QUit quantum information theory group

A new approach to Quantum Estimation: Theory and Applications

Giacomo Mauro D'Ariano

QUIT group, University of Pavia

Quantum Estimation: Theory and Practice

August 25-30, Perimeter Institute, Waterloo, Canada,

QUIT Group in Pavia


Chiara Macchiavello


Lorenzo Maccone


Massimiliano Sacchi


Paolo Perinotti

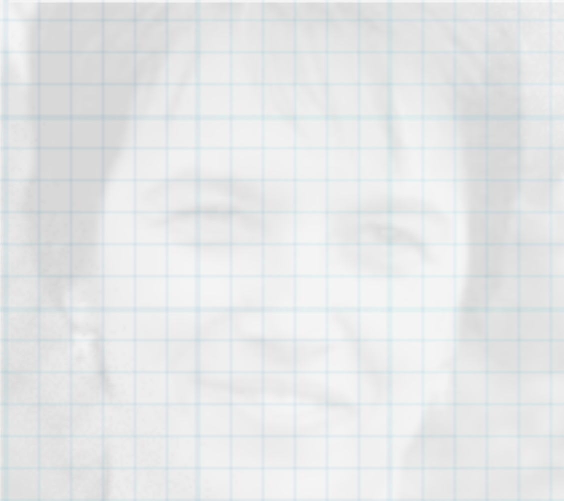

Giulio Chiribella


Daniele Magnani


Stefano Facchini


Alessandro Bisio

Theory of Quantum Combs in collaboration with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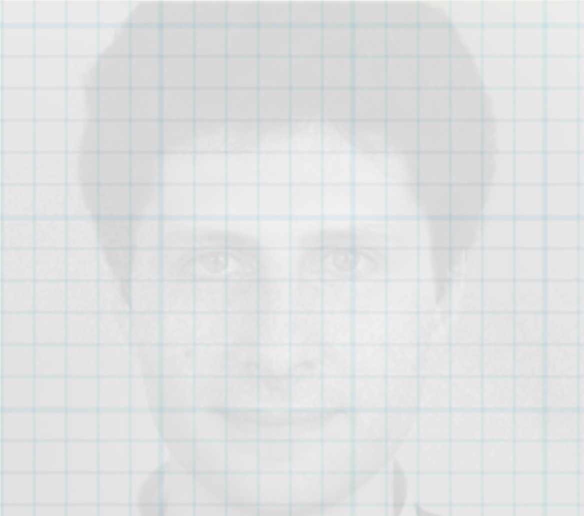
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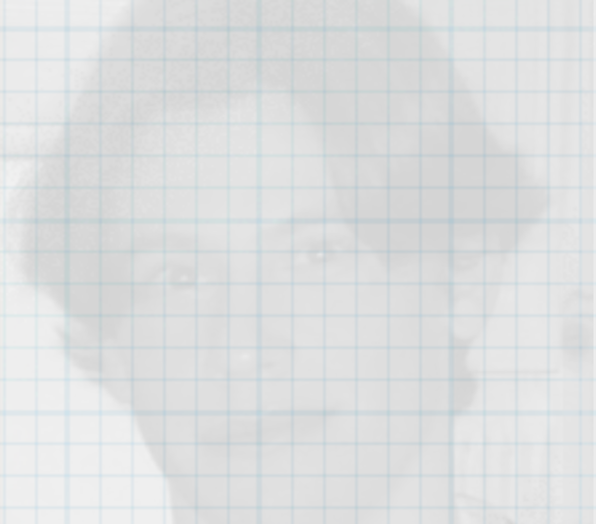
Alessandro Bisio

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Application to Optimal Quantum Tomography in collaboration with

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Giulio Chiribella

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Stefano Facchini

Alessandro Bisio

Outline

- New Quantum Estimation Theory, with multiple copies, and optimization of the setup
  - Convex optimization method based on the new notions of quantum comb and quantum tester
- Applications:
  - Optimal discrimination of unitary operators and quantum memory channels
  - Optimal process tomography
  - Cloning of processes, quantum learning, quantum strategies and algorithms, ...

Helstrom

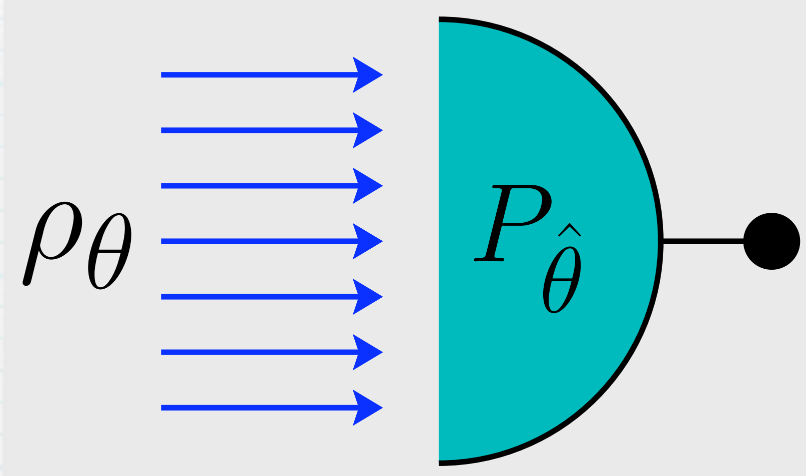
Quantum Estimation Theory

Quantum state ρ_θ parameterized by θ

Problem: estimate θ optimally according to the cost function $C(\theta, \hat{\theta})$

Mathematical formulation:

find the optimal POVM $P_{\hat{\theta}}$ minimizing the cost



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Helstrom

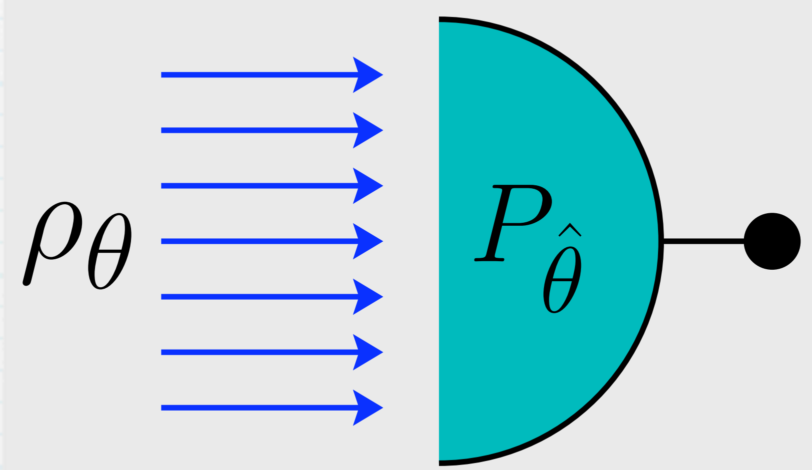
Quantum Estimation Theory

Practically interesting situation
(e.g. for the phase of an e.m. mode):

$$\theta \Longrightarrow \rho_\theta = U_\theta \rho U_\theta^\dagger$$

Then you want also to optimize ρ

Subtle issue: the optimal POVM for estimating θ depends on ρ



Interesting situation: the parameter to be estimated is encoded on a transformation---not on the state!

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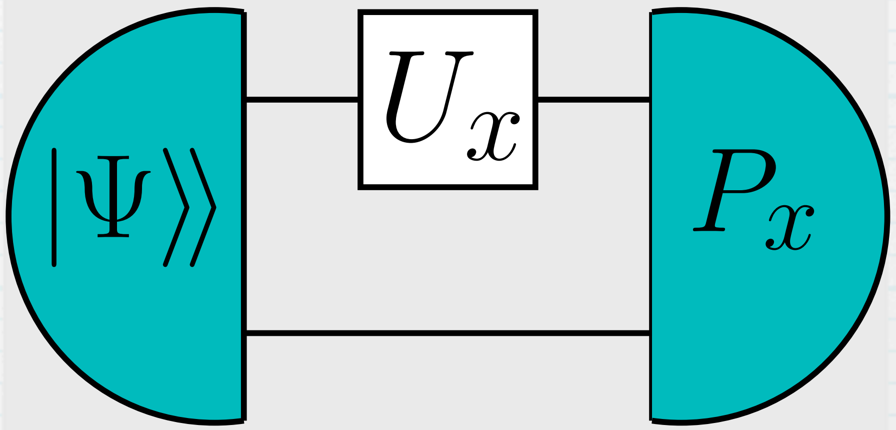
Quantum Estimation Theory

Problem: estimate x parameterizing the (unitary) transformation U_x optimally according to the cost function

Lesson that we learned from entanglement:

Find the optimal entangled state $|\Psi\rangle\rangle$ (with an any possible ancilla) along with the optimal joint POVM P_x

With the phase we were lucky!



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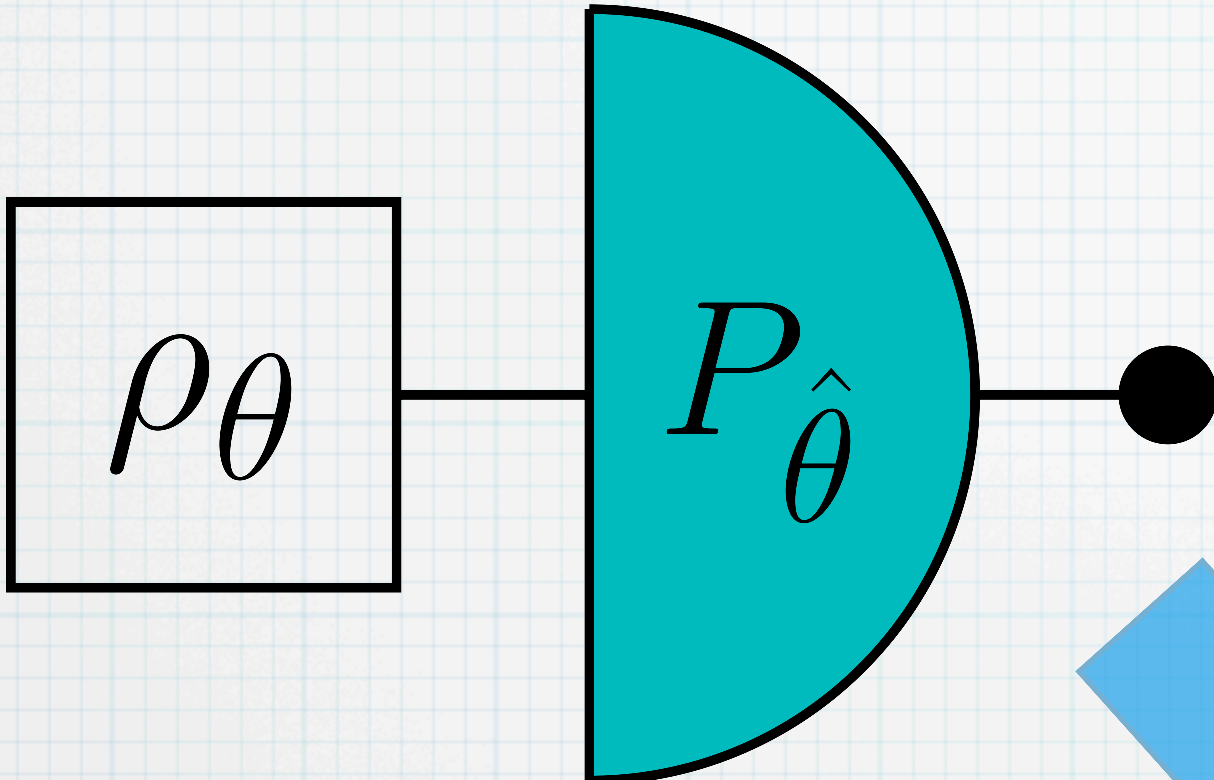
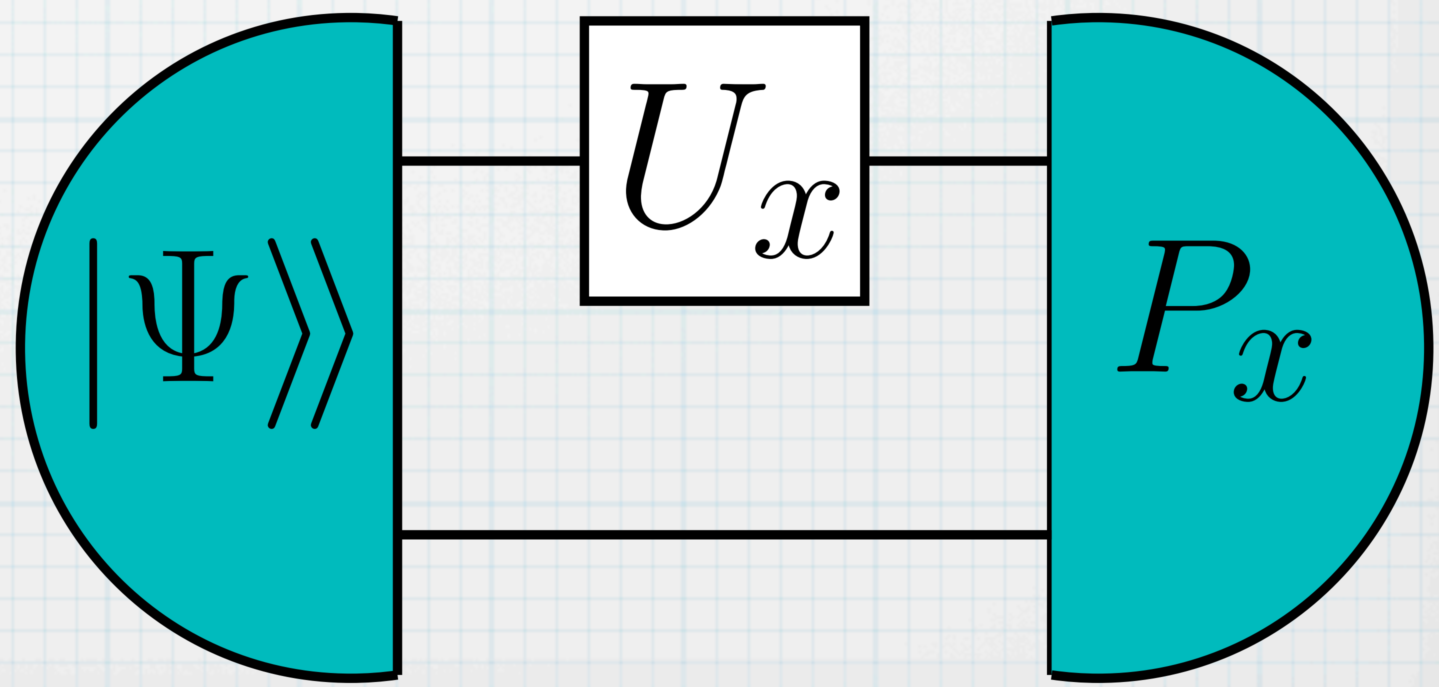
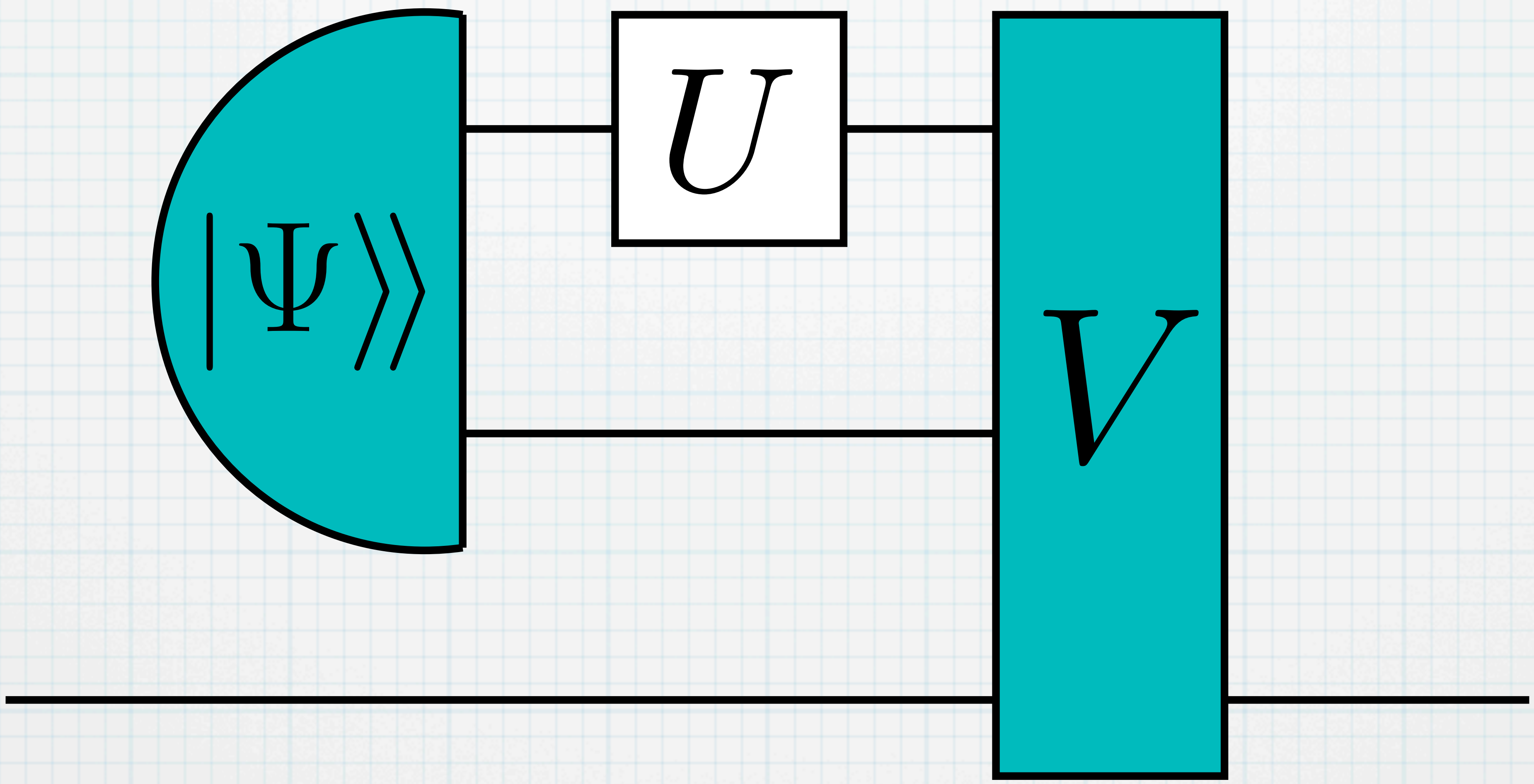
Quantum Estimation Theory



