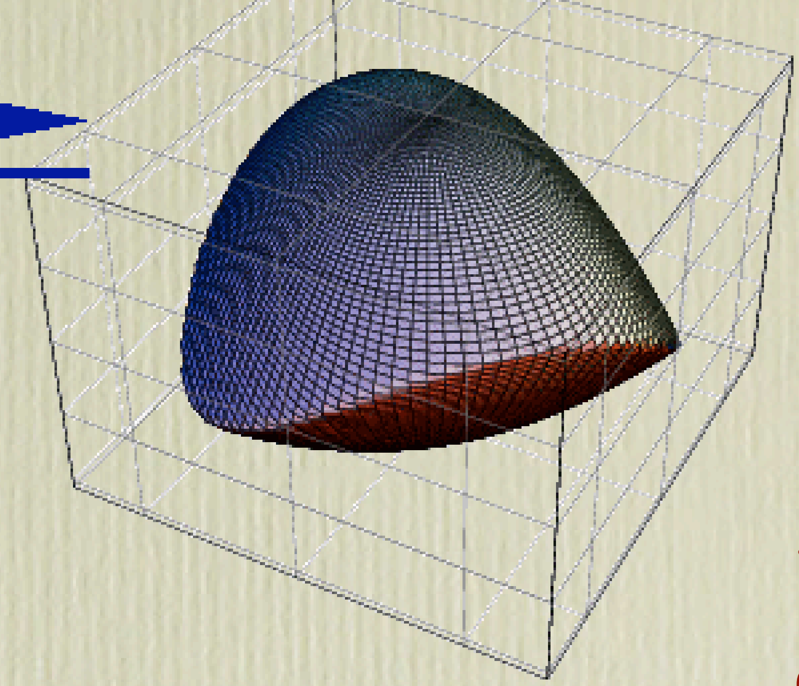
Tomography of operations

STATES ρ



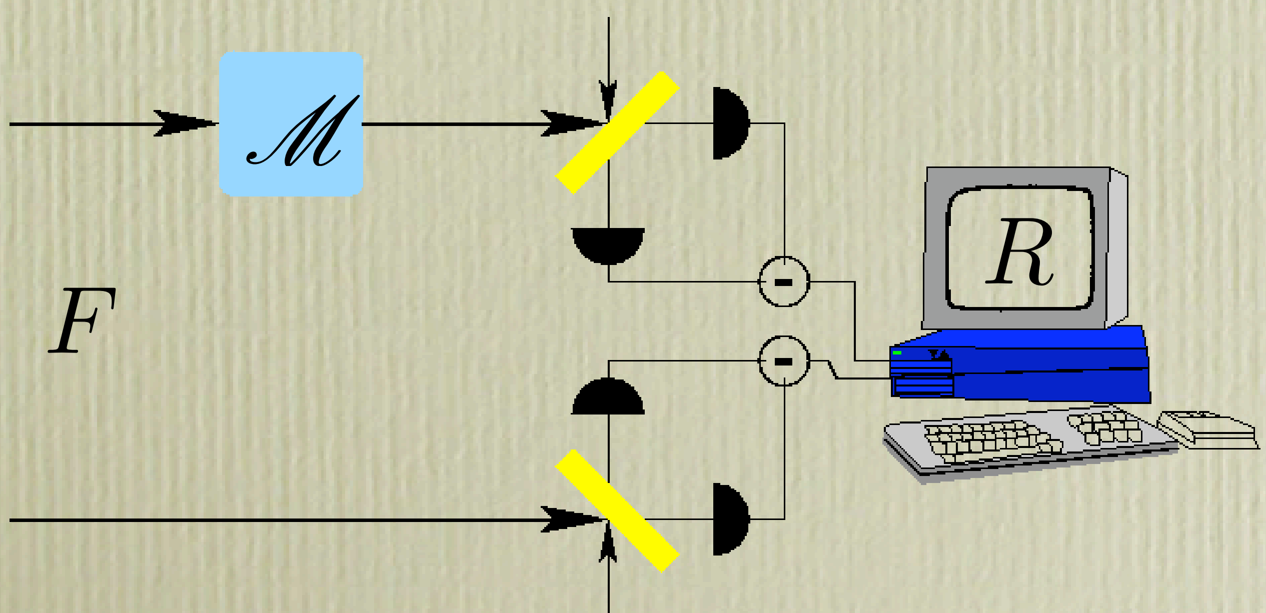
Tomography

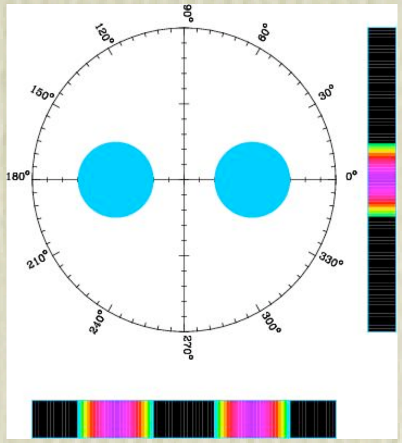
OPERATIONS $\mathscr{M}$

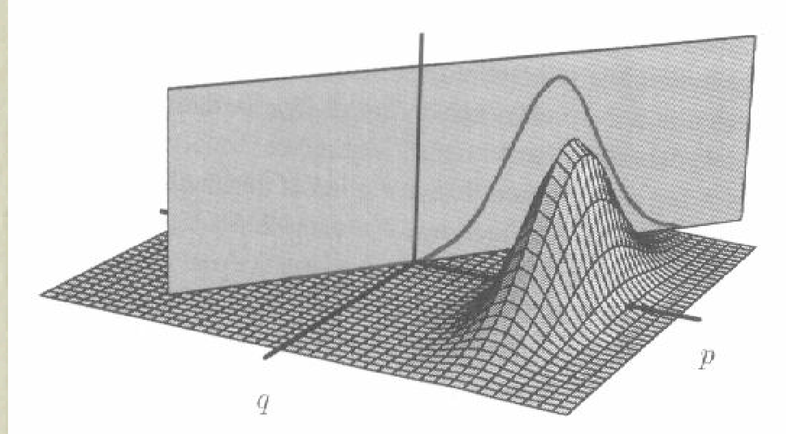


$$R \Longleftrightarrow \mathscr{M}$$

$$R = \mathscr{M} \otimes \mathscr{I}(F)$$

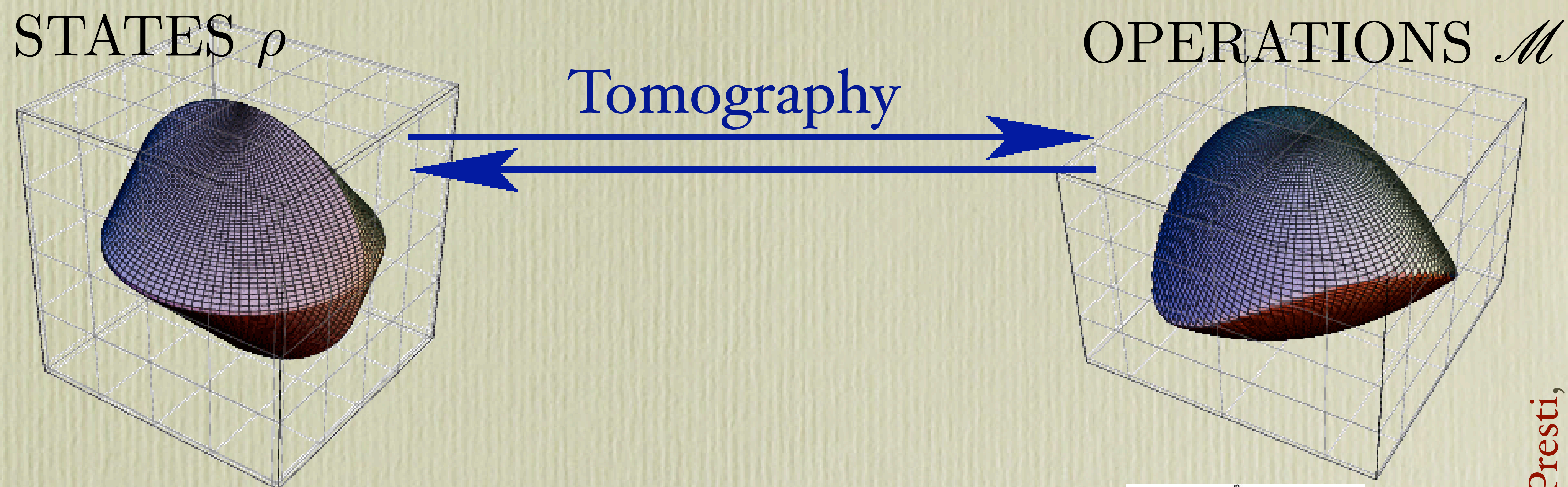






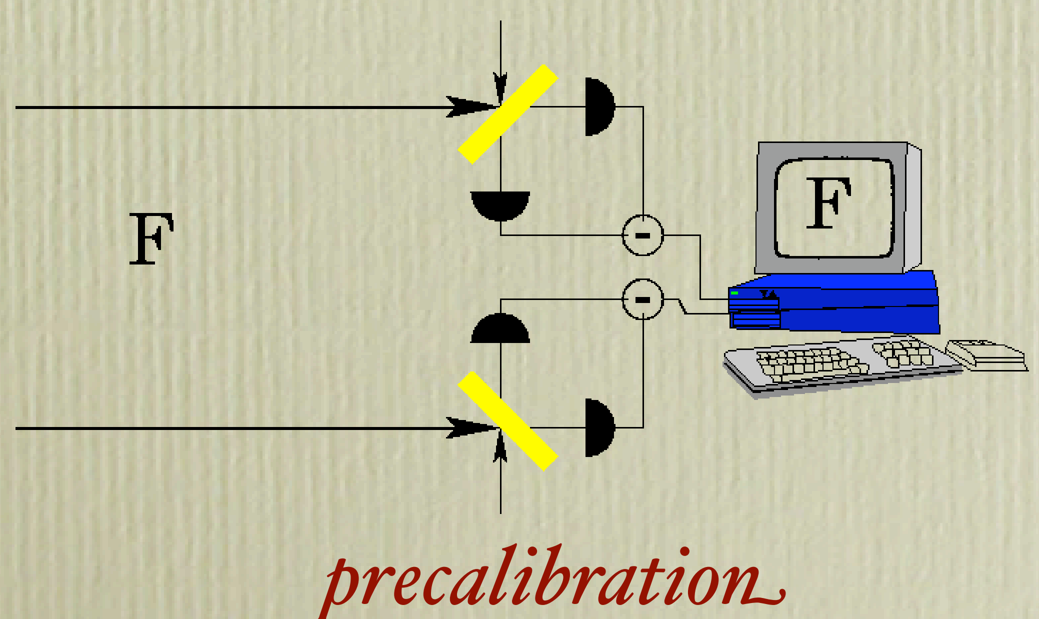
G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **91** 047902-(2003)