New scheme



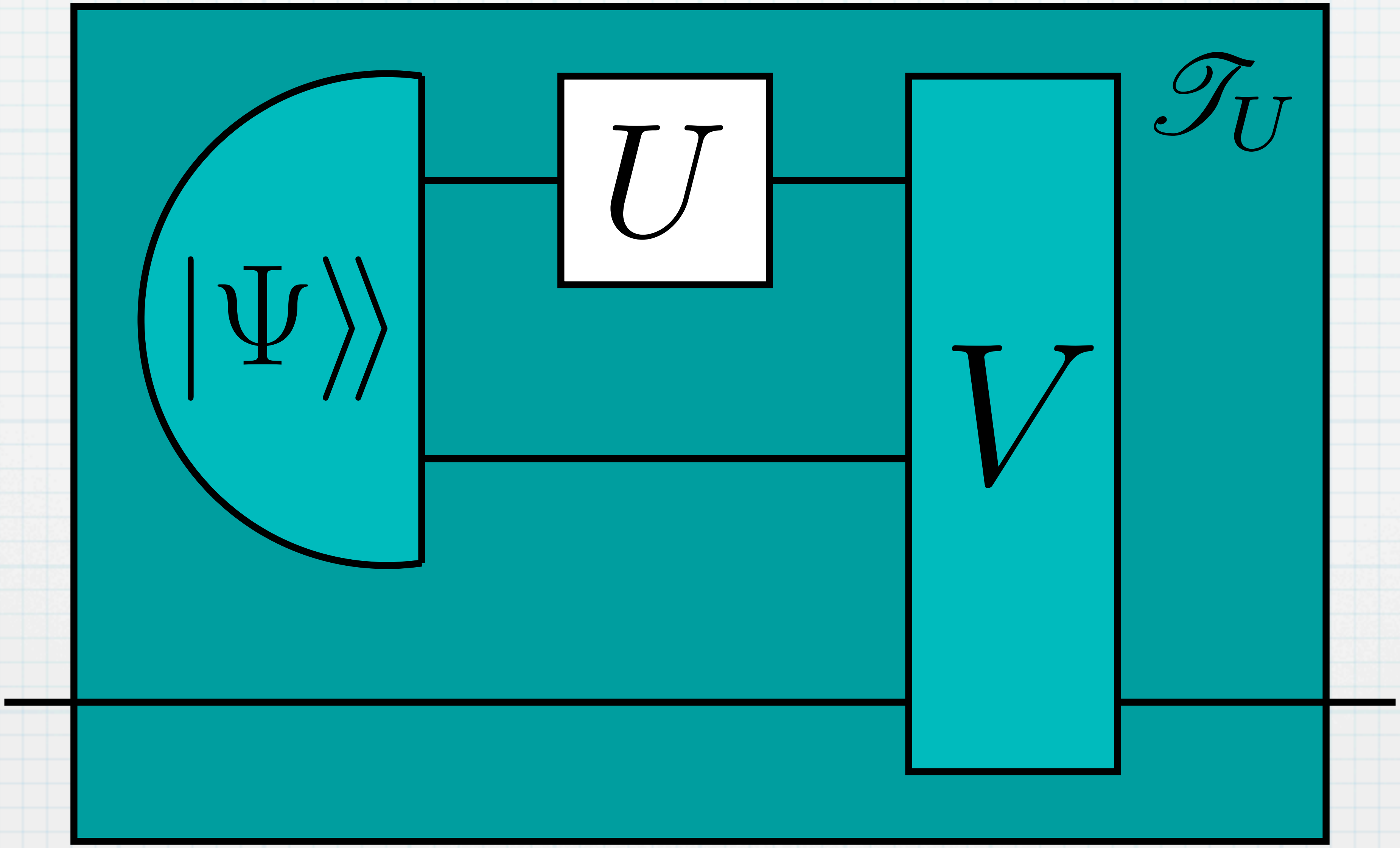
Quantum Feedback



- quantum feedback: perform a transformation $\mathscr{T}_U$ on a system which depends on an unknown unitary transformation U occurring on a (generally different) system (e.g. reference-frames realignment).

quantum information theory group

Quantum Feedback



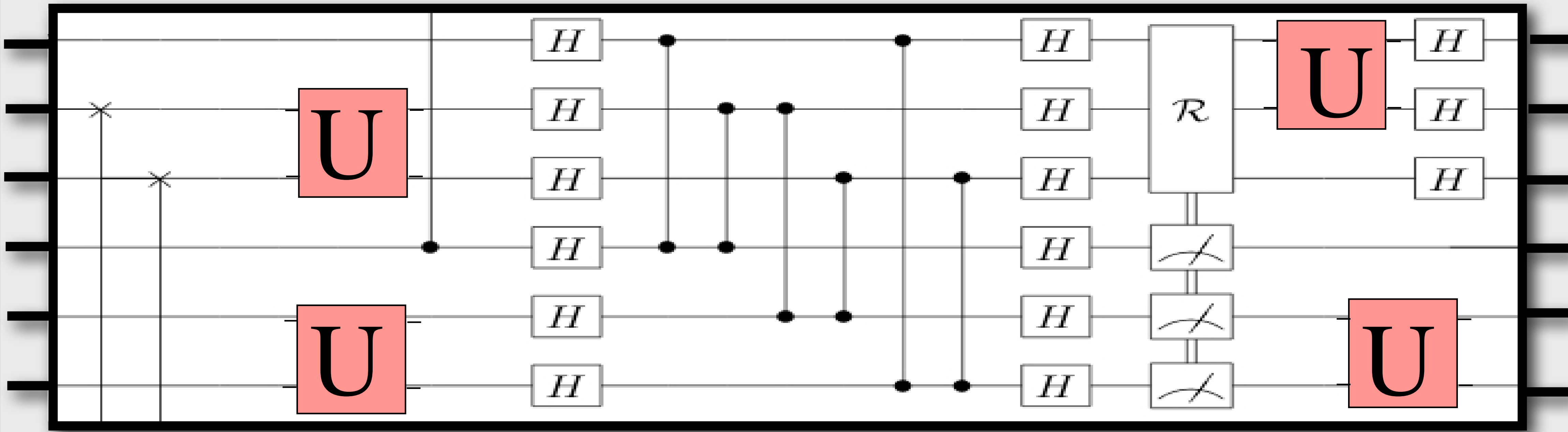
- quantum feedback: perform a transformation $\mathscr{T}_U$ on a system which depends on an unknown unitary transformation U occurring on a (generally different) system (e.g. reference-frames realignment).

Multiple copies

- For parameter estimation: repeat the estimation N times, gaining a precision factor $\sqrt{N}$
- However, you better use a coherent strategy, in which you perform a joint POVM
- and you want to do the same for the quantum feedback

What is the best that you can do?

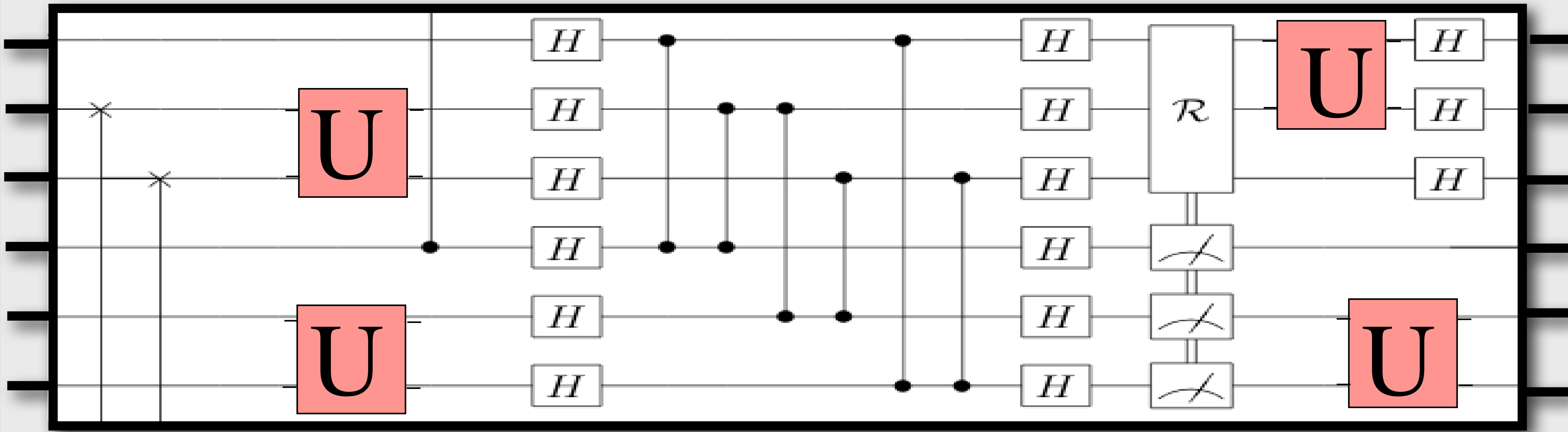
Use a Quantum Board!



General scheme: put the copies of the unknown unitary in a suitable quantum circuit which performs the desired transformation/estimation.

Quantum circuit board: input and output are themselves circuits that are slotted into the board.

Use a Quantum Board

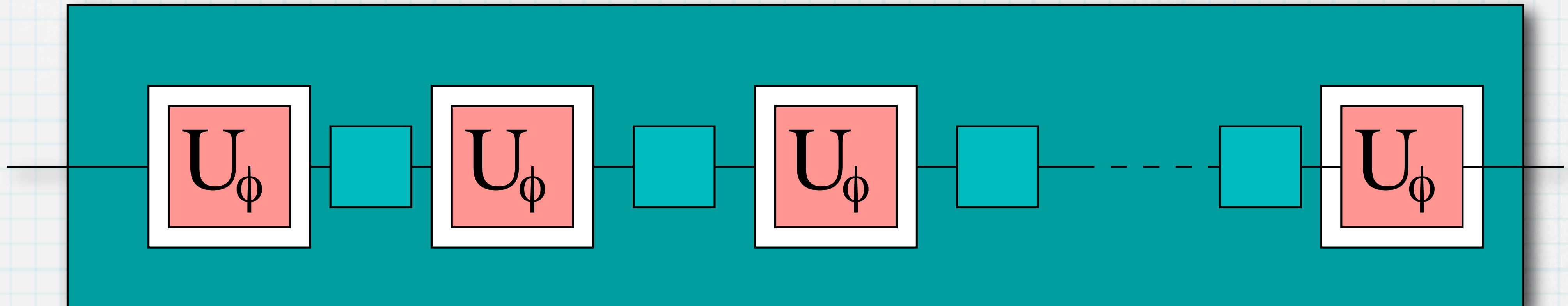


Formulation of the problem:

Optimize the quantum circuit board for all possible dispositions of the slots

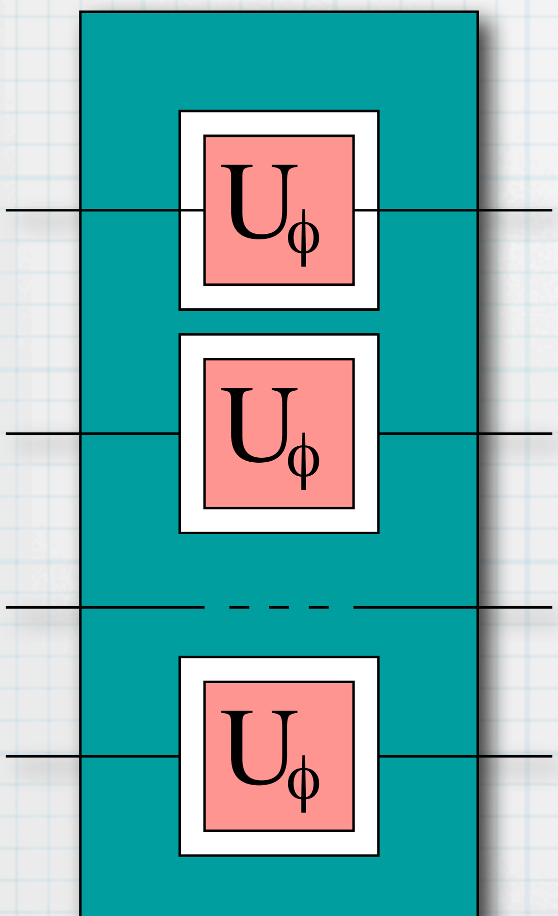
It looks a difficult problem ...

For example: what is the optimal board for phase estimation?



In sequence intercalated by some unitary?

For unitary discrimination:[Duan, Feng, Ying, PRL 98, 100503 (2007)]



In parallel over a joint entangled state?

For unitary discrimination: G.M.D'Ariano, P. Lo Presti, M. Paris, PRL 87, 270404 (2001); A. Acín, E. Jané, and G. Vidal, Phys. Rev. A 64, 050302 (2001)

Asymptotically: same sensitivity [Giovannetti, LLoyd, Maccone, PRL 96, 010401 (2006)]

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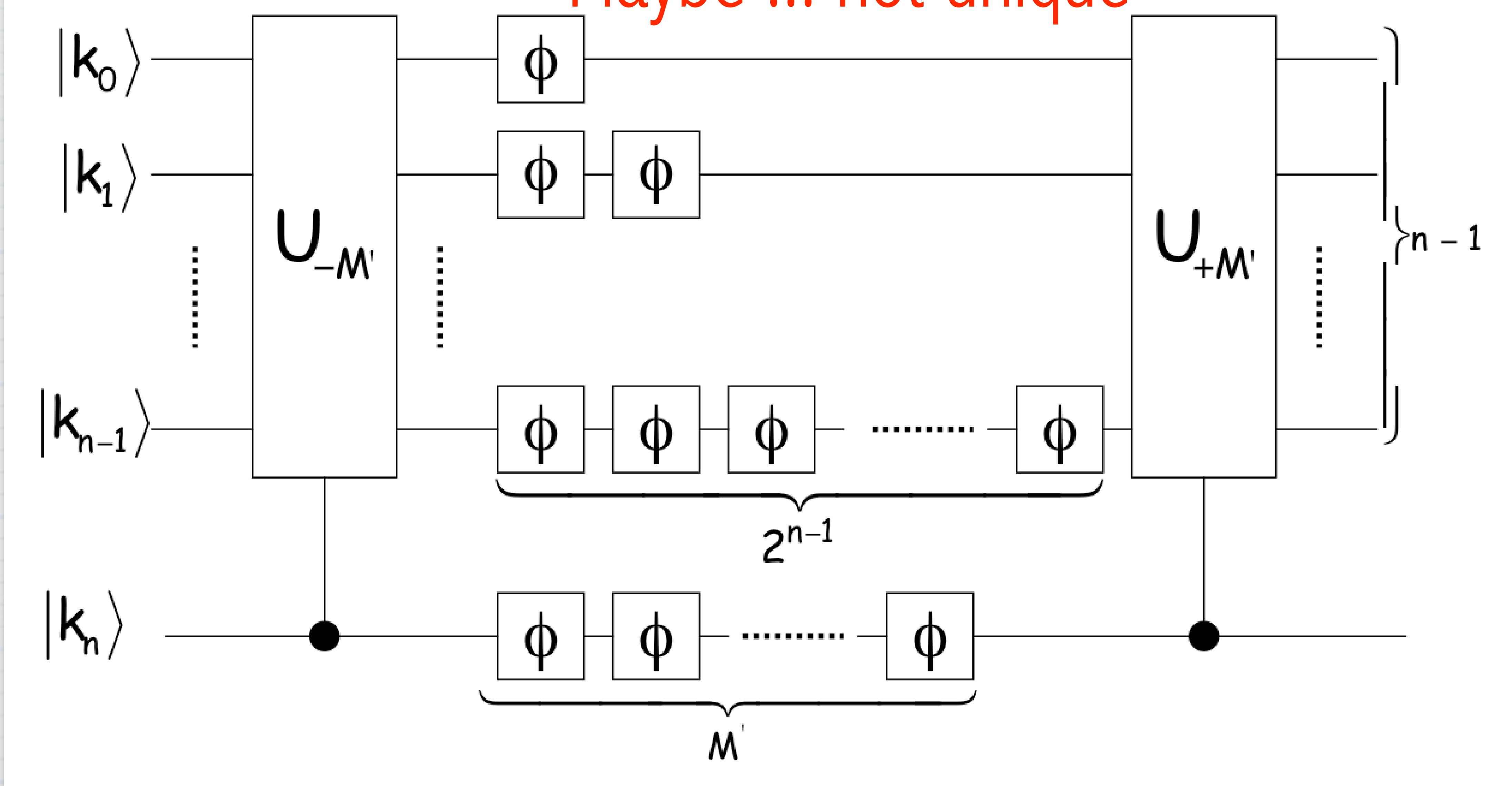
For example: what is the optimal board for phase estimation?

An optimal board architecture [van Dam, D'Ariano, Ekert, Macchiavello, Mosca, PRL 98, 090501 (2007)]

Maybe ... not unique

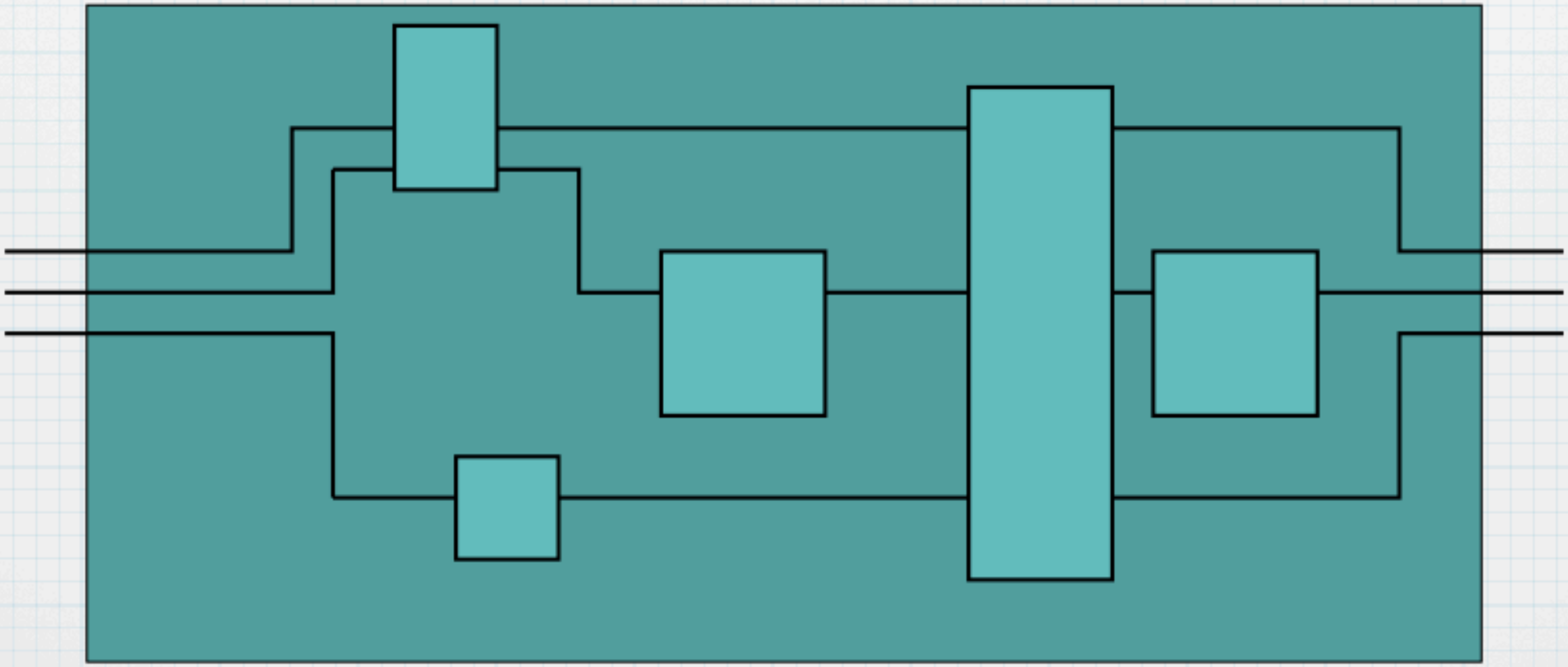


What is the mathematical formulation of the problem?

Quantum Channel

It can be regarded as an equivalence class of quantum circuits performing the same input-output transformation ...

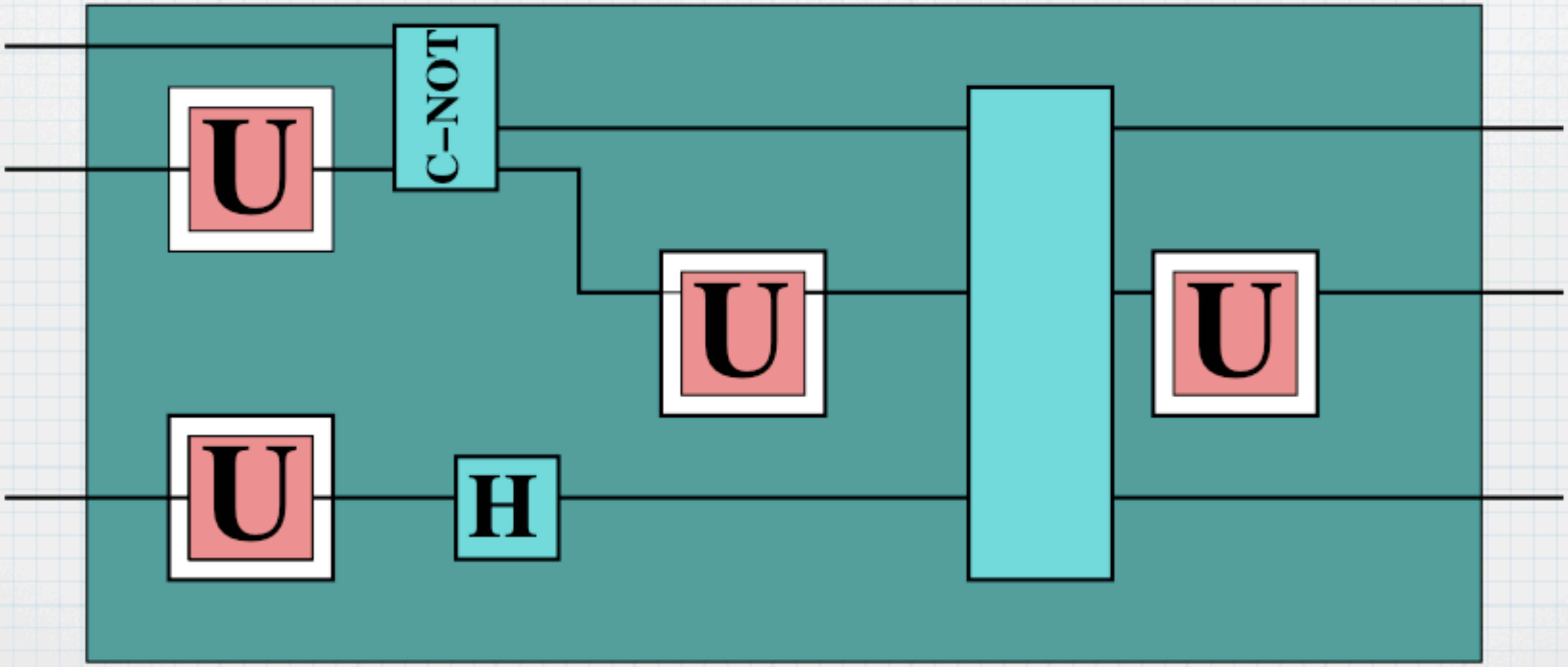
For a channel the input and the output are states



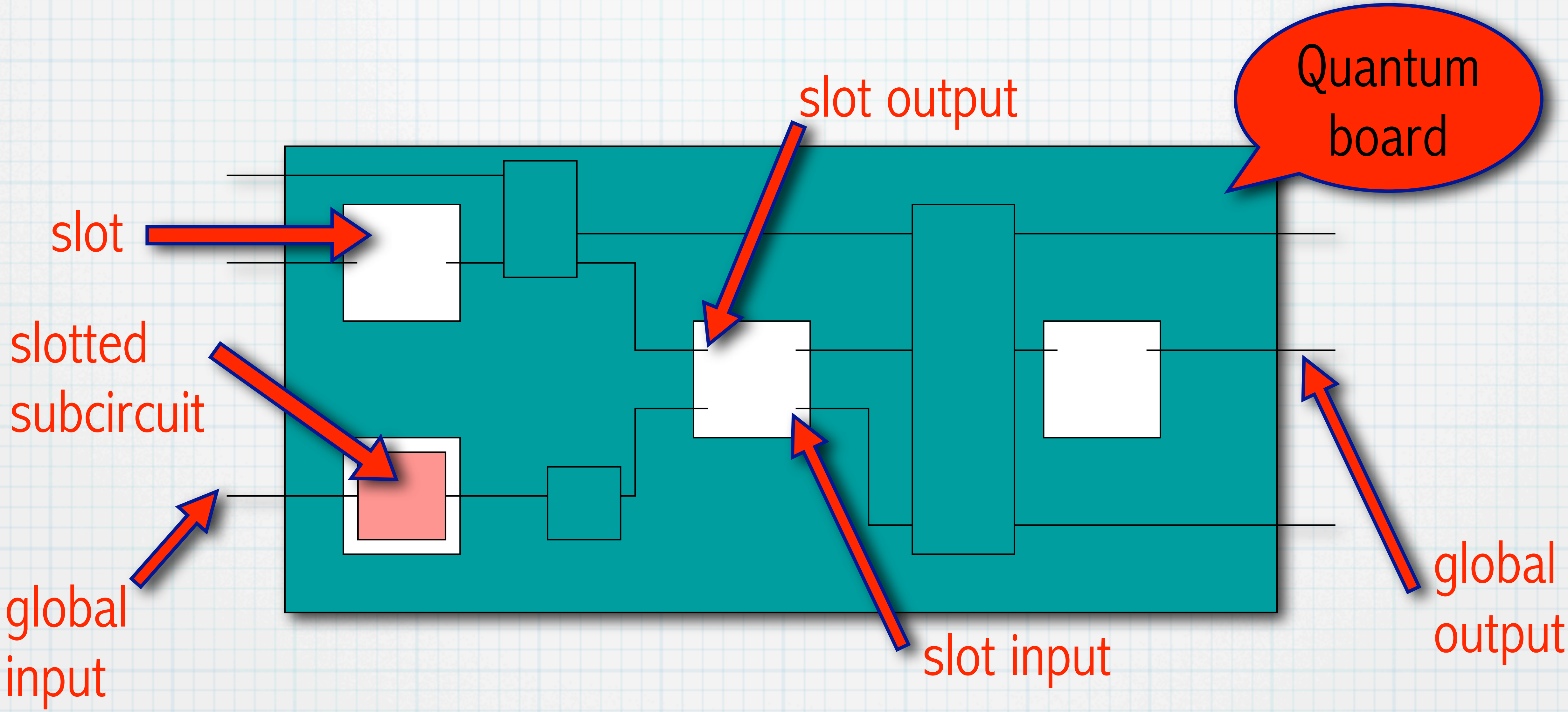
Quantum Board

Equivalence class of quantum circuits boards performing the same overall input-output transformation ...