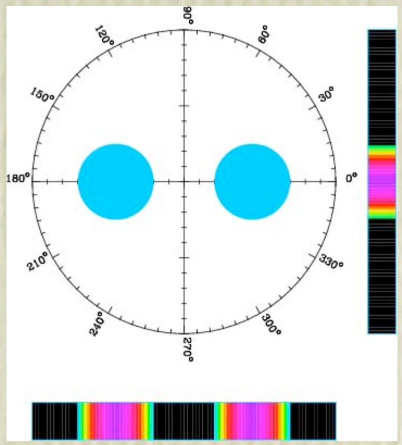
Tomography of operations

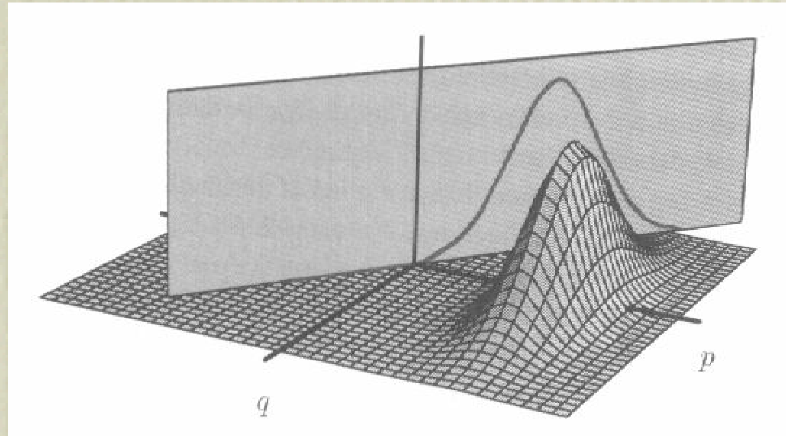


$$R \Longleftrightarrow \mathscr{M}$$

$$R = \mathscr{M} \otimes \mathscr{I}(F)$$

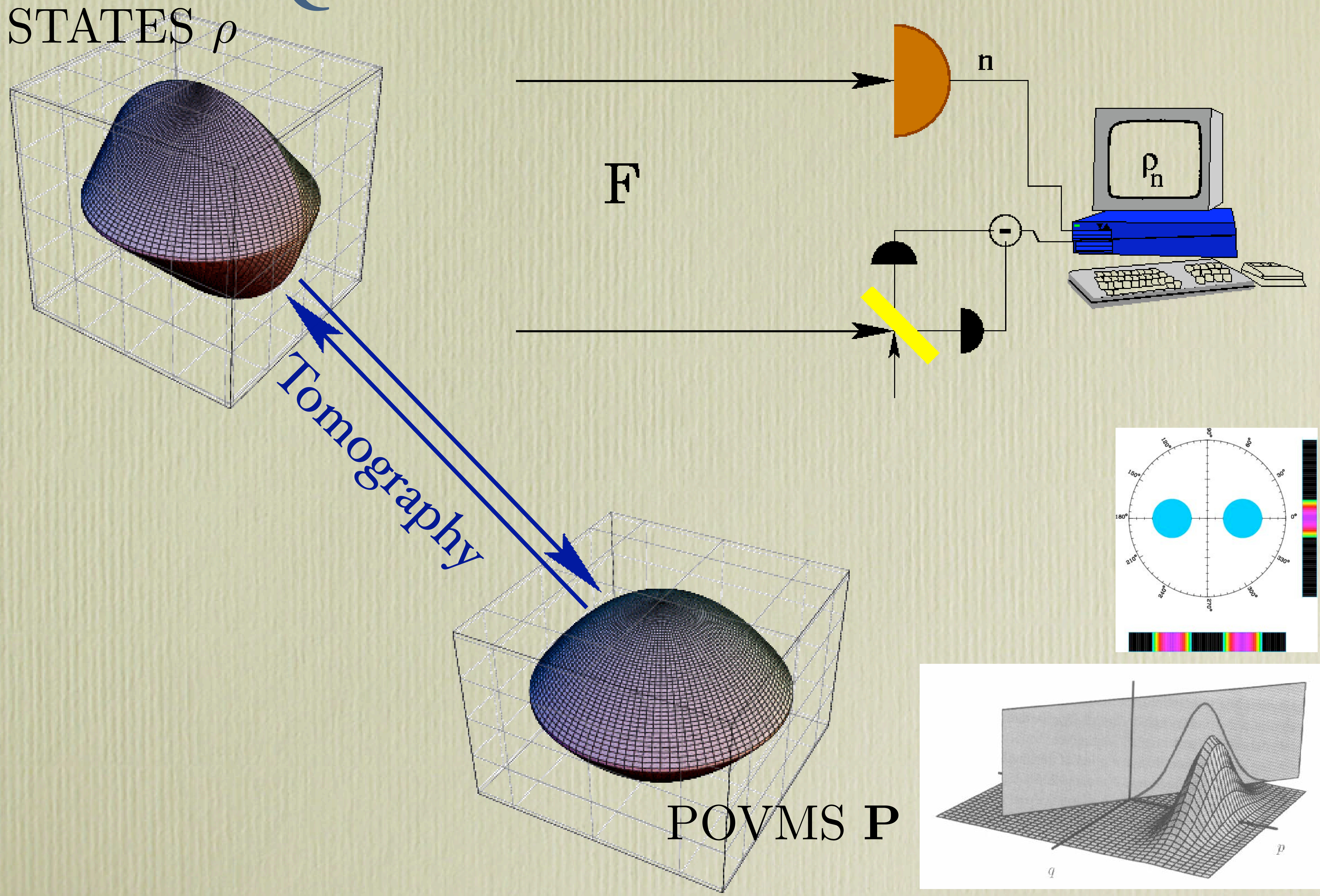




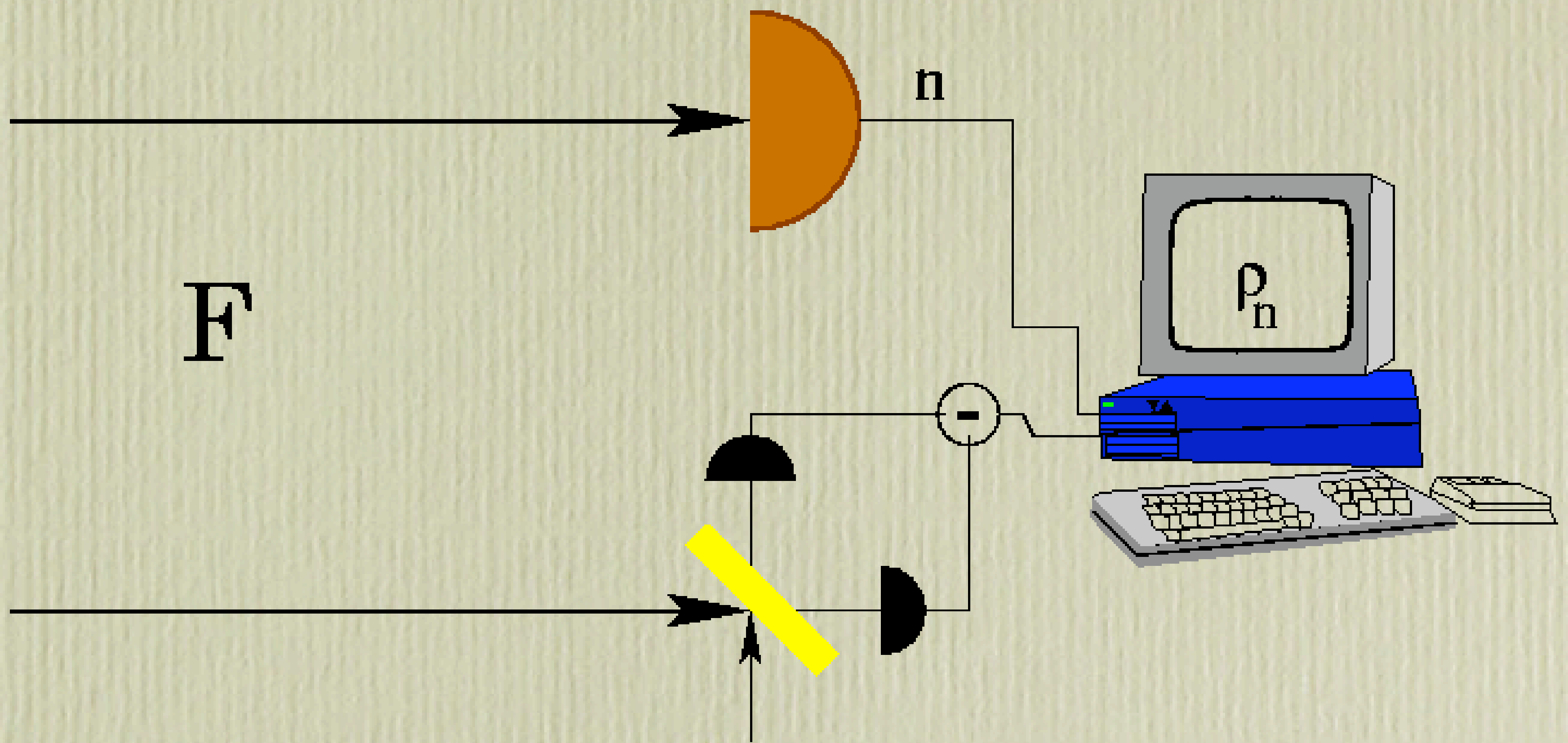


G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **91** 047902-(2003)

Quantum Calibration



Quantum Calibration



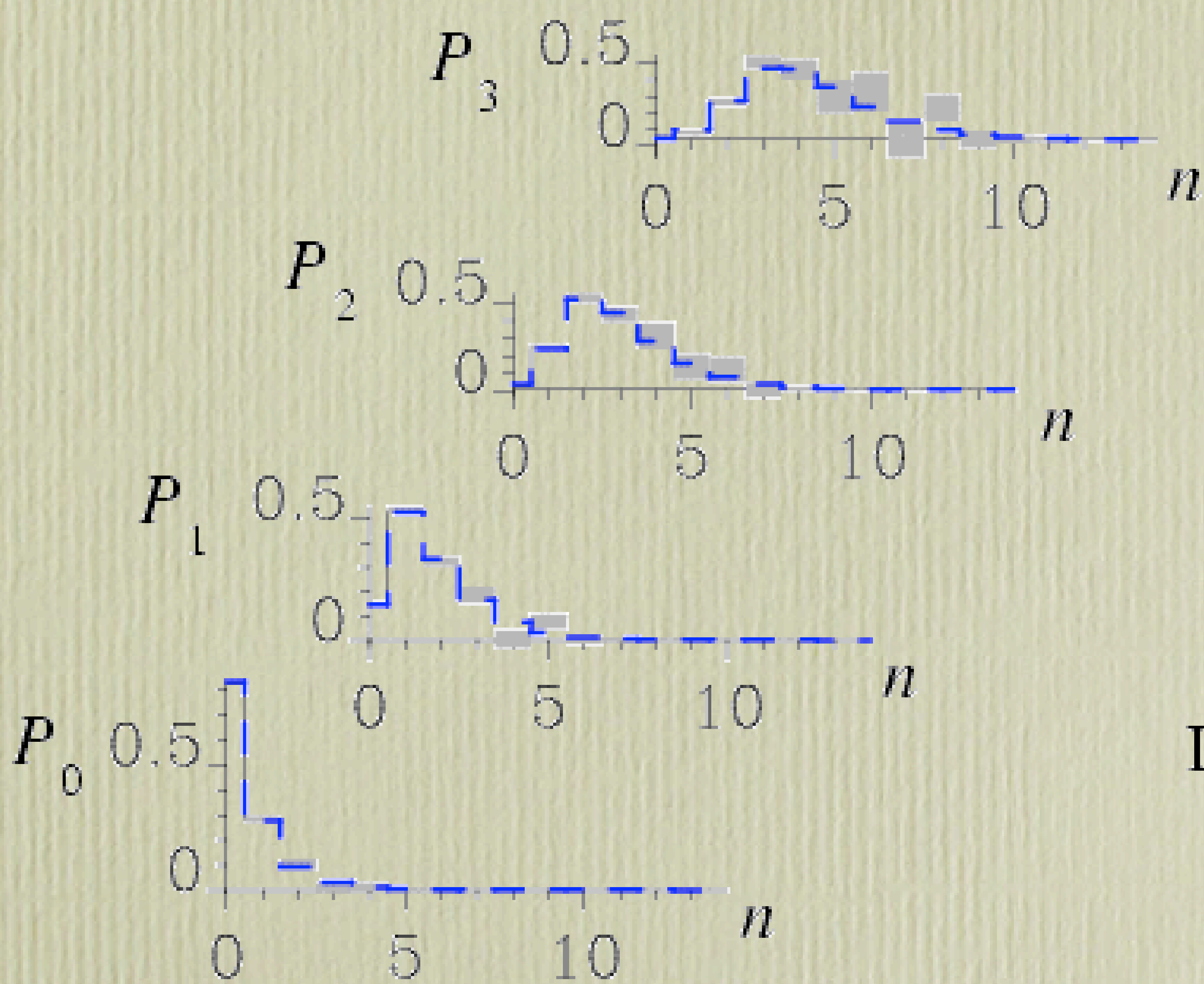
G. M. D'Ariano, P. Lo Presti, and L. Maccone, Phys. Rev. Lett. **93** 250407 (2004)

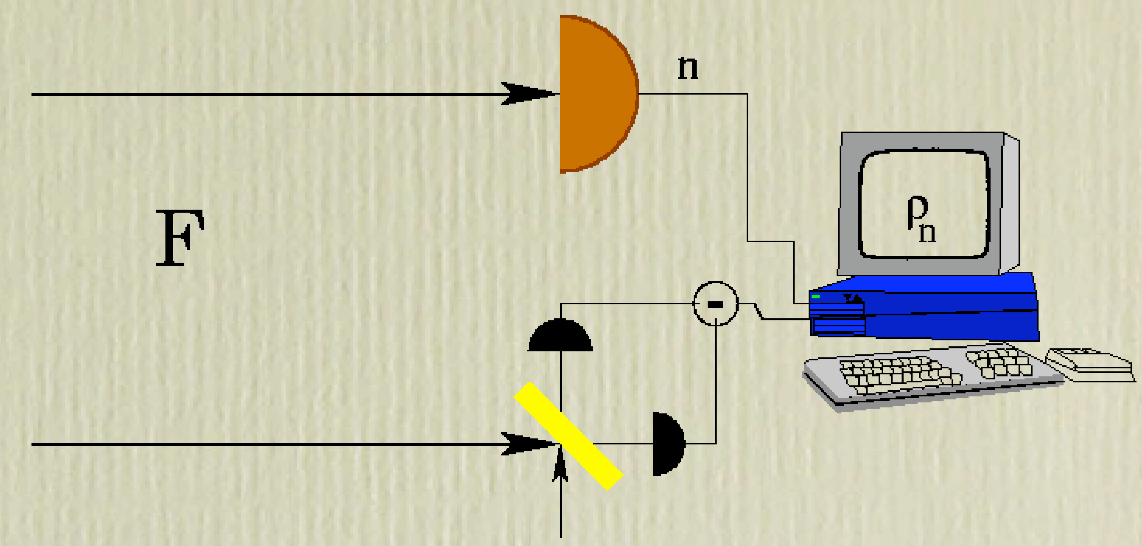
$$p_n\rho_n = \mathscr{F}(P_n), \quad P_n = \mathscr{F}^{-1}(p_n\rho_n),$$

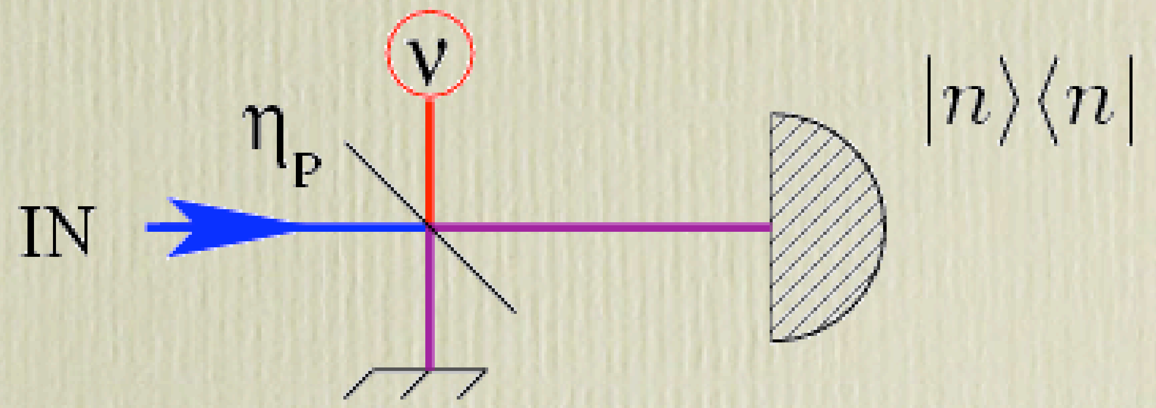
$$\mathscr{F}(X) = \mathrm{Tr}_2[(I\otimes X)F]$$

- p_n probability of the outcome n,
- ρ_n conditioned state, to be determined by quantum tomography,
- $\mathscr{F}$ associated map of the faithful state F.

Quantum Calibration

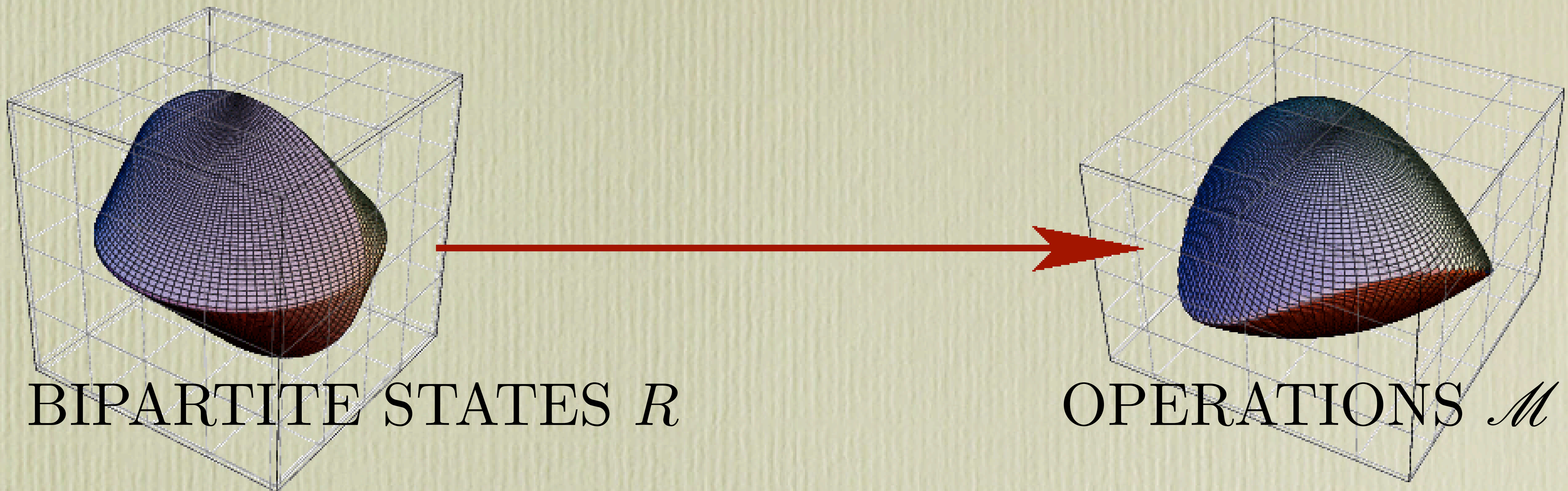






G. M. D'Ariano, P. Lo Presti, and L. Maccone, Phys. Rev. Lett. **93** 250407 (2004)

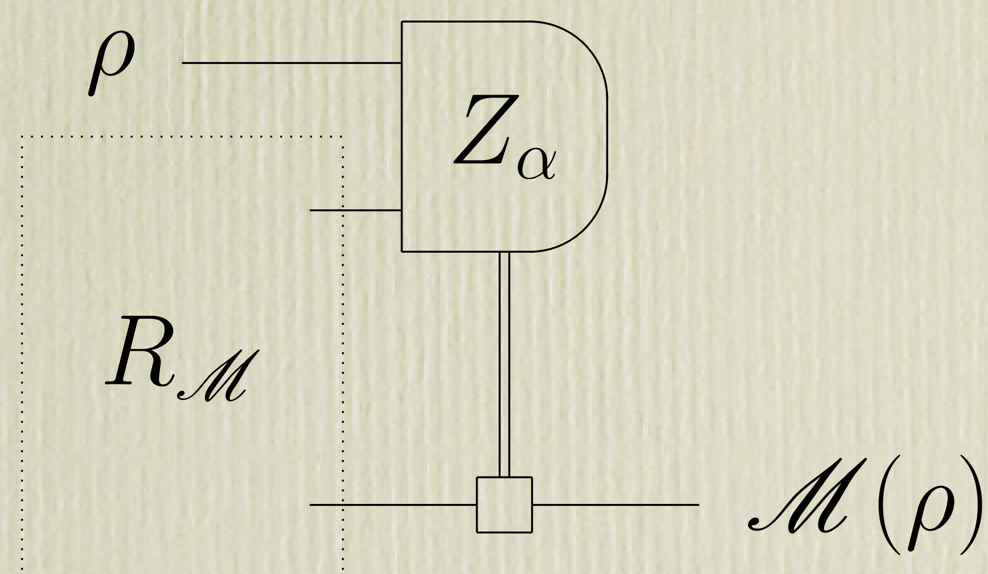
Programmability of operations



Probababilistic

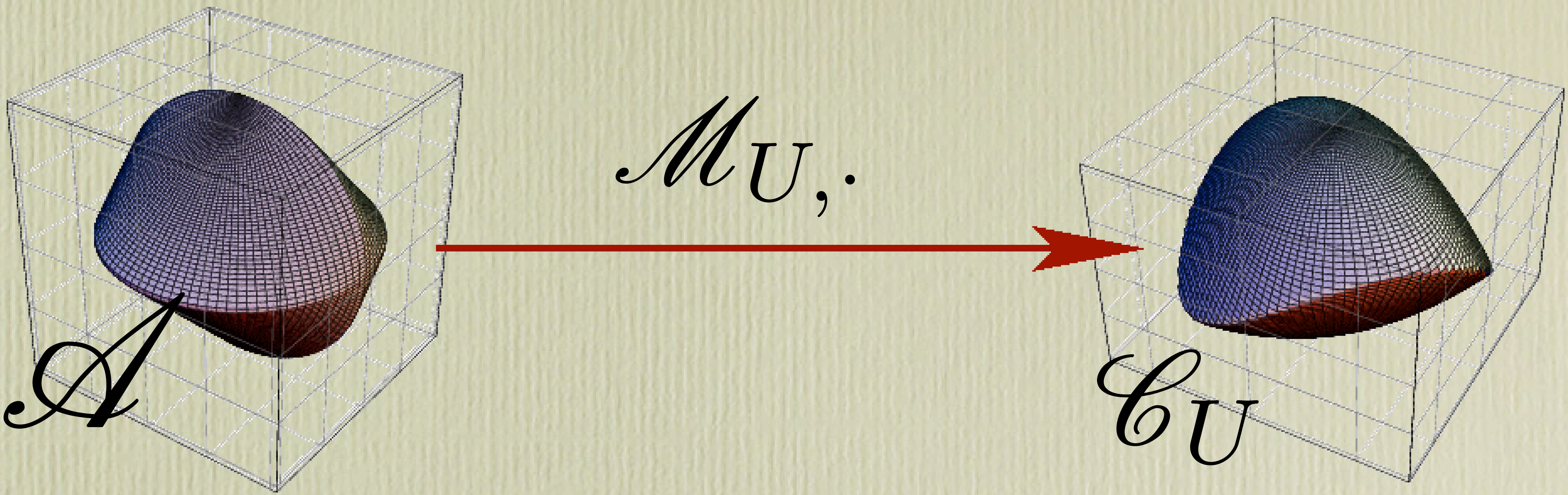
$$p\mathscr{M}(\rho) = \mathrm{Tr}_2[(I \otimes \rho^{\intercal})R_{\mathscr{M}}]$$
$$= \mathrm{Tr}_{23}[I \otimes |\Omega\rangle\!\rangle\langle\!\langle\Omega|)(R_{\mathscr{M}} \otimes \rho)]$$

$$R_{\mathscr{M}} = \mathscr{M} \otimes \mathscr{I}(I \otimes |\Omega\rangle\!\rangle\langle\!\langle\Omega|)$$



$$\Omega = \frac{1}{\sqrt{d}}I, \quad Z_0 = |\Omega\rangle\!\rangle\langle\!\langle\Omega|, \quad p = \frac{1}{d^2}$$

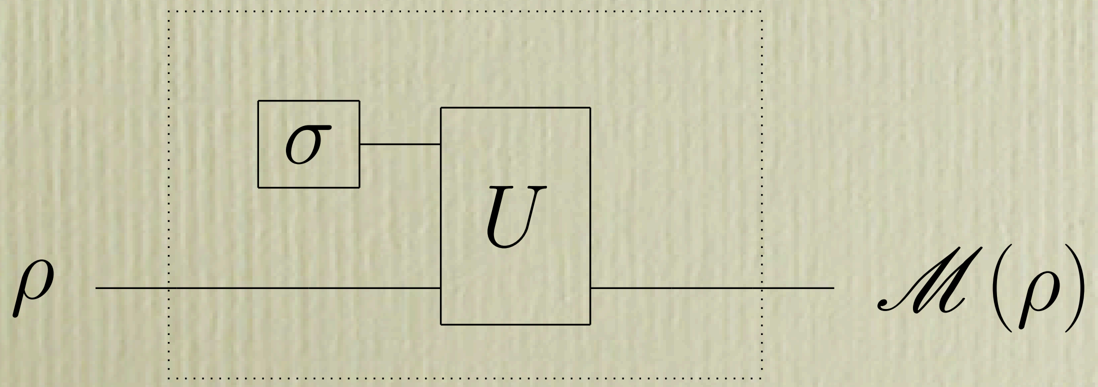
Programmability of operations



Deterministic

$$\mathscr{M}_{U,\sigma}(\rho) \doteq \mathrm{Tr}_2[U(\rho \otimes \sigma)U^\dagger]$$