But now, the input and the output are transformations



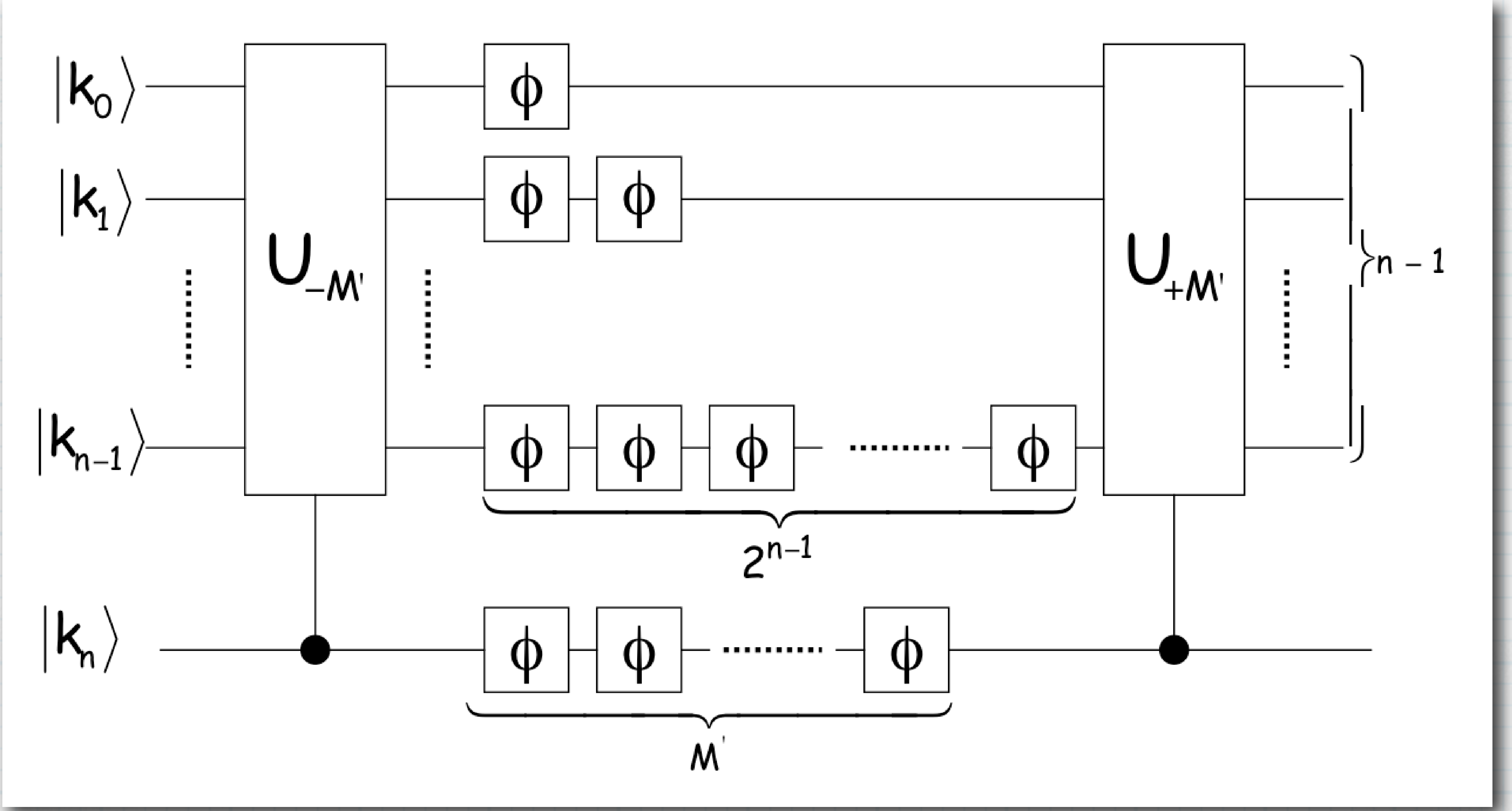
Quantum Board



Problem: what is the optimal board for given slots achieving a global input/output transformation optimally according to a given cost function?

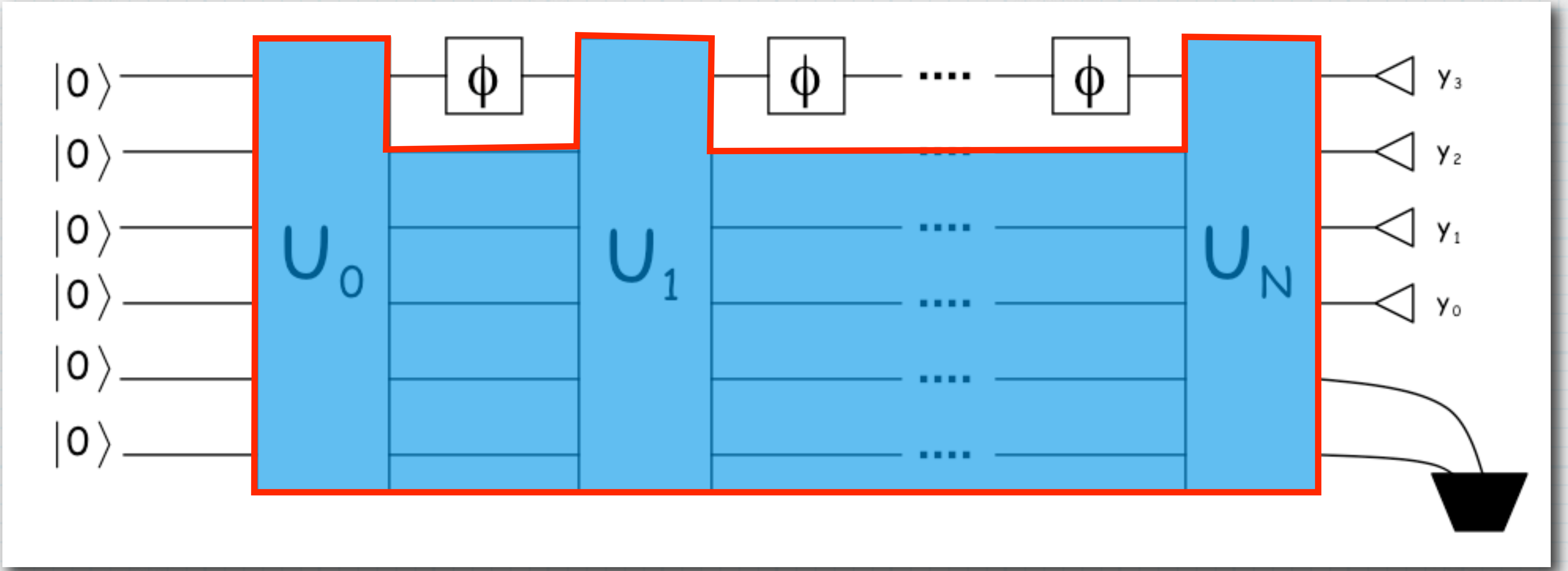
Quantum Board

[van Dam, D'Ariano, Ekert, Macchiavello, Mosca, PRL 98, 090501 (2007)]



Quantum Board

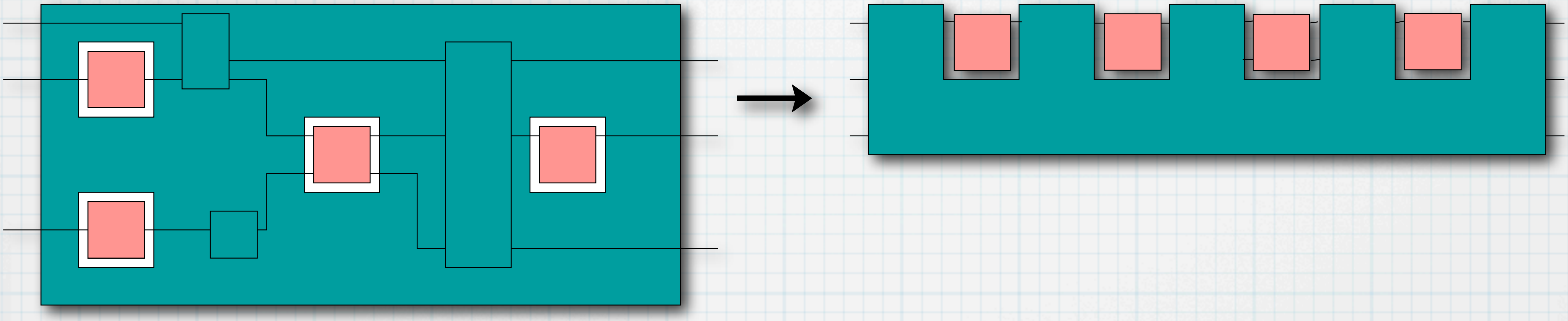
[van Dam, D’Ariano, Ekert, Macchiavello, Mosca, PRL 98, 090501 (2007)]



Quantum Combs

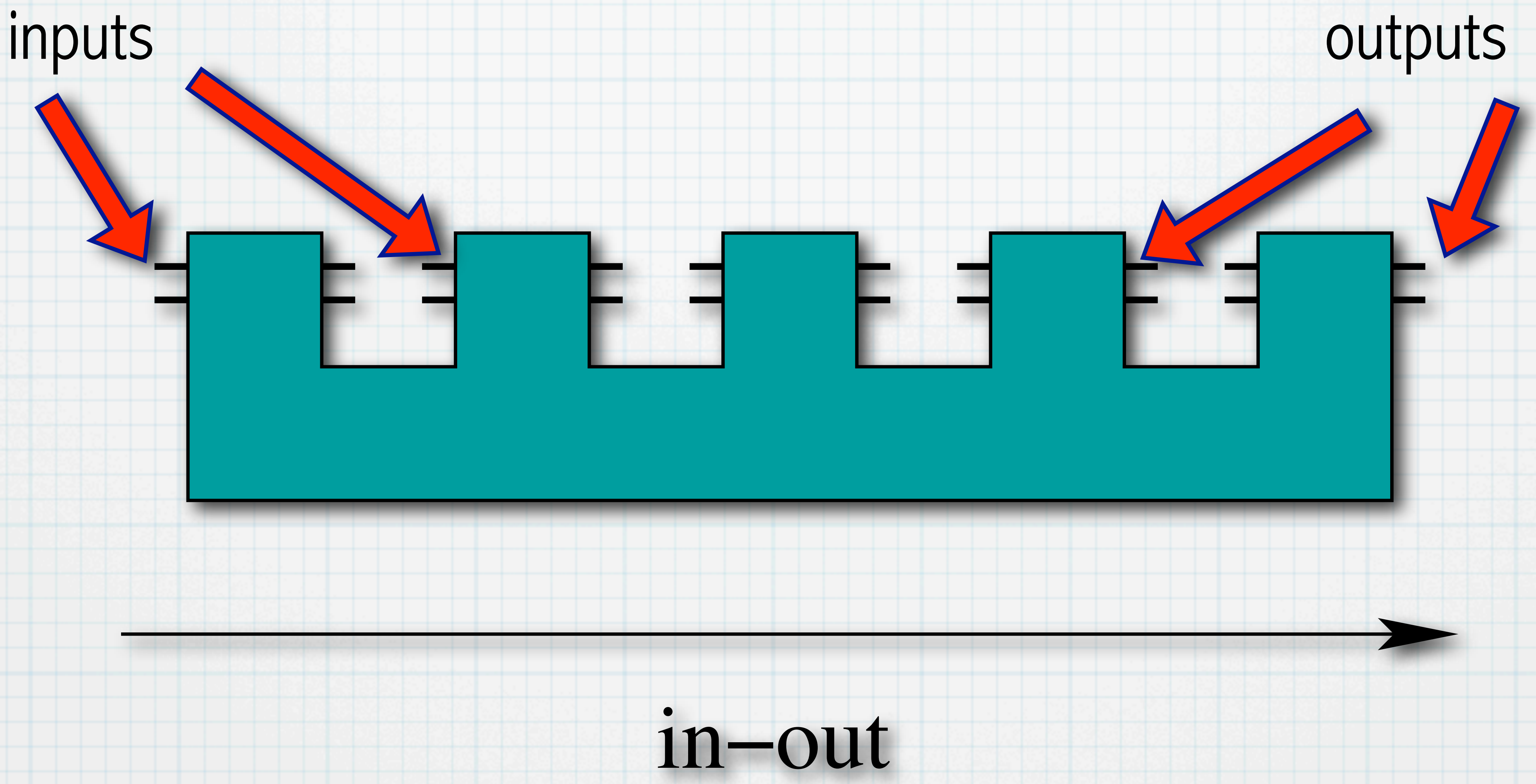
G.Chiribella, G.M.D'Ariano, P.Perinotti, PRL 101 060401 (2008)

All circuits-boards can be reshaped in form of "combs", with an ordered sequence of slots, each between two successive teeth



Quantum Combs

G.Chiribella, G.M.D'Ariano, P.Perinotti, PRL 101 060401 (2008)



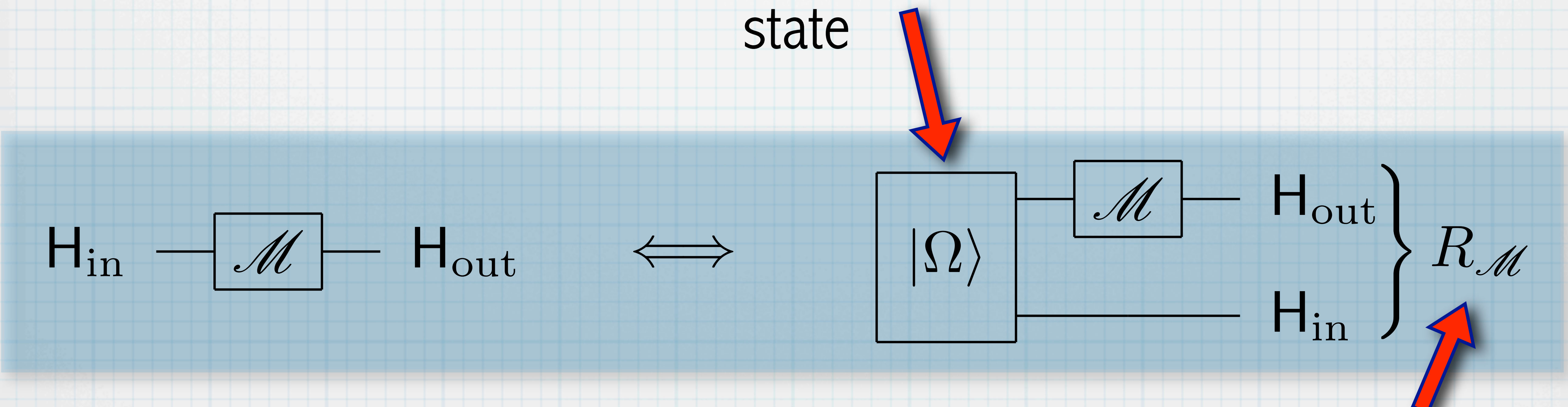
Pins = quantum systems with generally variable dimensions

How do we describe a quantum comb mathematically?

Channel: Choi representation

Mathematically the input-output transformation operated by a quantum circuit is a CP map, and is in one-to-one correspondence with a positive operator called "Choi-Jamiolkowski operator", which is nothing but the output state of the map applied locally to a maximally entangled state.

Max-entangled state

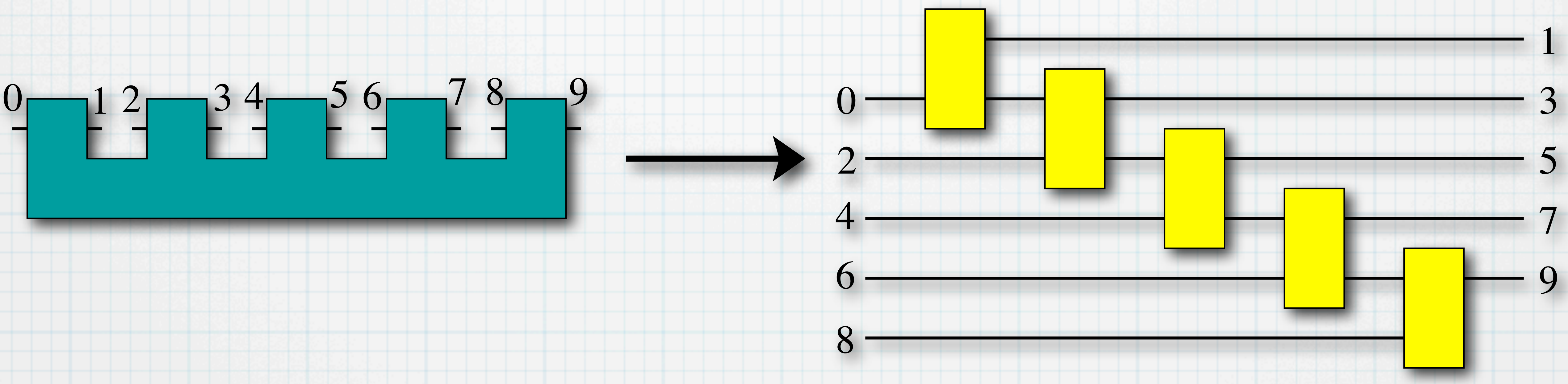


Choi-Jamiolkowski operator

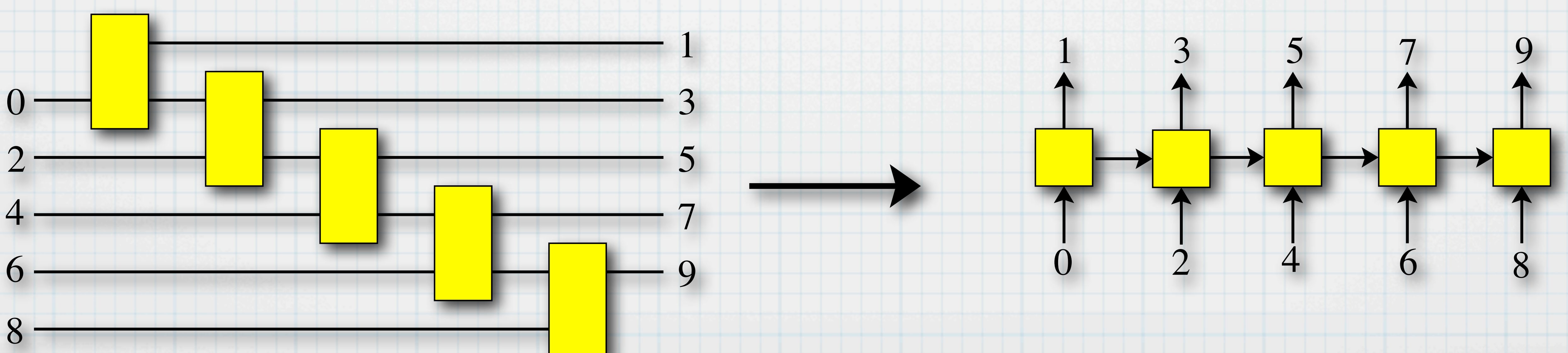
PRL 101 060401 (2008)

Causal networks

The quantum comb is equivalent to a causal network with all inputs on the left and all outputs on the right



The causal network is also equivalent to the stack of memory channels



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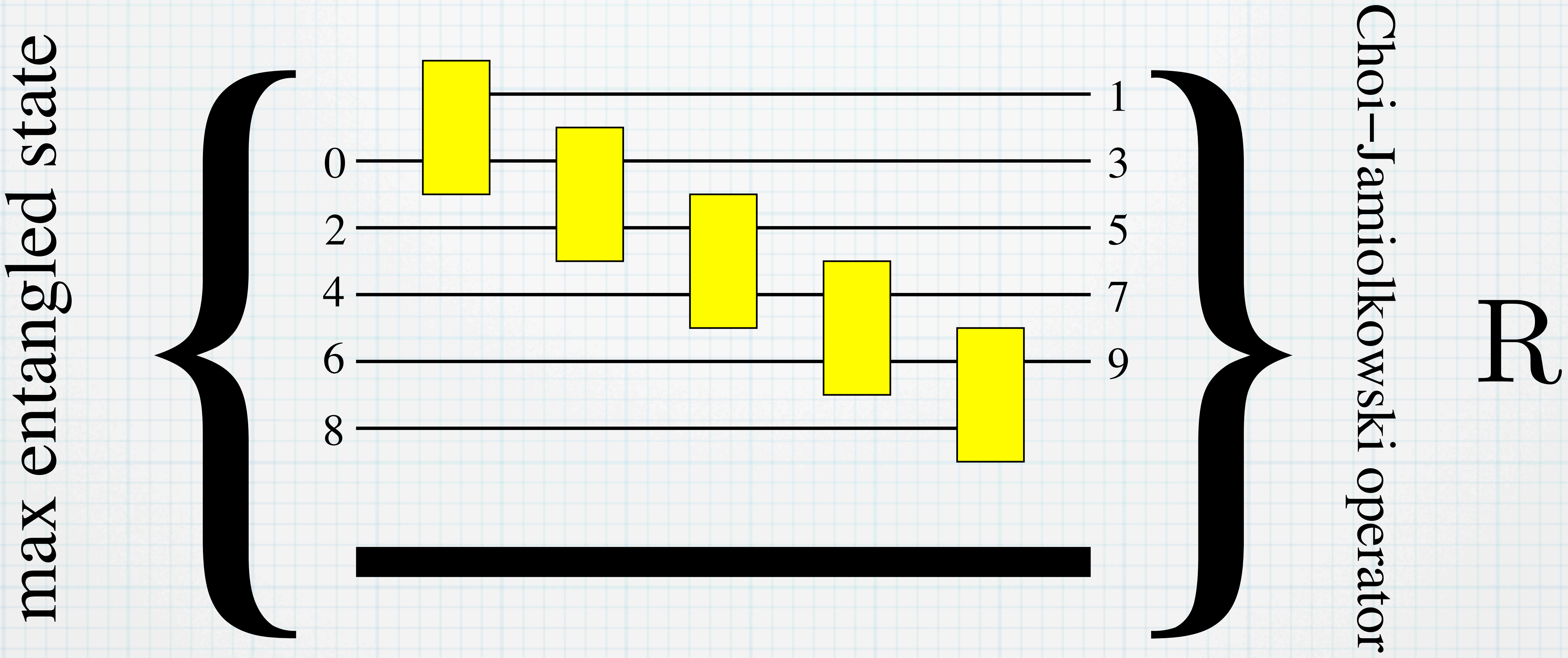
Choi representation



Causality constraints: (N+1 inputs/outputs)

$$\mathrm{Tr}_{2n+1}\left[R^{(n)}\right] = I_{2n} \otimes R^{(n-1)}, \quad n = 0, 1, N,$$

$$R^{(N)} \equiv R, \; R^{(-1)} = 1$$

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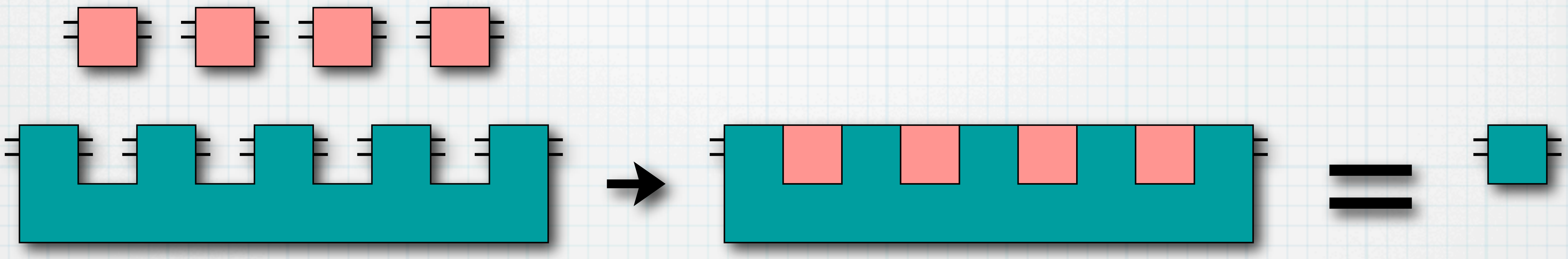
Supermaps

A quantum comb performs a transformation that is a generalization of the quantum operation: the so called "supermap"



A supermap sends a series of N channels to one channel.

Mathematically it is represented by a CP N-linear map which sends N Choi operators to one Choi operator, and with his own Choi operator satisfying the causality constraints.

(we can likewise consider probabilistic supermaps).

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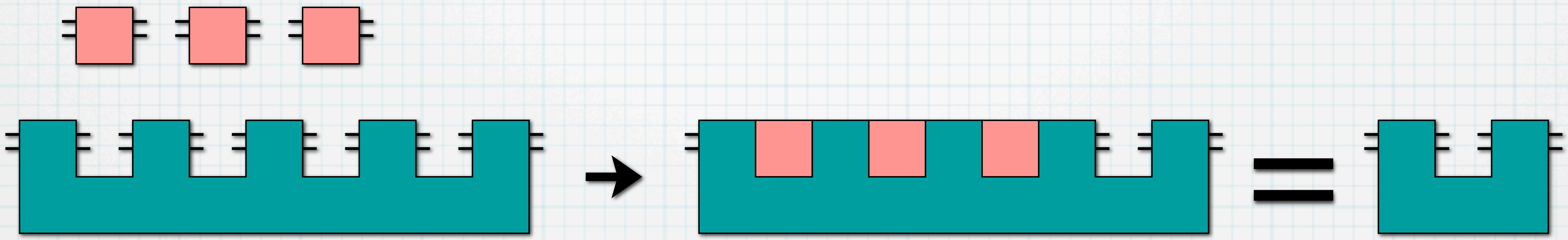
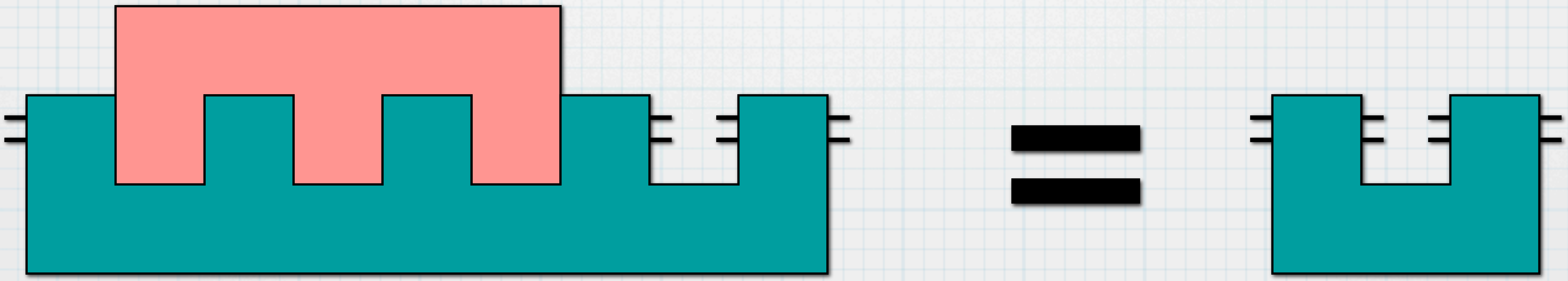
Supermaps

More generally, a quantum comb maps a series of channels into a comb



or, even more generally, a comb to a comb



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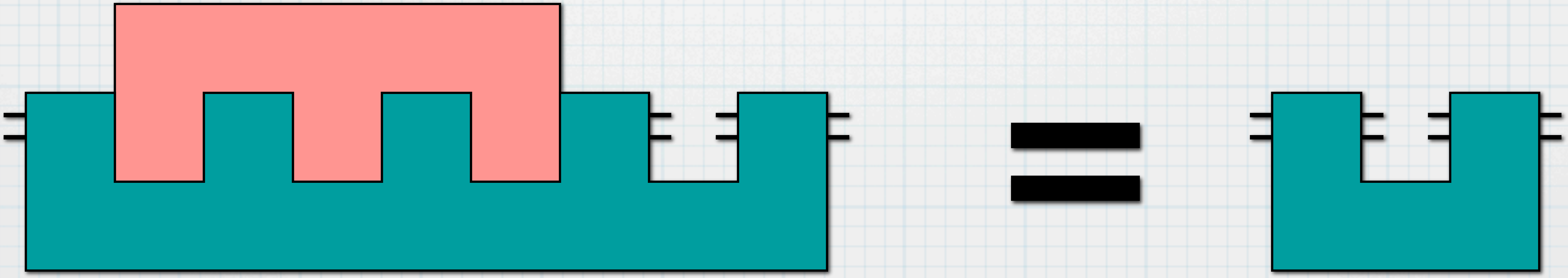
Supermaps

The notion of supermap is the last level of generalization, i.e. “super-supermaps” (mapping supermaps to supermaps) are still supermaps = quantum combs.



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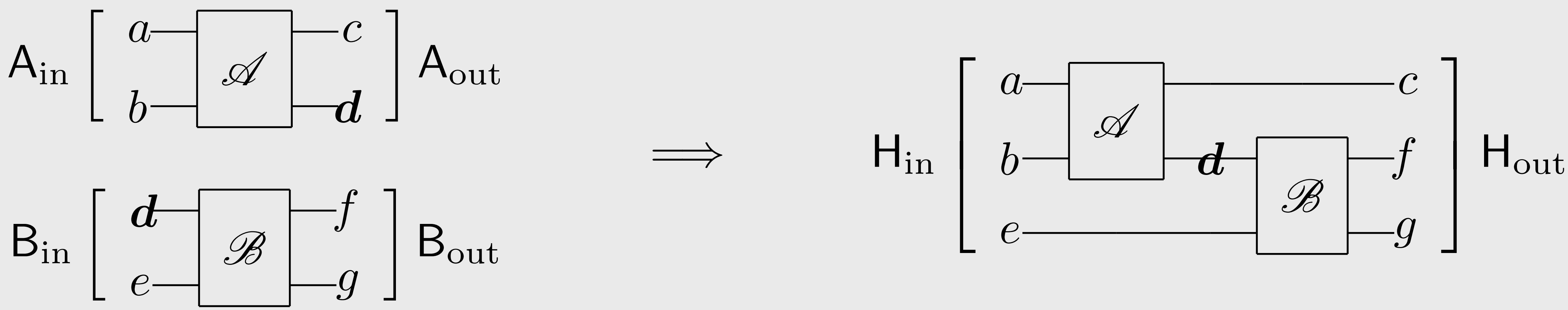
Link product



Choi-operator calculus

$$A \in \mathsf{B}(\mathsf{A}_{\mathrm{out}} \otimes \mathsf{A}_{\mathrm{in}}) = \mathsf{B}(\mathsf{H}_a \otimes \mathsf{H}_b \otimes \mathsf{H}_c \otimes \mathsf{H}_{\boldsymbol{d}}), \qquad \mathsf{J} \equiv \mathsf{H}_{\boldsymbol{d}}$$

$$B \in \mathsf{B}(\mathsf{B}_{\mathrm{out}} \otimes \mathsf{B}_{\mathrm{in}}) = \mathsf{B}(\mathsf{H}_{\boldsymbol{d}} \otimes \mathsf{H}_e \otimes \mathsf{H}_f \otimes \mathsf{H}_g)$$