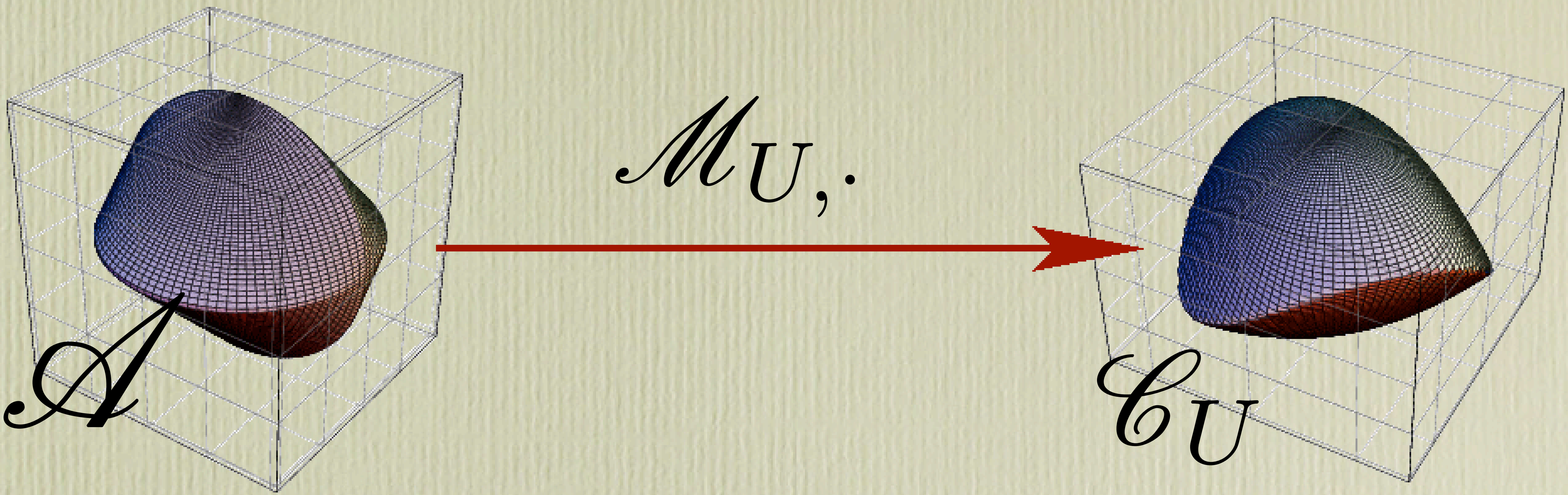
$$\mathscr{C}_U \doteq \mathscr{M}_{U,\mathscr{A}}$$



No go theorem (Nielsen-Chuang)

It is impossible to program all unitary channels with a single U and a finite-dimensional ancilla

Programmability of operations



Deterministic

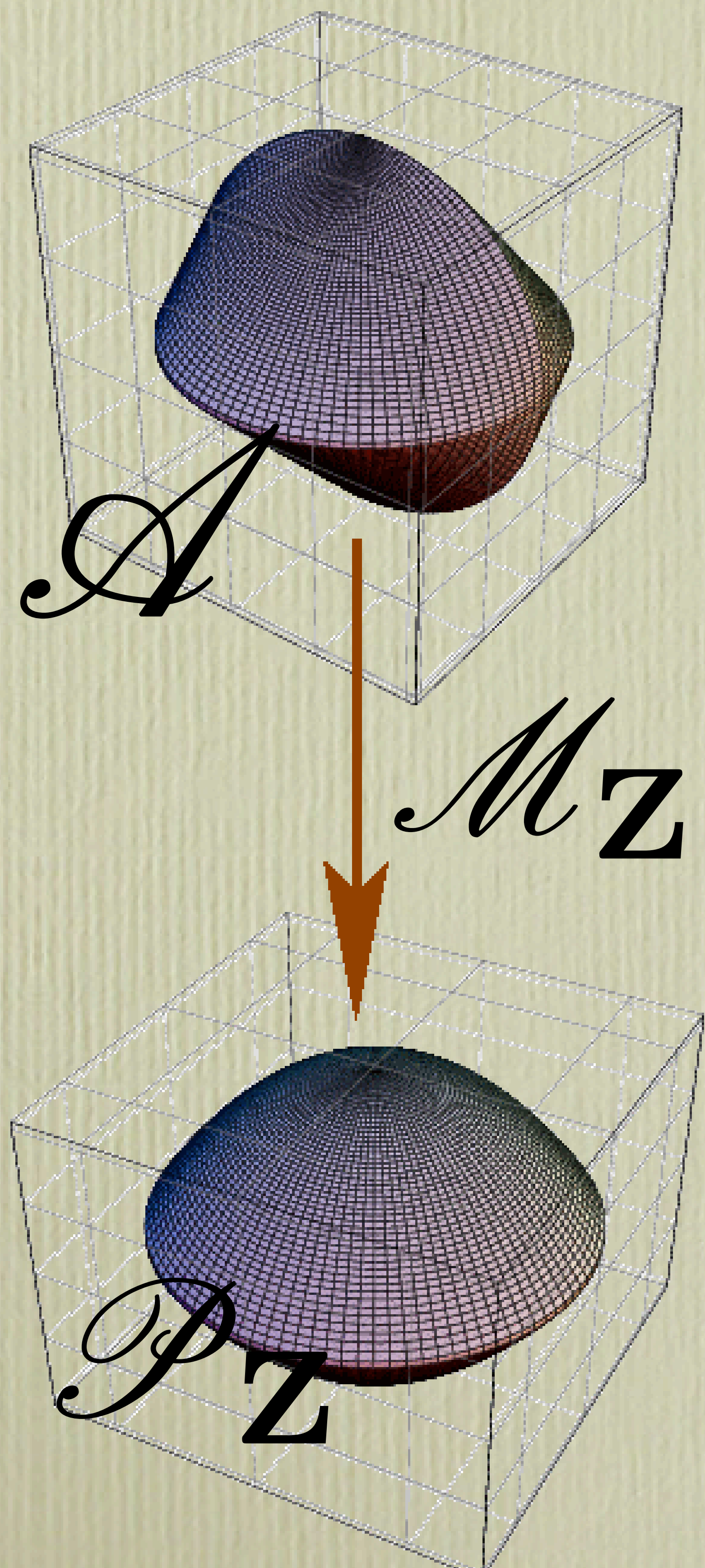
$$\mathscr{M}_{U,\sigma}(\rho) \doteq \mathrm{Tr}_2[U(\rho \otimes \sigma)U^\dagger]$$

$$\mathscr{C}_U \doteq \mathscr{M}_{U,\mathscr{A}}$$

Problem: *The "big U"*

For given $d = \dim(\mathcal{A})$ find the unitary operator U that maximizes the "size" of the convex set $\mathscr{C}_U$.

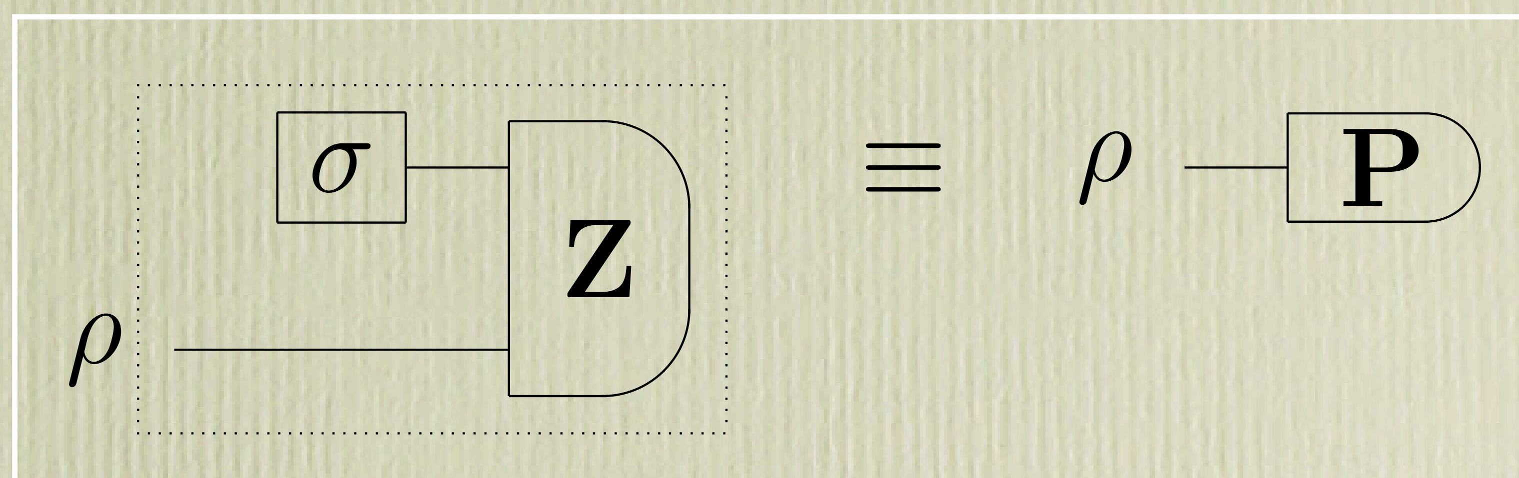
Programmability of POVMs



Deterministic

$$\mathscr{M}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}] = \mathbf{P}$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathscr{M}_{\mathbf{Z},\mathscr{A}}$$



Programmability of POVMs

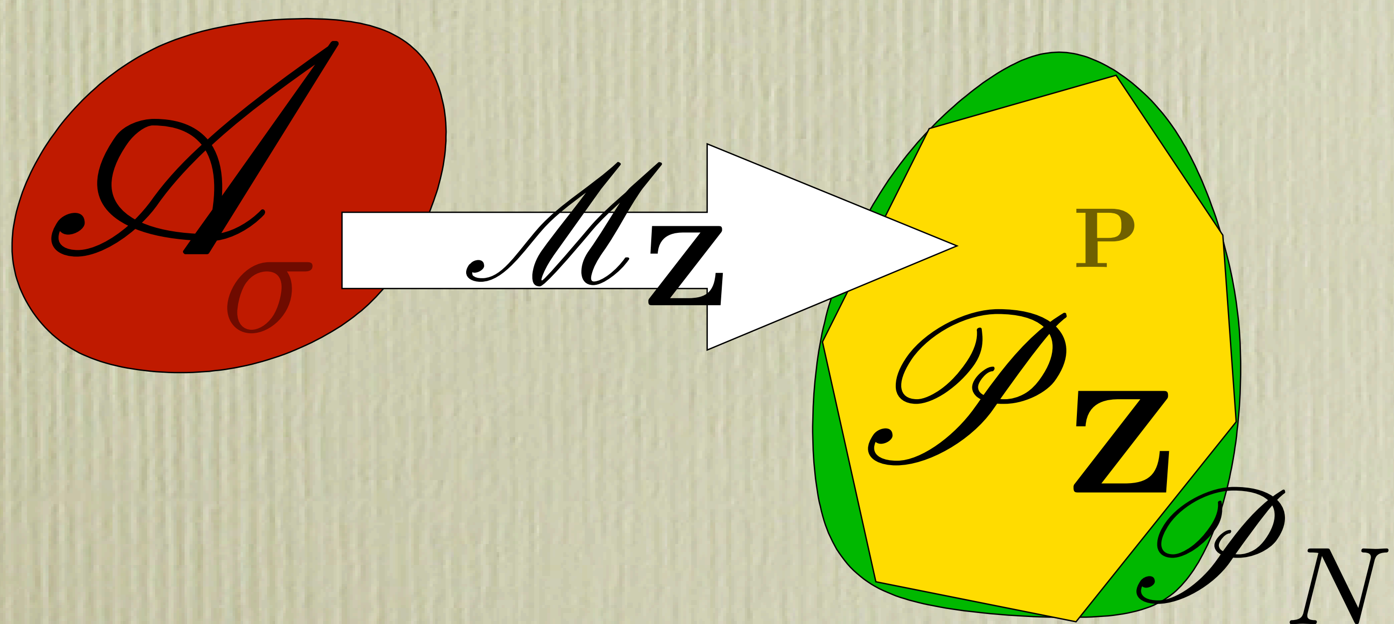
Deterministic

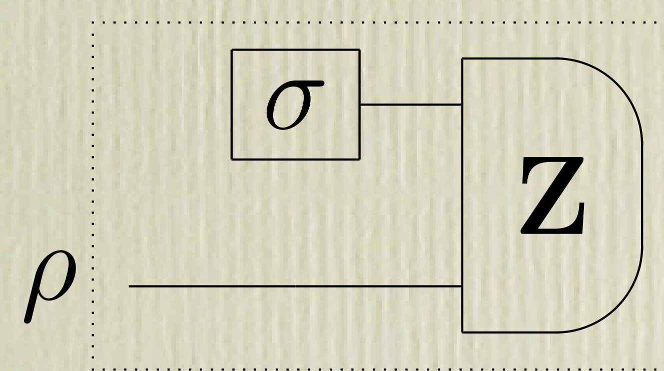
$$\mathscr{M}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}] = \mathbf{P}$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathscr{M}_{\mathbf{Z},\mathscr{A}}$$