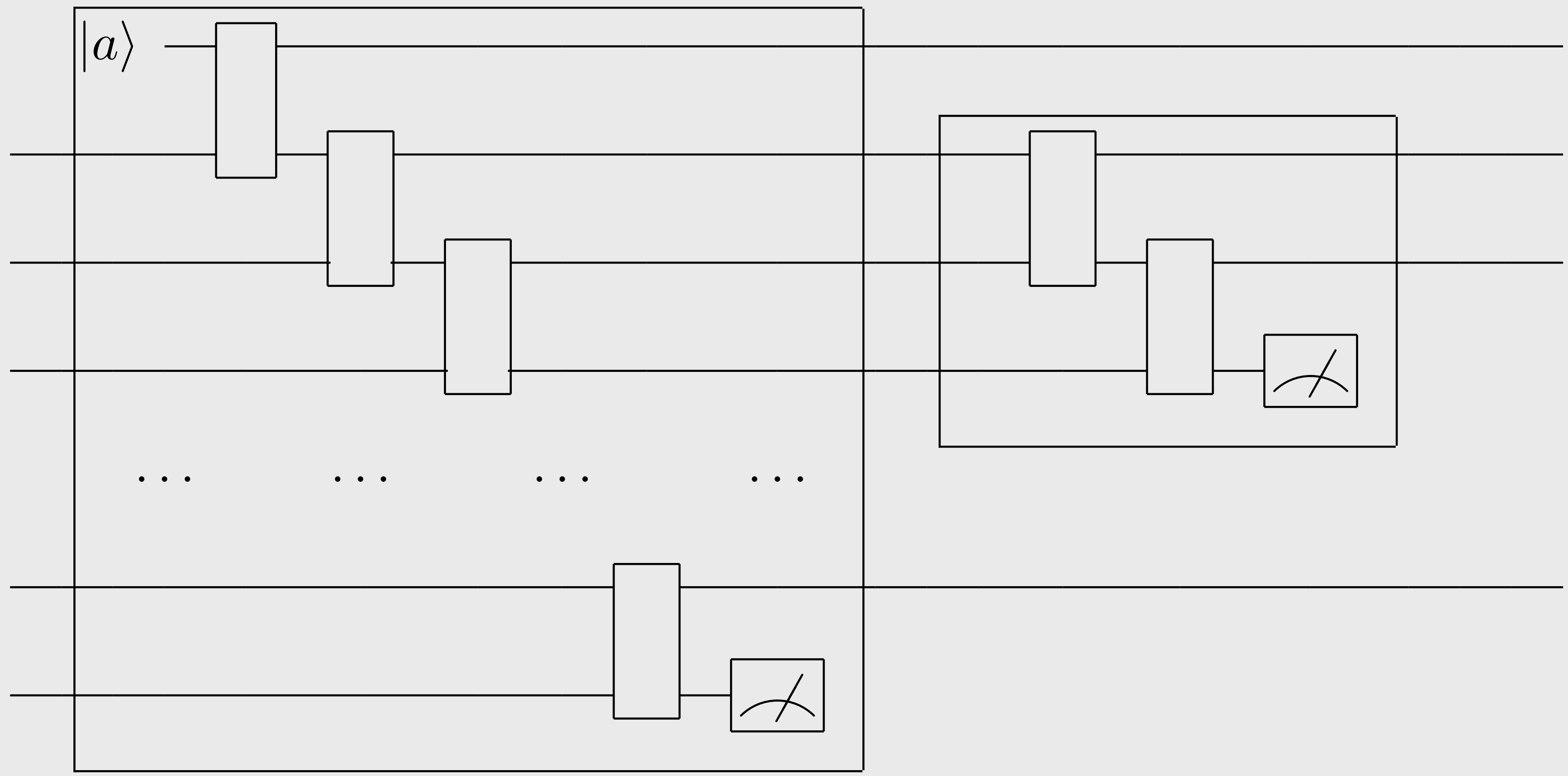
$$AB := (A \otimes I_{e,f,g})(I_{a,b,c} \otimes B)$$

$$A * B = \mathrm{Tr}_{\mathsf{J}}[A^{\theta_{\mathsf{J}}} B] \in \mathsf{B}(\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}})$$

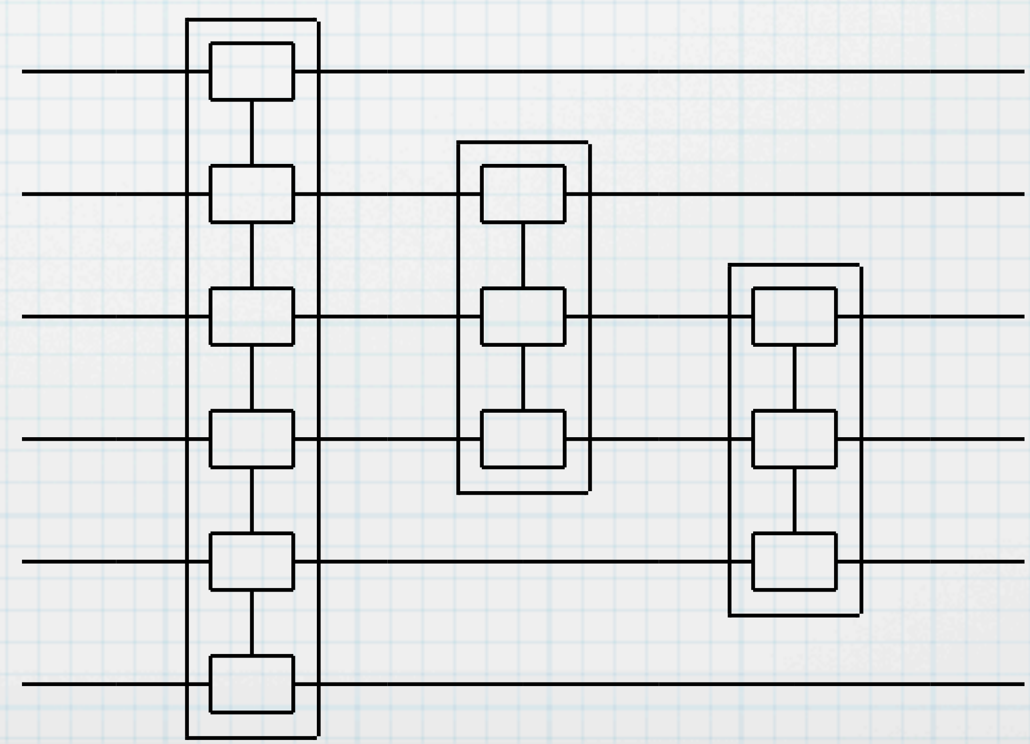
The link-product is commutative!

Link product

$$R_1 * R_2 \iff$$



$$R_1 * R_2 * R_3 \iff$$



Link product

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Special cases:

$$\mathscr{M}(\rho) = R_{\mathscr{M}} * \rho$$ quantum operation

$$\mathrm{Tr}[P^* \rho] = P * \rho$$ POVM

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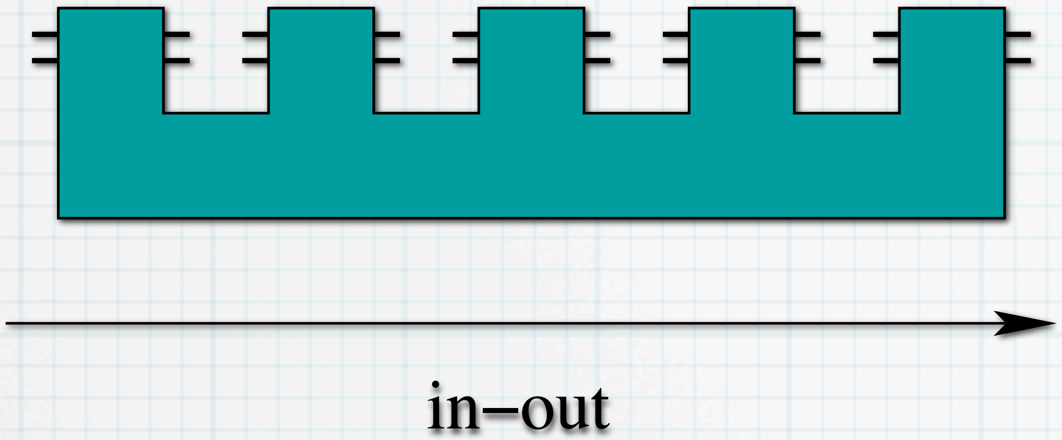
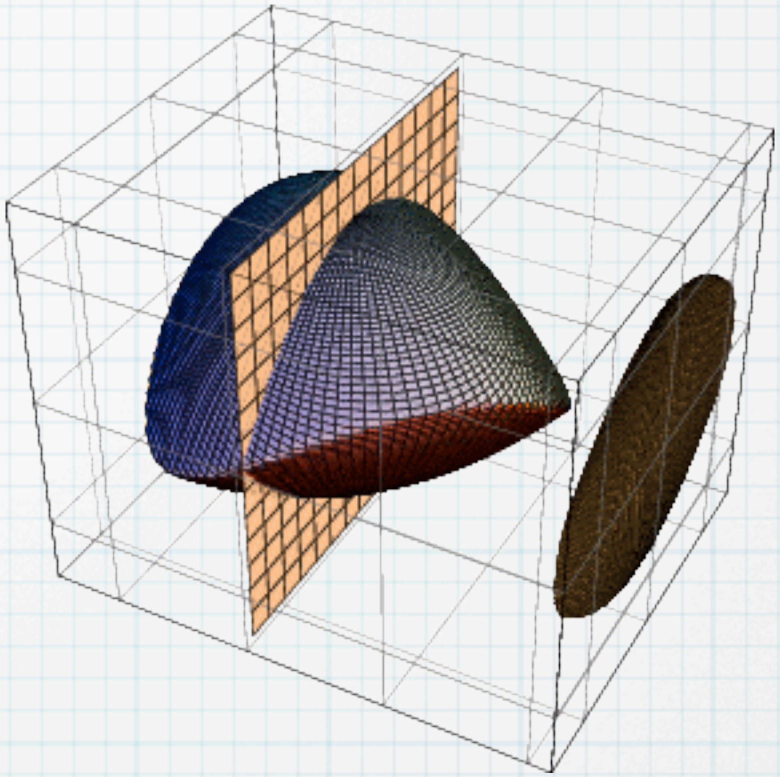
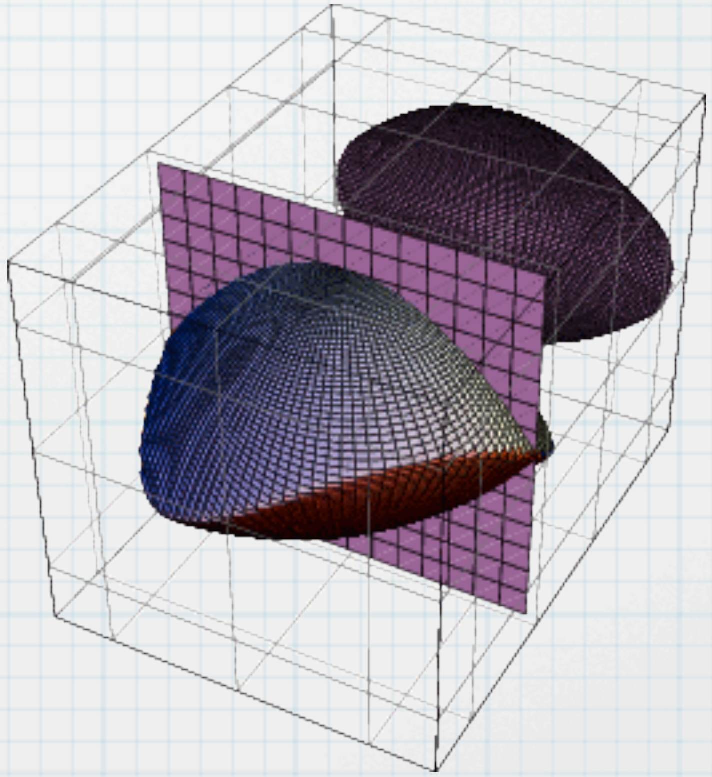
Circuits Architecture Optimization





- The Choi operators of a fixed input-output comb structure make a convex set
- Causality constraints correspond to a hyperplane section of the convex
- Group-covariance gives another linear constraint:



$$[R, V_g] = 0 \implies R = \bigoplus_j R_j \otimes \mathbb{1}_{m_j}$$

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The mathematical formulation is reduced to a convex problem!

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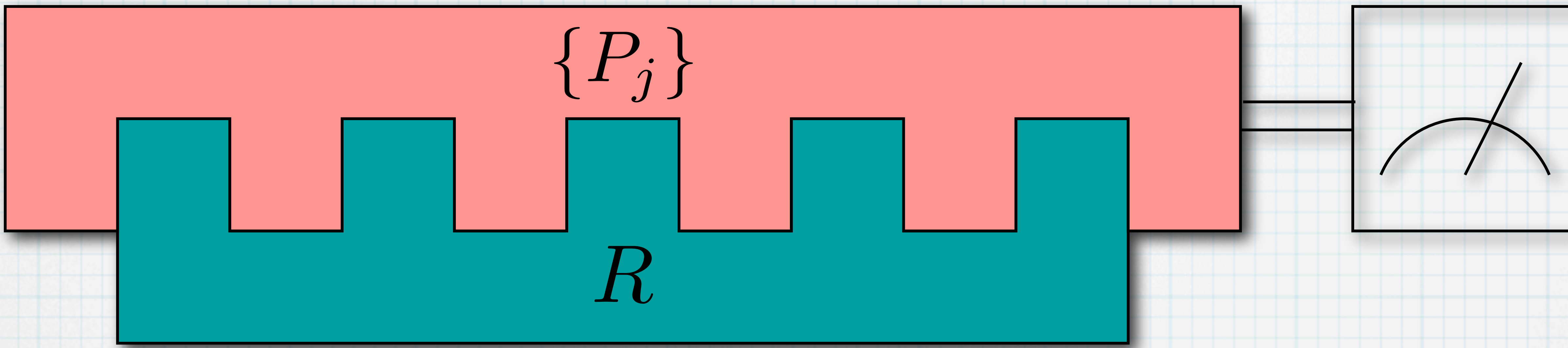
Quantum board testers



Tester Born rule:

$$\mathrm{Tr}[P_j R] = p_j, \qquad \sum_j P_j = \Xi$$

causality constraints:

$$\mathrm{Tr}_{2n+1}[\Xi^{(n)}] = I_{2n} \otimes \Xi^{(n-1)}, \quad n = 0, 1, \ldots, N$$
$$\Xi^{(N)} \equiv \Xi, \quad \mathrm{Tr}_1[\Xi^{(0)}] = 1$$

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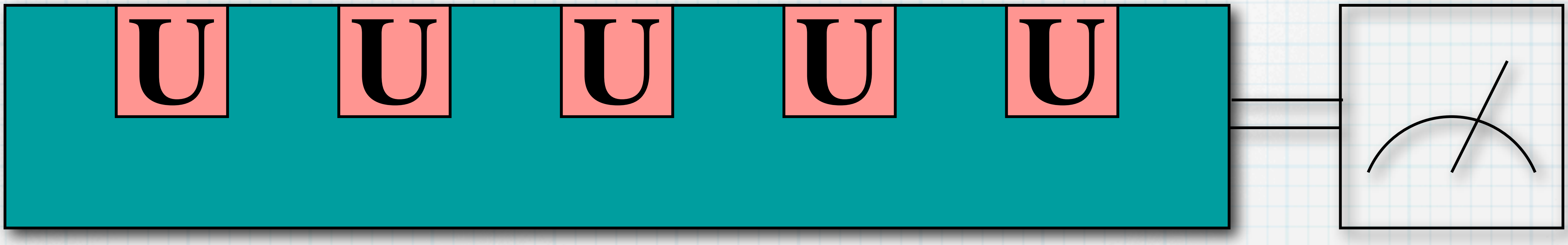
Estimating tester



Tester

Born rule:

$$\mathrm{Tr}[P_j R] = p_j, \qquad \sum_j P_j = \Xi$$

causality constraints:

$$\mathrm{Tr}_{2n+1}[\Xi^{(n)}] = I_{2n} \otimes \Xi^{(n-1)}, \quad n = 0, 1, \ldots, N$$
$$\Xi^{(N)} \equiv \Xi, \quad \mathrm{Tr}_1[\Xi^{(0)}] = 1$$

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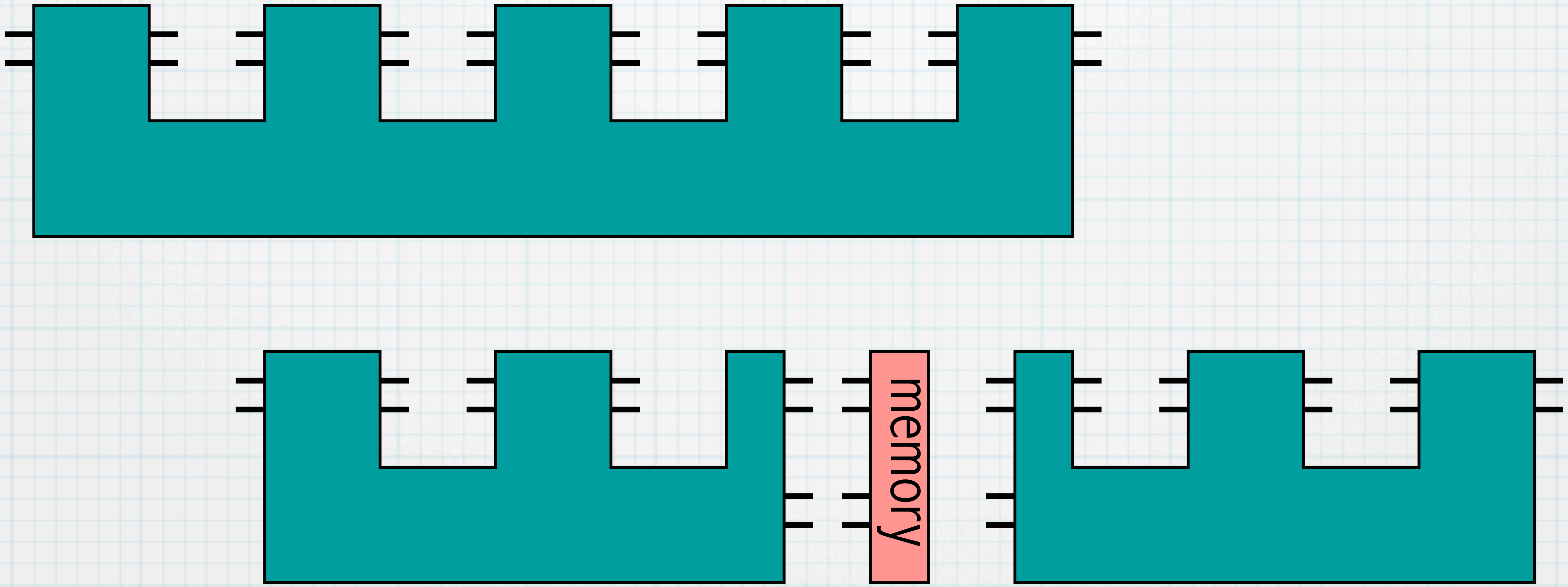
Using quantum memory

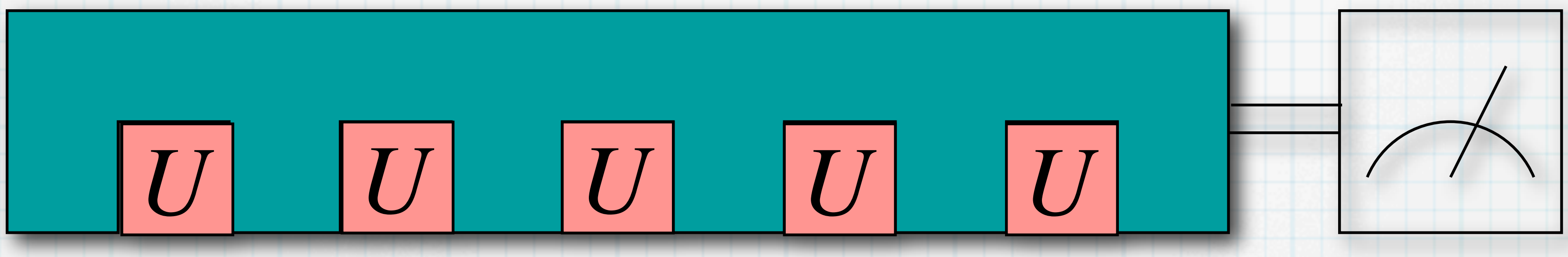
delay the use of subcircuits by breaking the comb into subcombs + quantum memory



Applications

Discrimination of unitaries

Optimal discrimination between two possible unitary operators U_1 U_2



The parallel strategy is already optimal!

Optimal discrimination of channels

Optimal estimation of unitaries

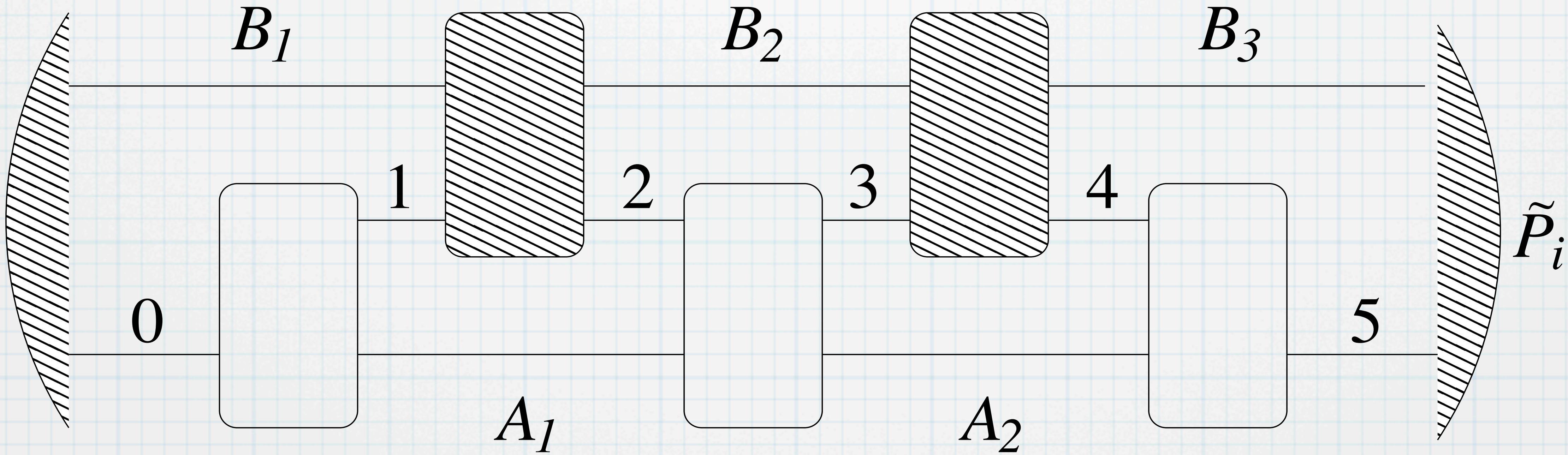
STILL OPEN PROBLEMS

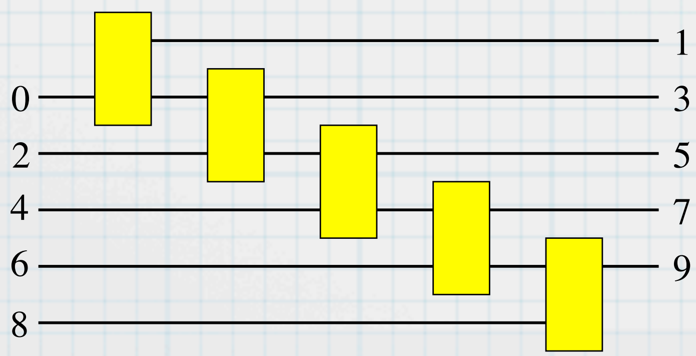
G.Chiribella, G.M.D'Ariano, P.Perinotti arXiv:0803.3237

Discrimination of memory channels

arXiv:0803.3237

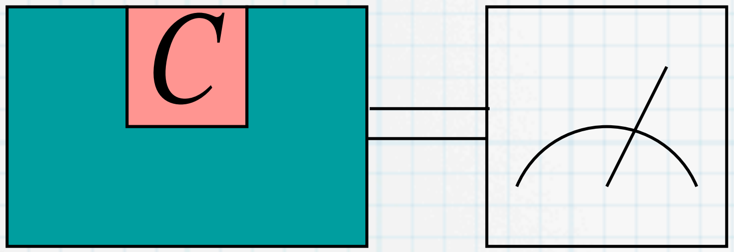
There are memory channels that can be discriminated perfectly with a single use by a quantum tester, and not conventionally





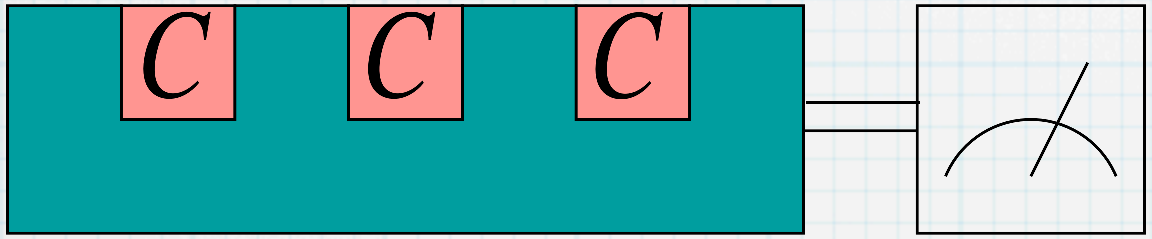
G.Chiribella, G.M.D'Ariano, P.Perinotti arXiv:0803.3237

Optimal tomographers

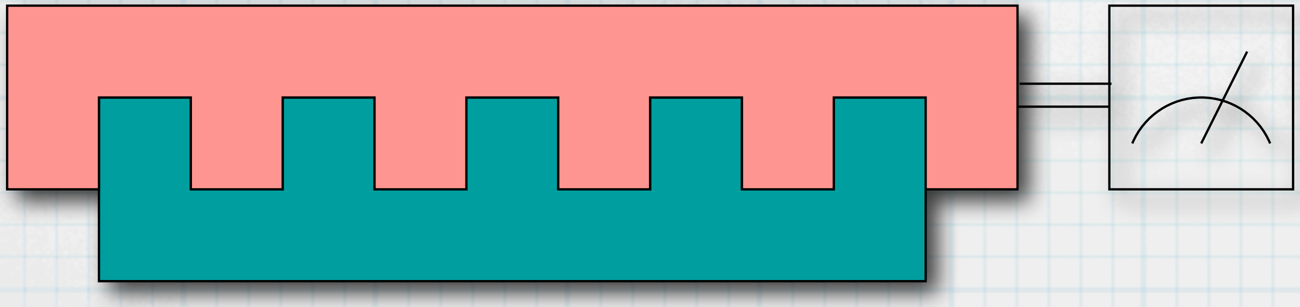


(d^4 outcomes)

Informationally complete tester



multiple uses



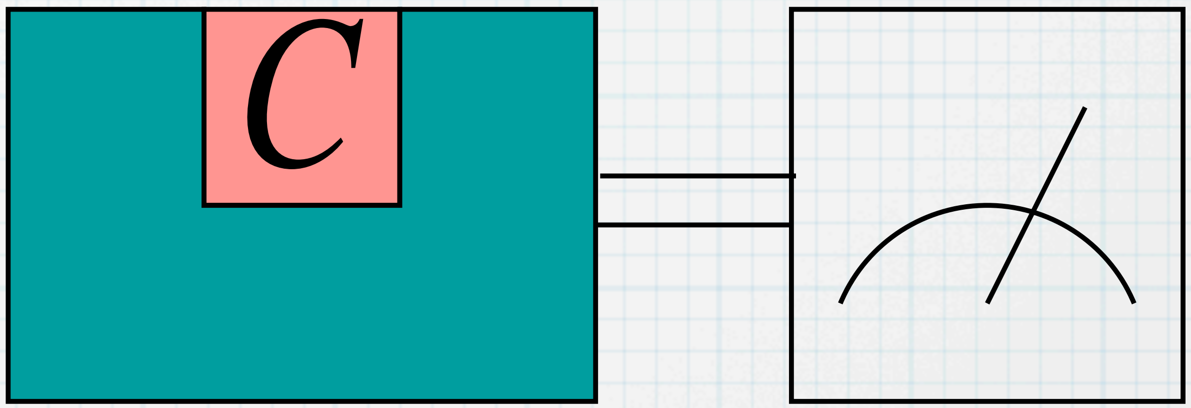
circuit board tomographer

arXiv: 0806.1172

Optimal tomography

- Prior distribution of channels corresponding to the depolarizing average channel
- Cost function = representation, (equally weighted orthonormal set of operators)
- Further selection:
  1) quantum operations,
  2) channels,
  3) unital channels

Use different in and out dimensions to unify: states, channels, and POVMs



A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, P. Perinotti arXiv: 0806.1172

Informationally complete POVM

Tomographing an unknown state ρ of a quantum system means performing a suitable POVM $\{P_i\}$ such that every expectation value can be evaluated from the probability distribution $p_i = \mathrm{Tr}[\rho P_i]$

In particular the expectation value of an operator A can be obtained when it is possible to expand A over the POVM as follows

$$A = \sum_i f_i[A] P_i$$

The expectation is then obtained as: $\langle A \rangle = \sum_i f_i[A] \langle P_i \rangle$

When the expansion holds for all operators of $\mathcal{B}(\mathsf{H})$, namely $\mathcal{B}(\mathsf{H}) = \mathsf{Span}\{P_i\}$ then the POVM is called informationally complete. This includes the case of the quorum of observables.

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Informationally complete POVM

Notation: associate operators to bipartite vectors as follows

$$A = \sum_{m,n=1}^{d} A_{mn} |m\rangle \langle n| \leftrightarrow |A\rangle\!\rangle = \sum_{m,n=1}^{d} A_{mn} |m\rangle |n\rangle$$

$$\langle\!\langle B|A\rangle\!\rangle = \mathrm{Tr}[B^\dagger A]$$

$$X = \sum_i \mathrm{Tr}[B_i^\dagger X] A_i \iff |X\rangle\!\rangle = \sum_i \langle\!\langle B_i|X\rangle\!\rangle |A_i\rangle\!\rangle$$

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Informationally complete POVM

Information-completeness of the POVM $\{P_i\}$ corresponds to invertibility of the frame operator:

$$F = \sum_i |P_i\rangle\rangle\langle\langle P_i|$$

The operator expansion can be written as follows

$$|A\rangle\rangle = \sum_i \langle\langle D_i|A\rangle\rangle|P_i\rangle\rangle$$

in terms of the dual frame $\{D_i\}$ satisfying the identity

$$\sum_i |P_i\rangle\rangle\langle\langle D_i| = I$$

Informationally complete POVM

The request for the POVM $\{P_i\}$ to be informationally complete can be relaxed if we have some prior information about the state ρ. If we know that the state belongs to a given subspace $\mathcal{V} \subseteq \mathcal{B}(\mathsf{H})$ the expectation value is

$$\langle A \rangle = \langle\langle \rho | A \rangle\rangle = \langle\langle \rho | Q_{\mathcal{V}} | A \rangle\rangle$$

$Q_{\mathcal{V}}$ orthogonal projector on $\mathcal{V}$, whence the set $\{P_i\}$ is required to span only $\mathcal{V}$.

Informationally complete POVM

Cost function

For the estimation of the expectation $\langle A\rangle$ of an observable A , optimality means minimization of the cost function given by the variance $\delta(A)$ of the random variable $\langle\langle D_i|A\rangle\rangle$ with probability distribution $\mathrm{Tr}[\rho P_i]$, namely

$$\delta(A) := \sum_i |\langle\langle D_i|A\rangle\rangle|^2 \,\mathrm{Tr}[\rho P_i] - |\,\mathrm{Tr}[\rho A]|^2.$$

Informationally complete POVM

Bayesian scheme

In a Bayesian scheme the state ρ is randomly drawn from an ensemble $\mathcal{S} = \{\rho_k, p_k\}$ of states ρ_k with prior probability p_k, with the variance averaged over $\mathcal{S}$, leading to

$$\delta_{\mathcal{S}}(A) := \sum_i |\langle\langle D_i | A \rangle\rangle|^2 \operatorname{Tr}[\rho_{\mathcal{S}} P_i] - \sum_k p_k |\operatorname{Tr}[\rho_k A]|^2$$

where $\rho_{\mathcal{S}} = \sum_k p_k \rho_k$

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Informationally complete POVM

Representation=cost function

A priori we can be interested in some observables more than other ones, and this can be specified in terms of a weighted set $\mathcal{G}=\{A_n,q_n\}$ of observables A_n with weights $q_n>0$.

Averaging over $\mathcal{G}$ we have

$$\delta_{\mathcal{S},\mathcal{G}}:=\sum_i \langle\!\langle D_i|G|D_i\rangle\!\rangle \operatorname{Tr}[\rho_{\mathcal{S}}P_i]-\sum_{k,n}p_k q_n|\operatorname{Tr}[\rho_k A_n]|^2$$

$$G=\textstyle\sum_n q_n|A_n\rangle\!\rangle\langle\!\langle A_n|$$

The weighted set $\mathcal{G}$ yields a representation of the state, given in terms of the expectation values.