No go theorem

It is impossible to program all observables with a single ***Z*** and a finite-dimensional ancilla





Programmability of POVMs

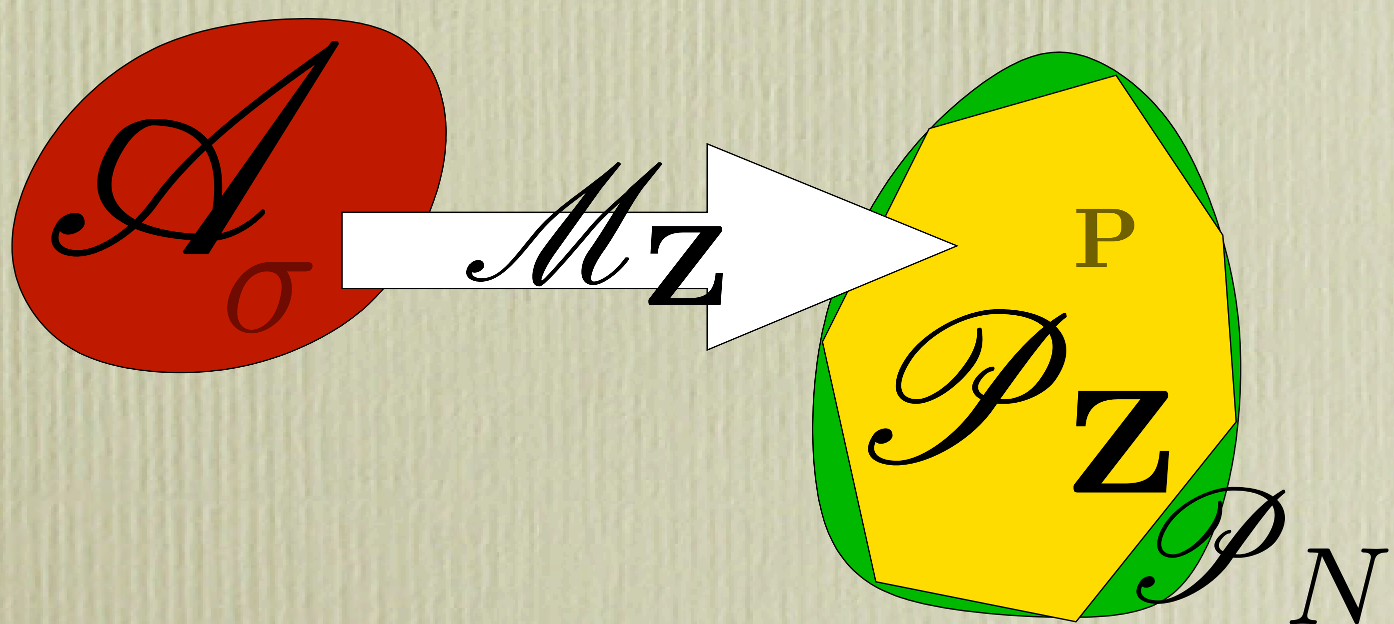
Deterministic

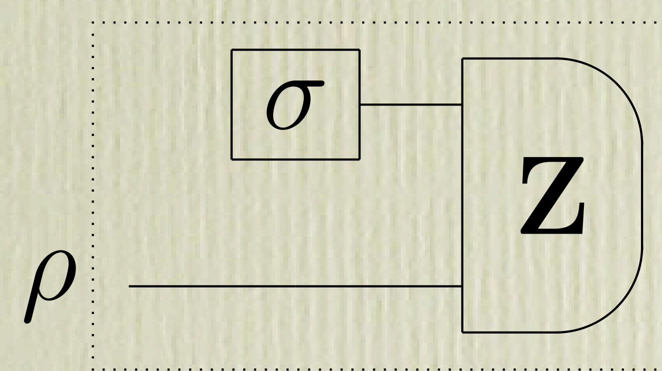
$$\mathscr{M}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}] = \mathbf{P}$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathscr{M}_{\mathbf{Z},\mathscr{A}}$$

Problem: *The "big* **Z**"

For given $d = \dim(\mathcal{A})$ and $N = |\mathbf{Z}| = |\mathbf{P}|$, find the observable $\mathbf{Z}$ that maximizes the "size" of the convex set $\mathscr{P}_{\mathbf{Z}}$.





Programmability of POVMs

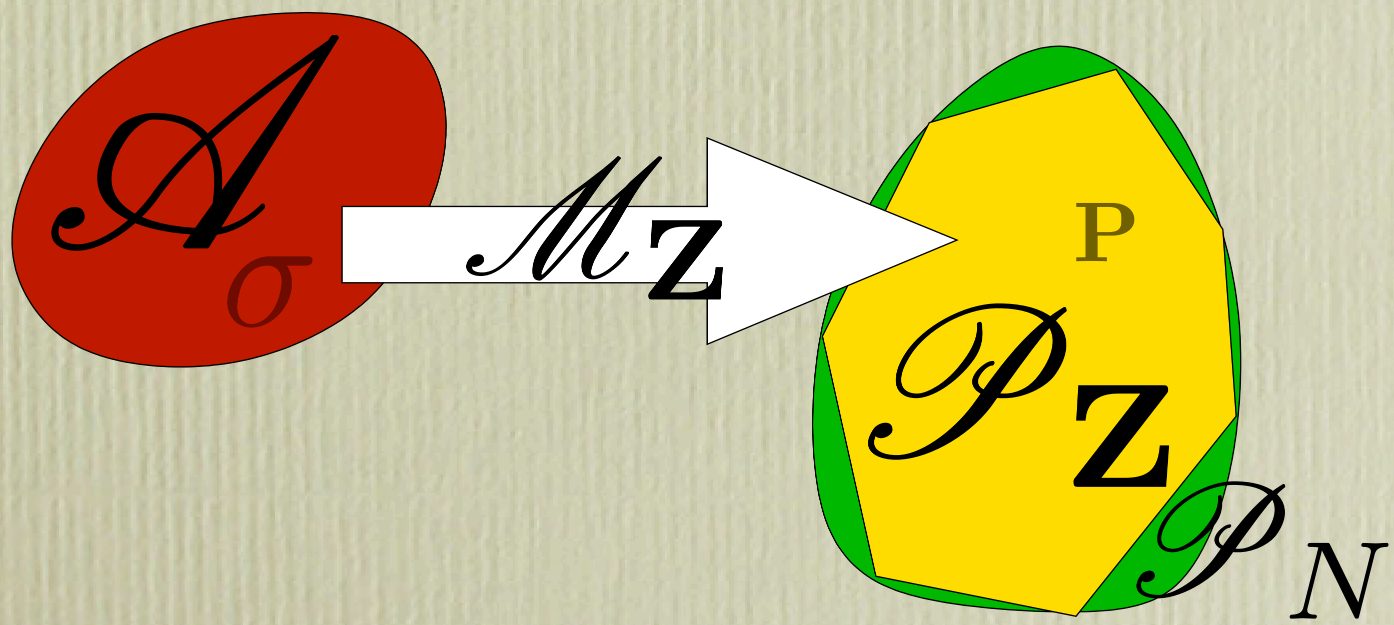
Deterministic

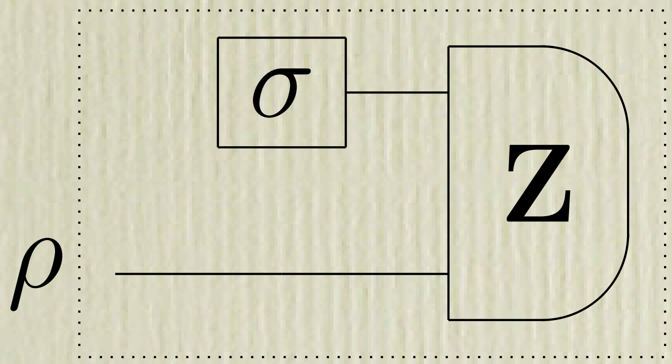
$$\mathscr{M}_{\mathbf{Z},\sigma} \doteq \mathrm{Tr}_2[(I \otimes \sigma)\mathbf{Z}] = \mathbf{P}$$

$$\mathscr{P}_{\mathbf{Z}} \doteq \mathscr{M}_{\mathbf{Z},\mathscr{A}}$$

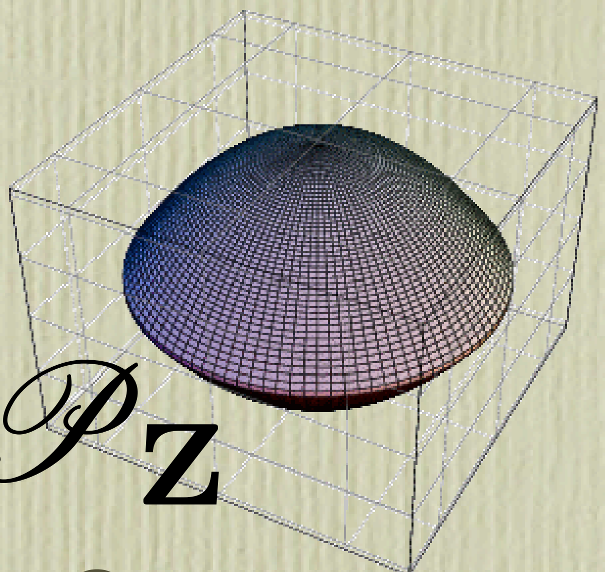
A "measure" of the green region can be given in terms of the **accuracy** ε^{-1} of the programmability

$$\varepsilon \doteq \max_{\mathbf{P}\in\mathscr{P}_N} \min_{\mathbf{Q}\in\mathscr{P}_{\mathbf{Z}}} \delta(\mathbf{P},\mathbf{Q})$$





Approximate programmability



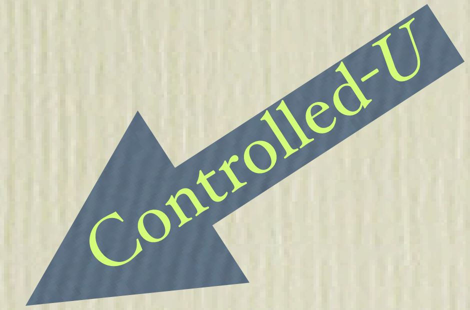
$\mathscr{P}_{\mathbf{Z}}$

programmability with **accuracy** ε^{-1}:

$$\varepsilon \doteq \max_{\mathbf{P}\in\mathscr{P}_N} \min_{\mathbf{Q}\in\mathscr{P}_{\mathbf{Z}}} \delta(\mathbf{P},\mathbf{Q})$$

$$\delta(\mathbf{P},\mathbf{Q}) = \max_{\rho} \sum_i |\,\mathrm{Tr}[\rho(P_i - Q_i)]|$$

Using a joint observable $\mathbf{Z}$ of the form



$$Z_i = U^\dagger(|\psi_i\rangle\langle\psi_i| \otimes I_A)U, \qquad U = \sum_{k=1}^{\dim(\mathcal{A})} W_k \otimes |\phi_k\rangle\langle\phi_k|$$

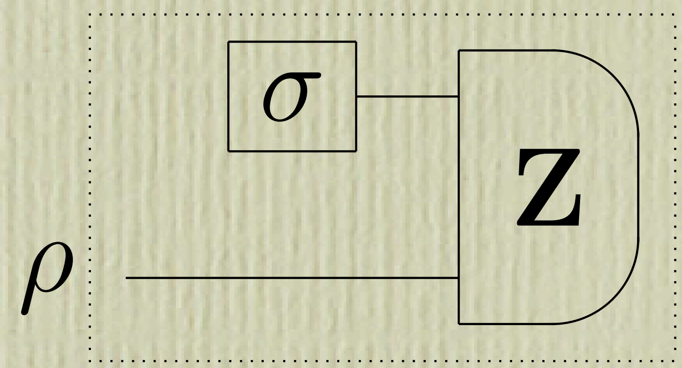
with $\{\psi_i\}$ and $\{\phi_k\}$ orthonormal sets and W_k unitary, we can program observables with accuracy ε^{-1} using an ancilla with **polynomial** growth

$$\dim(\mathcal{A}) \leqslant \kappa(N)\left(\frac{1}{\varepsilon}\right)^{N(N-1)}$$

G. M. D'Ariano, **P. Perinotti**, submitted to Phys. Rev. Lett. (in press: quant-ph0410169)

Approximate programmability

For qubits: *linear* growth!



Program for the observable $\mathbf{P} = \{U_g^{(1/2)}|\pm\tfrac{1}{2}\rangle\langle\pm\tfrac{1}{2}|U_g^{(1/2)\dagger}\}$

$$\sigma = U_g^{(j)}|jj\rangle\langle jj|U_g^{(j)\dagger}$$

in dimension $\dim(\mathcal{A}) = 2j+1$, with joint observable

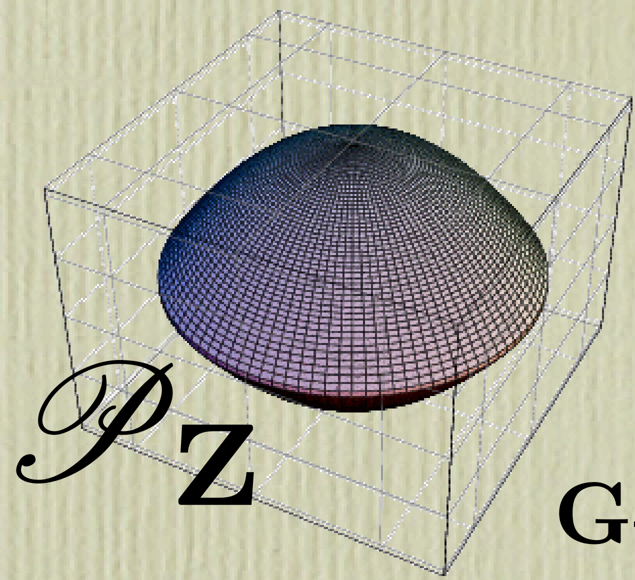
$$\mathbf{Z} = \{\Pi^{(j\pm\frac{1}{2})}\}$$

gives the programmability accuracy

$$\varepsilon = \delta(\mathbf{P},\mathbf{Q}) = \frac{2}{2j+1} \quad \Rightarrow \quad \dim(\mathcal{A}) = 2\varepsilon^{-1}$$

G. M. D'Ariano, **P. Perinotti**, submitted to Phys. Rev. Lett. (in press: quant-ph0410169)

Exact programmability



Covariant measurements are exactly programmable

G. M. D'Ariano, **P. Perinotti**, Phys. Lett. A **329** 188 (2004)

G-covariant POVM densities (Holevo theorem)

$$P_g \,\mathrm{d}\, g = U_g \xi U_g^\dagger \,\mathrm{d}\, g, \qquad g \in \mathbf{G}$$

programmable as

$$P_g = \mathrm{Tr}_2[(I \otimes \sigma) F_g], \qquad \xi = V \sigma^\intercal V^\dagger$$

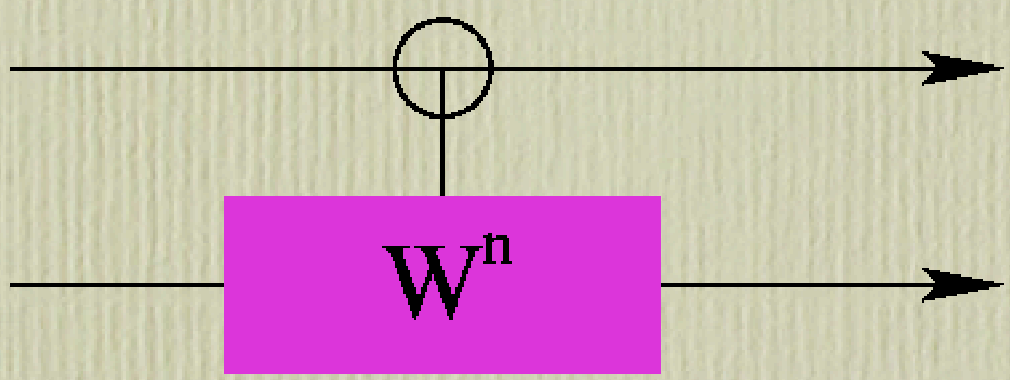
with covariant Bell POVM density

$$F_g = (U_g \otimes I) |V\rangle\!\rangle \langle\!\langle V| (U_g^\dagger \otimes I)$$

Bell from local observables

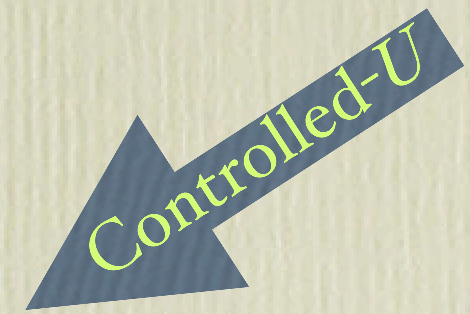
G. M. D'Ariano and P. Perinotti, *On the realization of Bell observables,* Phys. Lett A **329** 188-192 (2004)

Unitary operator U connecting the Bell observable with local observables



$$U(|m\rangle \otimes |n\rangle) = \frac{1}{\sqrt{d}} |U_{m,n}\rangle\!\rangle$$

of the controlled-U form

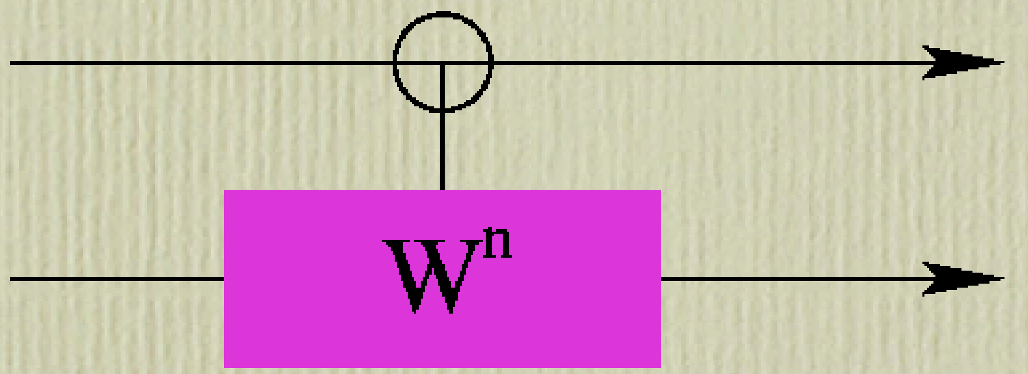


$$U = \sum_n |n\rangle\langle n| \otimes W^n$$

e. g. for projective d-dimensional UIR of the Abelian group $\mathbf{G} = \mathbf{Z}_d \times \mathbf{Z}_d$

$$U_{m,n} = Z^m W^n, \quad Z = \sum_j \omega^j |j\rangle\langle j|, \quad W = \sum_k |k\rangle\langle k \oplus 1|, \quad \omega = e^{\frac{2\pi i}{d}}.$$

Bell from local observables



Unitary operator U connecting the Bell observable with local observables

$$U(|m\rangle \otimes |n\rangle) = \frac{1}{\sqrt{d}} |U_{m,n}\rangle\rangle$$

Problem: *The "Bell-izing U's"*

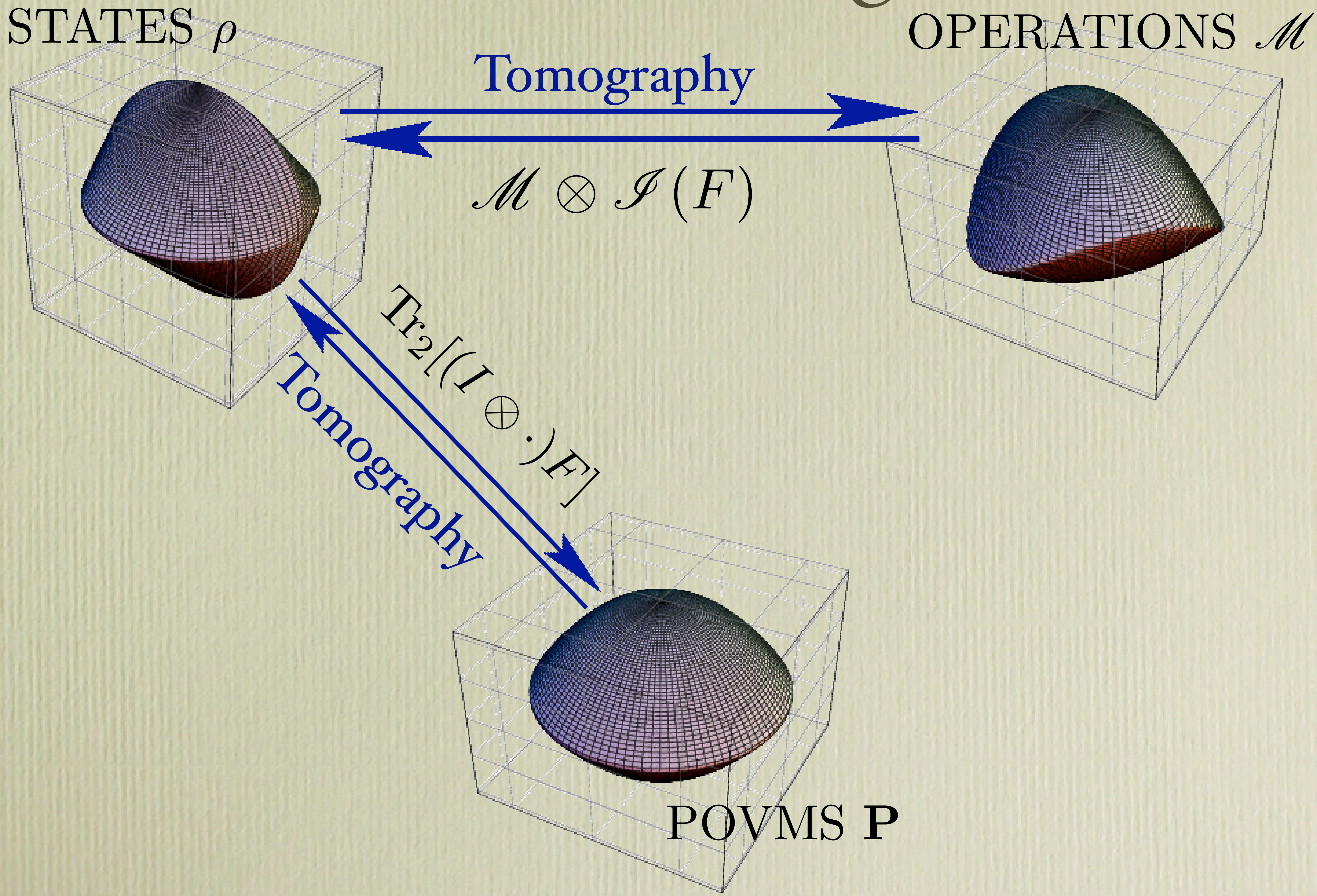
Find the unitary operators U that connect a fixed separable orthonormal basis to any Bell orthonormal basis

Problem: *The "Bell basis classification"*

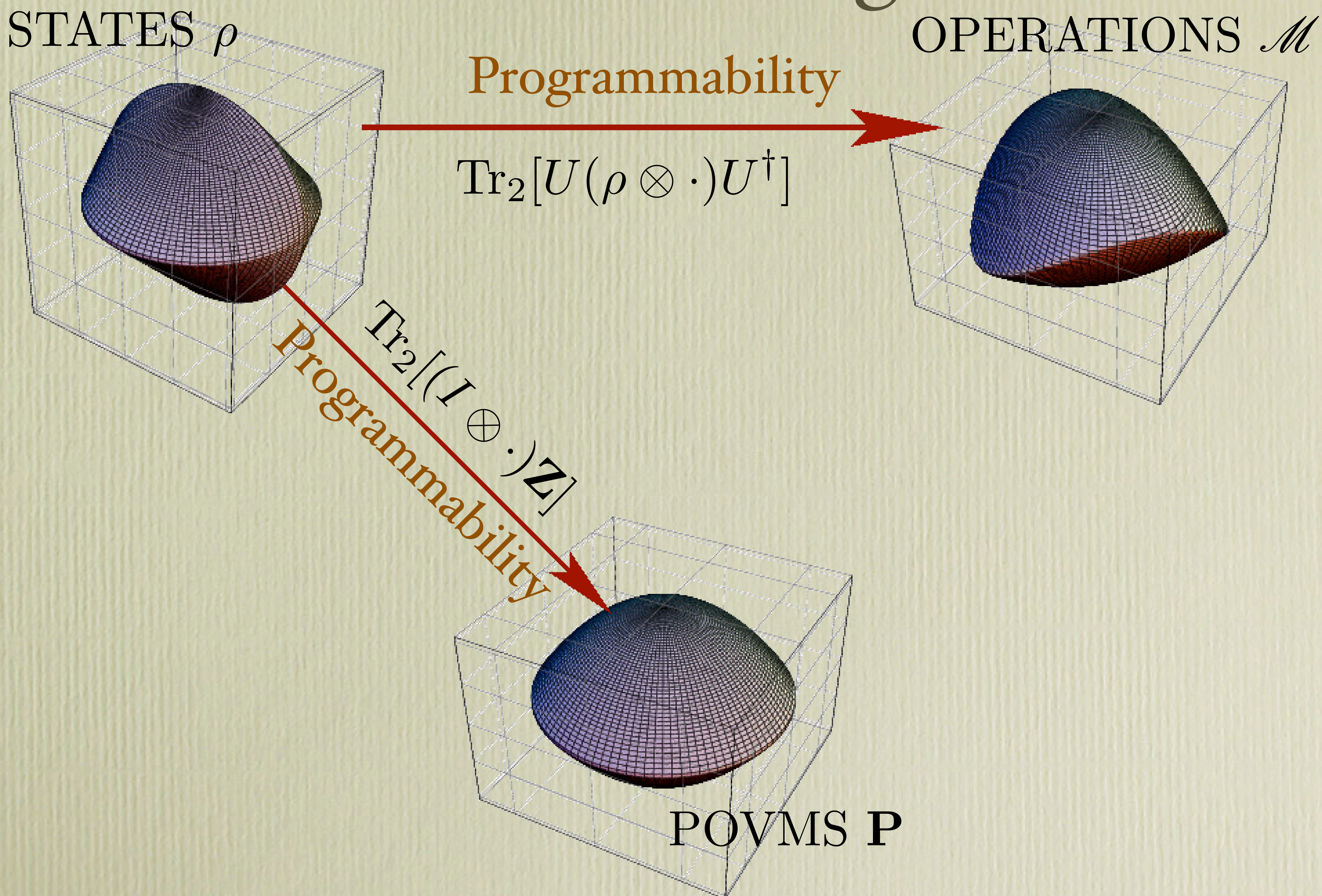
Classify all Bell orthonormal basis.

Equivalently: classify all orthonormal basis of unitary operators.

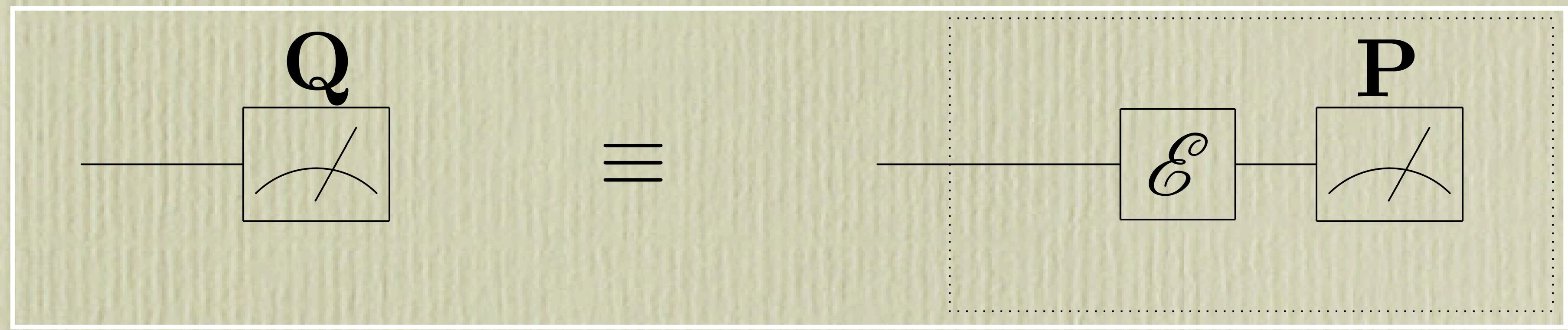
Summarizing



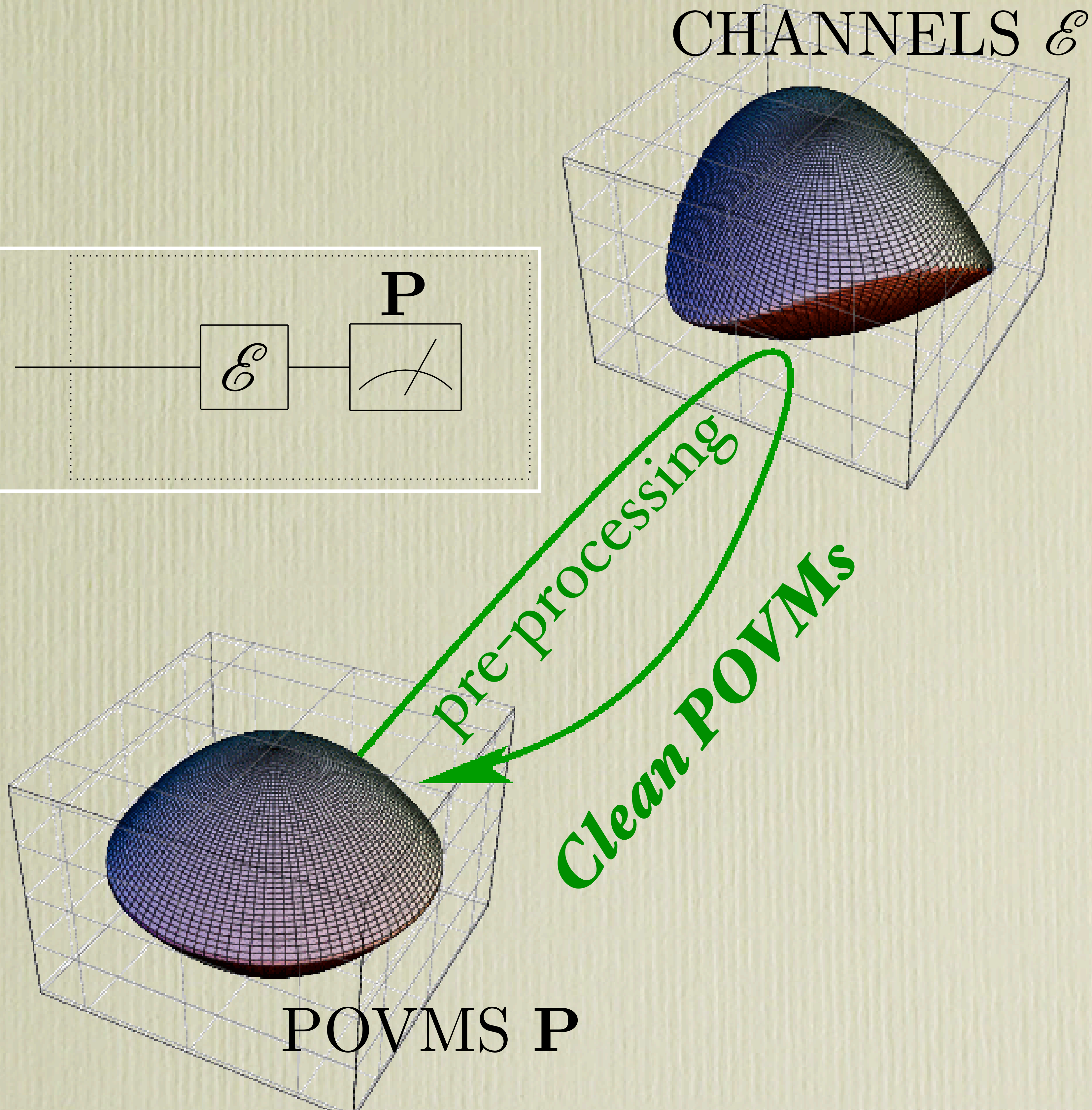
Summarizing



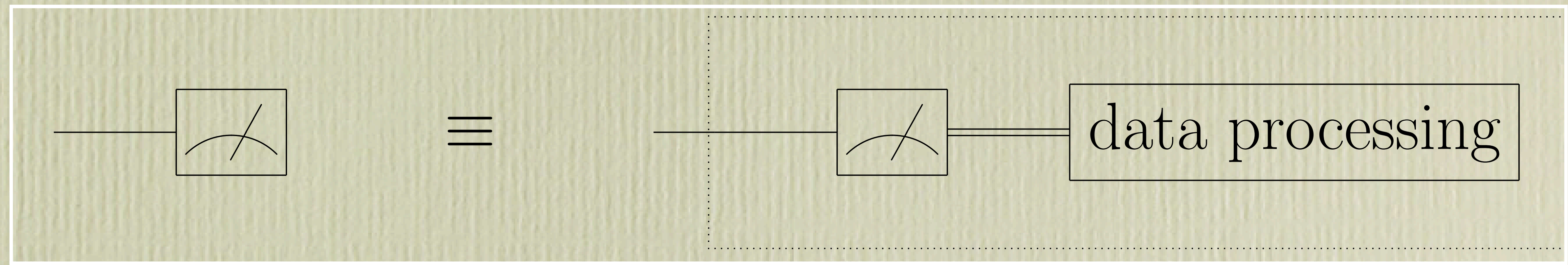
Pre and Post-processing of POVM's

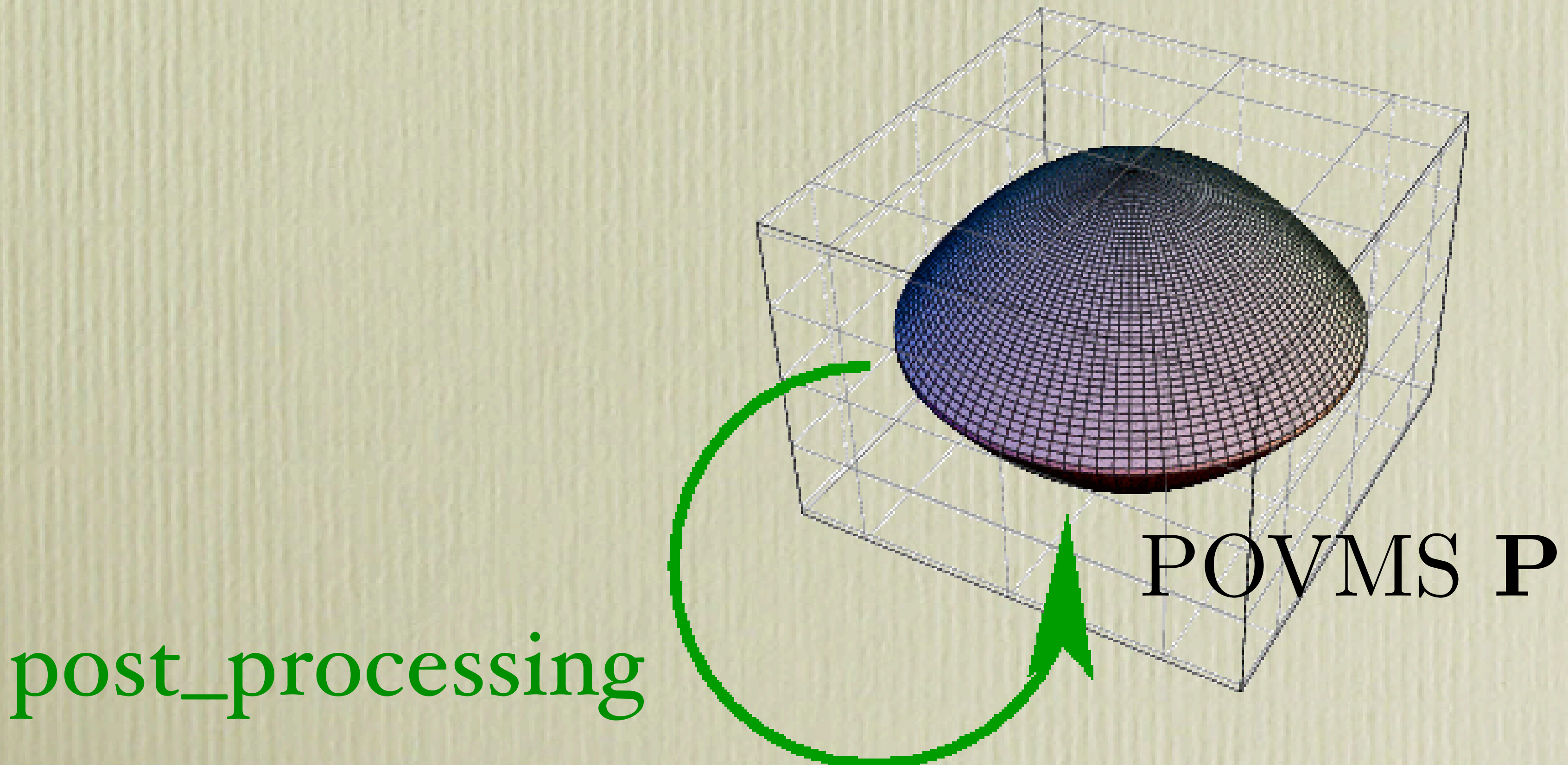


$$\mathbf{Q} = \mathscr{E}(\mathbf{P})$$

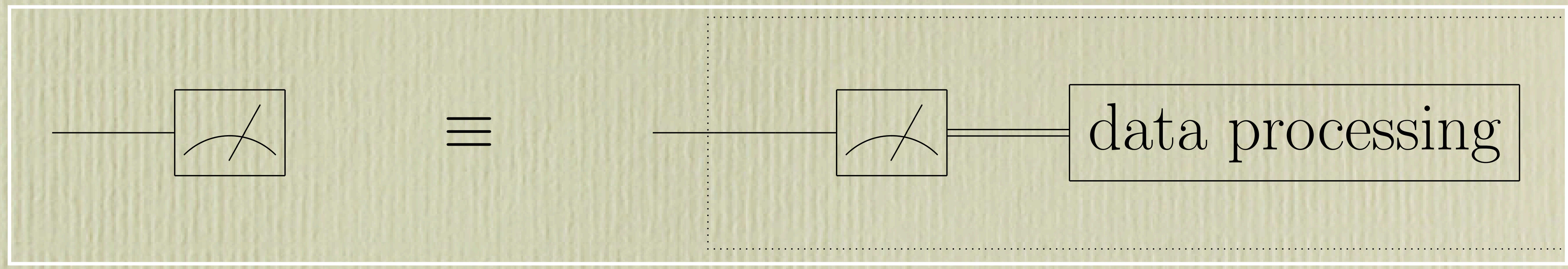


Pre and Post-processing of POVM's

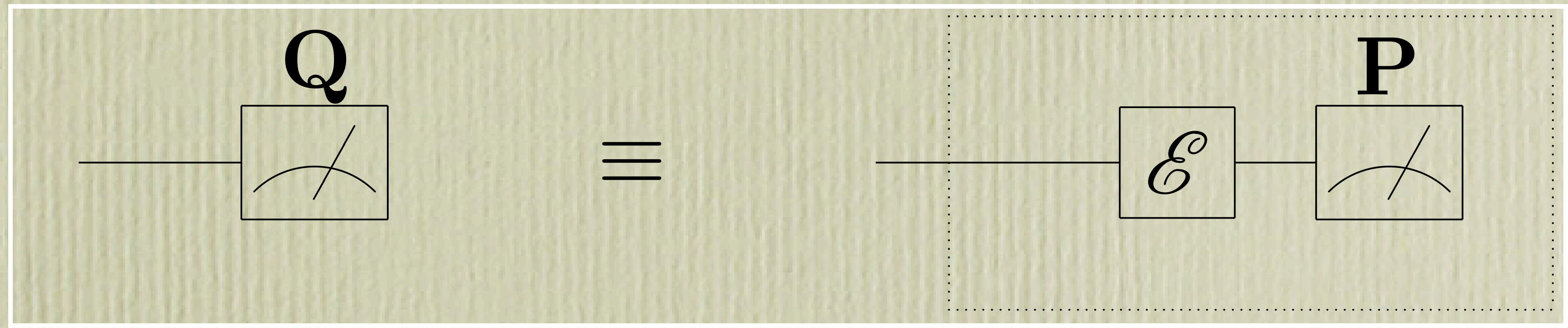




Pre and Post-processing of POVM's

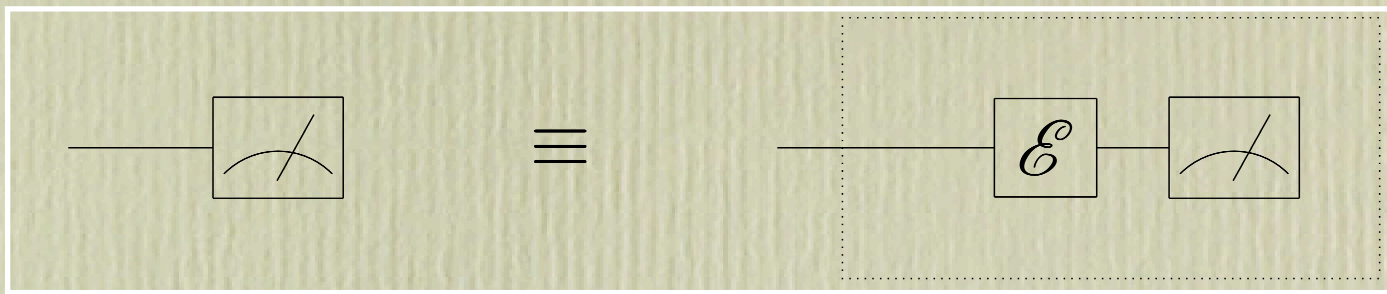


Post-processing is *classical*



Pre-processing is *quantum*

Clean POVM's



F. Buscemi, G. M. D'Ariano, M. Keyl, P. Perinotti, R. Werner (unpublished)

- A quantum channel transforms POVM's into POVM's, generally irreversibly.
- This poses the following problem: *which POVM's are "undisturbed", namely they are not irreversibly connected to another POVM?*
- We will call such POVM's *"clean"*.

Clean POVM's

Pre-ordering: *cleanness*

For two POVM's $\mathbf{P}$ and $\mathbf{Q}$ we define $\mathbf{P} \succ \mathbf{Q}$ iff there exists a channel $\mathscr{E}$ such that

$$\mathbf{Q} = \mathscr{E}(\mathbf{P}),$$

and we will say that the POVM $\mathbf{P}$ is *cleaner* than the POVM $\mathbf{Q}$.

We will say that $\mathbf{P} \simeq \mathbf{Q}$ if both $\mathbf{Q} \succ \mathbf{P}$ and $\mathbf{P} \succ \mathbf{Q}$ hold.

We call a POVM $\mathbf{P}$ "clean" iff for any POVM $\mathbf{Q}$ such that $\mathbf{Q} \succ \mathbf{P}$ one has $\mathbf{P} \simeq \mathbf{Q}$.

Clean POVM's

Two false conjectures

- *Cleanness equivalence* coincides with *unitary equivalence*: false!
- *Cleanness* coincides with *extremality*: false!

Clean POVM's

Main result

Theorem. For $N < d$ outcomes there are no clean POVM's. For $N = d$ the set of clean POVM's coincides with the set of observables.

Other results

- All rank-one POVM's are clean.
- For $d = 2$, $\mathbf{P} \simeq \mathbf{Q}$ iff $\mathbf{P}$ is unitarily equivalent to $\mathbf{Q}$.
- For $\mathbf{A}$ and $\mathbf{B}$ effects, $\mathbf{A} \succ \mathbf{B}$ iff

$$[\lambda_m(A), \lambda_M(A)] \supseteq [\lambda_m(B), \lambda_M(B)].$$

- If the POVM $\mathbf{Q}$ is infocomplete then every $\mathbf{P}$ such that $\mathbf{P} \succeq \mathbf{Q}$ is infocomplete, too.
- For infocomplete POVM's cleanness-equivalence is the same as unitary equivalence.

Clean POVM's

- For $\mathbf{A}$ and $\mathbf{B}$ effects, $\mathbf{A} \succ \mathbf{B}$ iff

$$[\lambda_m(A), \lambda_M(A)] \supseteq [\lambda_m(B), \lambda_M(B)].$$

Cleaness-equivalence is different from unitary-equivalence.

Proof: Consider two effects with different spectrum and the same spectral interval...

Clean POVM's

There are clean POVM's that are not extremal ...

e. g. a rank-one POVM with $N > d^2$

... and extremal POVM's that are not clean

e. g. any extremal POVM with $N < d$ outcomes, such as for $d = 3$, $\mathbf{P} = \{Z_0, Z_1\}$ with

$$Z_0 = |0\rangle\langle 0|, \qquad Z_1 = |1\rangle\langle 1| + |2\rangle\langle 2|.$$

Clean POVM's

Question: What does it mean that there are extremal POVM's that are not clean?

Answer: sometimes we need to give-up some "amount of information" for the "quality of the information".

Maximizing the "information" from the measurement is not necessarily compatible with the achievement of the minimal "cost".

Once the measurement is performed, no classical post-processing can achieve the same result of a quantum pre-processing.

Only for infocomplete measurement we have the same information available for a purpose that is decided *a posteriori*.

Cleanness under post-processing

We can define a *cleanness* for post-processing analogously to pre-processing.

Theorem. A POVM is clean under post-processing iff it is rank-one.

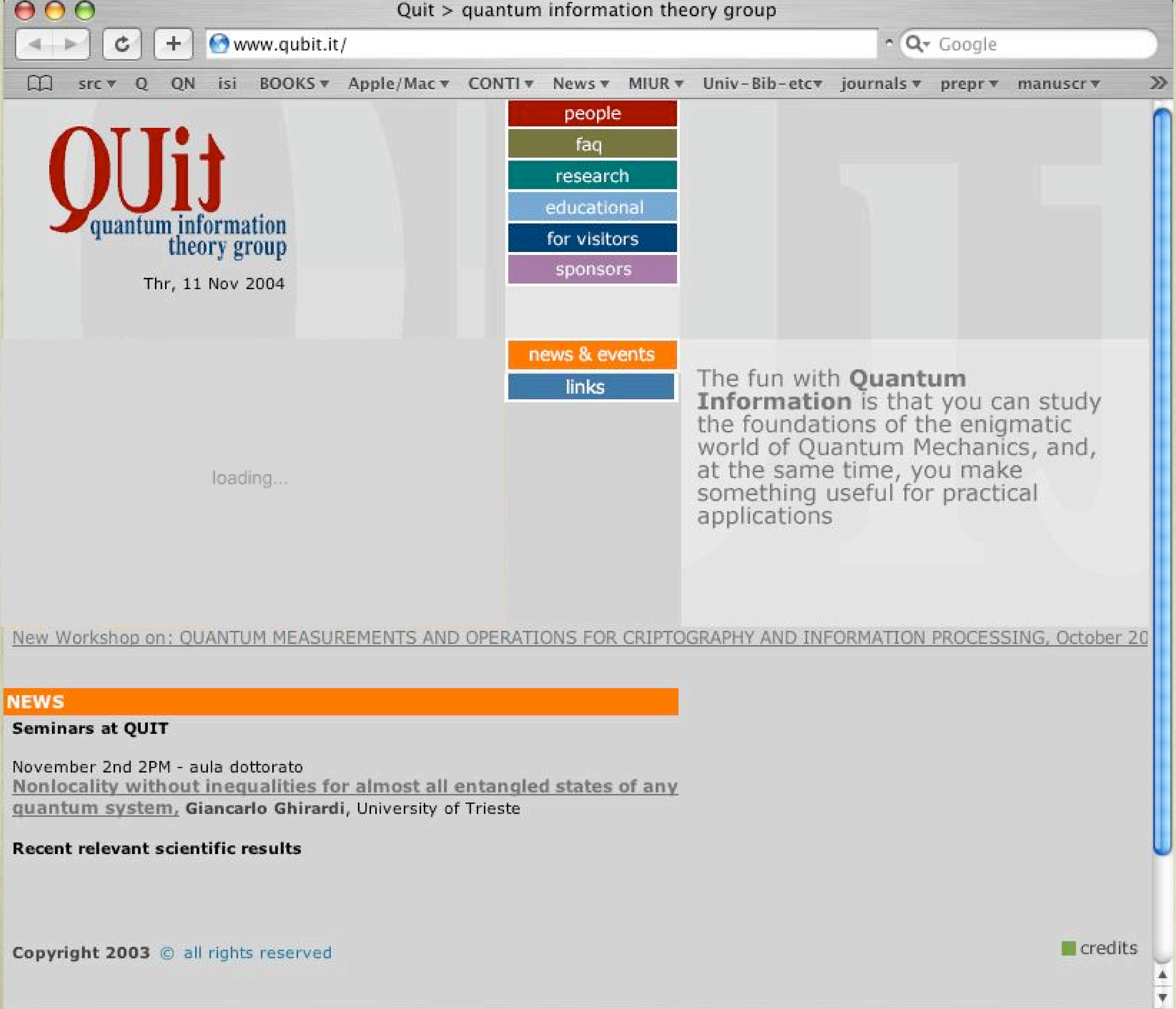
Rank-one POVM's are clean under pre- and post-processing

Both observables and rank-one infocomplete POVM's are:
1) extremal; 2) clean under post-processing;
3) clean under pre-processing

Main open problems

- "The big *U*"
- "The big **Z**"
- The "*Bell-izing*" U
- Classification of Bell POVM's
- Complete classification of clean POVM's and *cleanness* ordering

All problems are unsolved even for d=2!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