The representation is faithful when $\{A_n\}$ is an operator frame, e. g. when it is made of the dyads $|i\rangle\langle j|$ corresponding to the matrix $\langle j|\rho|i\rangle$

Informationally complete POVM

Notice that only the first term of $\delta_{\mathcal{S},\mathcal{G}}$ depends on $\{P_i\}$ and $\{D_i\}$. If $\rho_i \in \mathcal{V}$ for all states $\rho_i \in \mathcal{S}$, the second term of the variance becomes

$$\eta = \sum_i \langle\langle D_i | Q_{\mathcal{V}} G Q_{\mathcal{V}} | D_i \rangle\rangle \operatorname{Tr}[\rho_{\mathcal{S}} P_i]$$

$$G = \sum_n q_n |A_n\rangle\rangle\langle\langle A_n|$$

Process tomography

Keep variable input and output Hilbert spaces H_{in} and H_{out}

Advantage:

Usual state-tomography: H_{in} one-dimensional

POVM tomography: H_{out} one-dimensional

Process tomography

Quantum operation $\mathcal{T} : \mathcal{B}(\mathsf{H}_{in}) \longrightarrow \mathcal{B}(\mathsf{H}_{out})$

General procedure to get information on $\mathcal{T}$:

i) Prepare a state $\rho \in \mathcal{B}(\mathsf{H}_{in} \otimes \mathsf{H}_A)$

ii) Measure a POVM $\{P_i\}$ over the state $(\mathcal{T} \otimes \mathcal{I}_A)(\rho)$

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Process tomography

Using the Choi-Jamiolkowski isomorphism:

$$\mathcal{T}(\rho) = \mathrm{Tr}_{in}[(I_{out} \otimes \rho^T) R_{\mathcal{T}}], \quad R_{\mathcal{T}} = \mathcal{T} \otimes \mathcal{I}_{in}(|I\rangle\rangle\langle\langle I|)$$

the probability distribution $p_i = \mathrm{Tr}[(\mathcal{T} \otimes \mathcal{I}_A)(\rho) P_i]$ becomes

$$\mathrm{Tr}[\mathrm{Tr}_{in}[(I_A \otimes R_{\mathcal{T}})(\rho^{\theta_{in}} \otimes I_{out})] P_i] = \mathrm{Tr}[R_{\mathcal{T}} \Pi_i^{(\rho)}]$$

where

$$\Pi_i^{(\rho)} = \{\mathrm{Tr}_A[(\rho \otimes I_{out})(I_{in} \otimes P_i^{\theta_{out}})]\}^T$$

θ partial transposition, T transposition

New type of Born rule

Process tomography

Using a tester $\{\Pi_i\}$:

$$\sum_i \Pi_i = I \otimes \sigma, \qquad \mathrm{Tr}[\sigma] = 1$$

The tester Born rule can be written in terms of the usual one as follows

$$p_i = \mathrm{Tr}[R_{\mathcal{T}} \Pi_i] = \mathrm{Tr}[\mathcal{T} \otimes \mathcal{I}(\nu) P_i]$$

with

$$\nu = |\sqrt{\sigma}\rangle\rangle\langle\langle\sqrt{\sigma}|, \qquad P_i = (I \otimes \sigma^{-1/2})\Pi_i(I \otimes \sigma^{-1/2})$$

Process tomography

Tester Born rule: $p_i = \mathrm{Tr}[R_{\mathcal{T}} \Pi_i]$

The tester method allows a straightforward generalization of the tomographic method from states to transformation.

Tomography-ing a quantum operation means using a suitable tester $\{\Pi_i\}$ such that the expectation value of any other possible measurement can be inferred by the probability distribution $p_i = \mathrm{Tr}[R_{\mathcal{T}} \Pi_i]$

Notion of info-complete tester:

$\{\Pi_i\}$ is an operator frame for $\mathcal{B}(\mathsf{H}_{out} \otimes \mathsf{H}_{in})$, namely

$$A = \sum_i \langle\!\langle \Delta_i | A \rangle\!\rangle \Pi_i \qquad A \in \mathcal{B}(\mathsf{H}_{out} \otimes \mathsf{H}_{in})$$

Process tomography

We take:

$$\dim(\mathsf{H}_{in}) = \dim(\mathsf{H}_{out}) = d$$

Cost function as the variance averaged over the prior distribution of quantum operations $\mathcal{E} = \{R_k, p_k\}$, and over the representation $\mathcal{G} = \{A_n, q_n\}$

$$\delta_{\mathcal{E},\mathcal{A}} := \sum_i \langle\langle \Delta_i | G | \Delta_i \rangle\rangle \operatorname{Tr}[R_{\mathcal{E}} \Pi_i] - \sum_{k,n} p_k q_n |\operatorname{Tr}[R_k A_n]|^2$$

Prior averaged channel: max-depolarizing $R_{\mathcal{E}} = d^{-1} I \otimes I$

Representation: $G = I$ corresponding e.g. to $\{A_n\}$ o.n.b.

The relevant cost becomes: $\eta = \sum_i \langle\langle \Delta_i | \Delta_i \rangle\rangle d^{-1} \operatorname{Tr}[\Pi_i]$

Process tomography

Due to symmetry of the prior, one can take a tester which is unitarily covariant and have the same cost function.

$$\Pi_{i,g,h} := (U_g \otimes V_h)\Pi_i(U_g^\dagger \otimes V_h^\dagger)$$
$$\Delta_{i,g,h} := (U_g \otimes V_h)\Delta_i(U_g^\dagger \otimes V_h^\dagger)$$

Normalization:

$$\sum_i \int dg dh\, \Pi_{i,g,h} = d^{-1} I \otimes I$$

namely one has $\sigma = d^{-1} I$, and one can choose $\nu = d^{-1} |I\rangle\rangle\langle\langle I|$

The condition that the covariant tester is informationally complete w.r.t. the subspace of transformations to be tomographed will be verified after the optimization.

Process tomography

The tester normalization condition becomes:

$$\sum_i \mathrm{Tr}[\Pi_i] = d$$

A lengthy calculation leads to the optimal dual, corresponding to the optimal variance

$$\eta = \mathrm{Tr}[\tilde{X}^{-1}]$$

where

$$\tilde{X} = \sum_i \int dg dh \, \frac{d|\Pi_{i,g,h}\rangle\langle\Pi_{i,g,h}|}{\mathrm{Tr}[\Pi_{i,g,h}]} = \int dg dh \, W_{g,h} X W_{g,h}^\dagger$$

with

$$W_{g,h} = U_g \otimes U_g^* \otimes V_h \otimes V_h^*$$

$$X = \textstyle\sum_i d|\Pi_i\rangle\rangle\langle\langle\Pi_i| / \mathrm{Tr}[\Pi_i]$$

Process tomography

Using the Schur lemma we obtain:

$$\tilde{X} = P_1 + AP_2 + BP_3 + CP_4$$

$$P_1 = \Omega_{13} \otimes \Omega_{24} \qquad P_2 = (I_{13} - \Omega_{13}) \otimes \Omega_{24}$$

$$P_3 = \Omega_{13} \otimes (I_{24} - \Omega_{24}) \quad P_4 = (I_{13} - \Omega_{13}) \otimes (I_{24} - \Omega_{24})$$

where $\Omega = |I\rangle\!\rangle\langle\!\langle I|/d$ and

$$A = \frac{1}{d^2 - 1} \left\{ \sum_i \frac{\mathrm{Tr}[(\mathrm{Tr}_2[\Pi_i])^2]}{\mathrm{Tr}[\Pi_i]} - 1 \right\}$$

$$B = \frac{1}{d^2 - 1} \left\{ \sum_i \frac{\mathrm{Tr}[(\mathrm{Tr}_1[\Pi_i])^2]}{\mathrm{Tr}[\Pi_i]} - 1 \right\}$$

$$C = \frac{1}{(d^2 - 1)^2} \left\{ \sum_i \frac{d\,\mathrm{Tr}[\Pi_i^2]}{\mathrm{Tr}[\Pi_i]} - (d^2 - 1)(A + B) - 1 \right\}$$

One has

$$\mathrm{Tr}[\tilde{X}^{-1}] = 1 + (d^2 - 1) \left(\frac{1}{A} + \frac{1}{B} + \frac{(d^2 - 1)}{C} \right)$$

Process tomography

If the ensemble of transformations is contained in a subspace $\mathcal{V} \subseteq \mathcal{B}(\mathsf{H}_{\text{out}} \otimes \mathsf{H}_{\text{in}})$, the cost function becomes $\eta = \mathrm{Tr}[\tilde{X}^{\ddagger} Q_{\mathcal{V}}]$, where $\tilde{X}^{\ddagger}$ denotes the Moore-Penrose pseudoinverse.

We consider the three relevant cases:

- Quantum operations: $\mathcal{Q} = \mathcal{B}(\mathsf{H}_{\text{out}} \otimes \mathsf{H}_{\text{in}})$
- General chennels: $\mathcal{C} = \{\mathsf{R} \in \mathcal{Q},\ \mathrm{Tr}_{\text{out}}[\mathsf{R}] = \mathsf{I}_{\text{in}}\}$
- Unital channels: $\mathcal{U} = \{R \in \mathcal{Q},\ \mathrm{Tr}_{out}[R] = I_{in}, \mathrm{Tr}_{in}[R] = I_{out}\}$

We have:

$$Q_{\mathcal{C}} = P_1 + P_2 + P_4, \qquad Q_{\mathcal{U}} = P_1 + P_4$$

Optimal process tomography

W.l.g. we can take the "seeds" $\{\Pi_i\}$ as rank-one:

$$\Pi_i = \alpha_i |\Psi_i\rangle\!\rangle\langle\!\langle\Psi_i| \qquad \sum_i \alpha_i = d$$

The cost function is:

$$\eta_{\mathcal{Q}} = \mathrm{Tr}[\tilde{X}^{-1}] = 1 + (d^2-1)\left(\frac{2}{A} + \frac{(d^2-1)^2}{1-2A}\right)$$

$$\eta_{\mathcal{C}} = \mathrm{Tr}[\tilde{X}^{\ddagger} Q_{\mathcal{C}}] = 1 + (d^2-1)\left(\frac{1}{A} + \frac{(d^2-1)^2}{1-2A}\right)$$

$$\eta_{\mathcal{U}} = \mathrm{Tr}[\tilde{X}^{\ddagger} Q_{\mathcal{U}}] = 1 + (d^2-1)\left(\frac{(d^2-1)^2}{1-2A}\right)$$

The optimal values are obtained minimizing w.r.t. A

Optimal process tomography

Optimal costs (compare with Scott, J. Phys. A 41 055308 (2008) for unital channels and quantum operations):

$$\eta_{\mathcal{Q}} \geq d^6 + d^4 - d^2$$
$$\eta_{\mathcal{C}} \geq d^6 + (2\sqrt{2} - 3)d^4 + (5 - 4\sqrt{2})d^2 + 2(\sqrt{2} - 1)$$
$$\eta_{\mathcal{U}} \geq (d^2 - 1)^3 + 1.$$

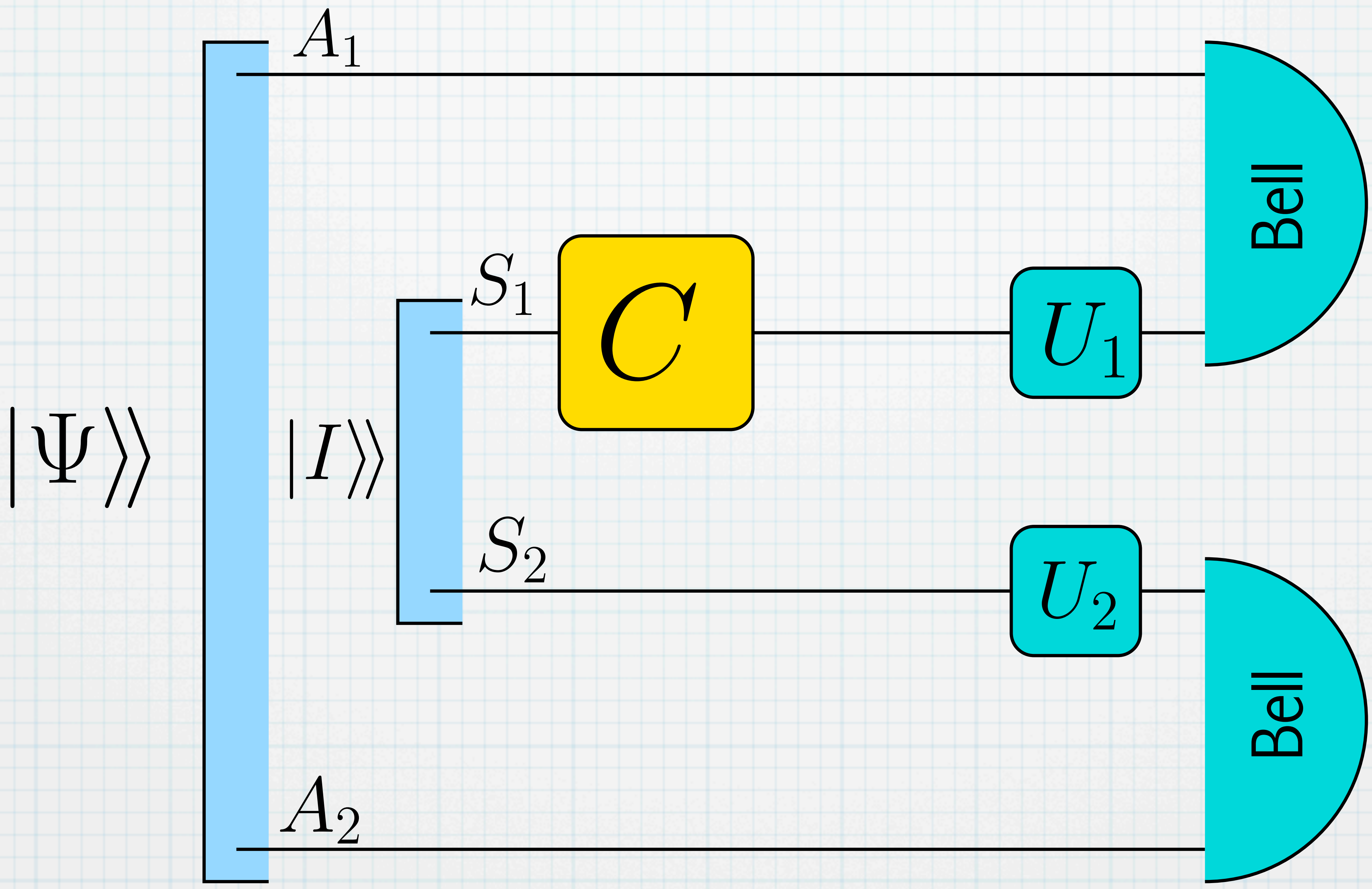
Bounds achieved by a single seed: $\Pi_0 = d|\Psi\rangle\!\rangle\langle\!\langle\Psi|$

$$\Psi = [d^{-1}(1-\beta)I + \beta\,|\psi\rangle\,\langle\psi|]^{\frac{1}{2}}$$

- Quantum operations: $\beta = \sqrt{(d+1)/(d^2+1)}$
- General channels: $\beta = [(d-1)(2+\sqrt{2}(d^2-1))]^{-1/2}$
- Unital channels: $\beta = 0$

arXiv: 0806.1172

Optimal tomography



$$\Psi = [d^{-1}(1-\beta)I + \beta\,|\psi\rangle\,\langle\psi|]^{\frac{1}{2}}$$

$\beta = \sqrt{(d+1)/(d^2+1)}$ quantum operations

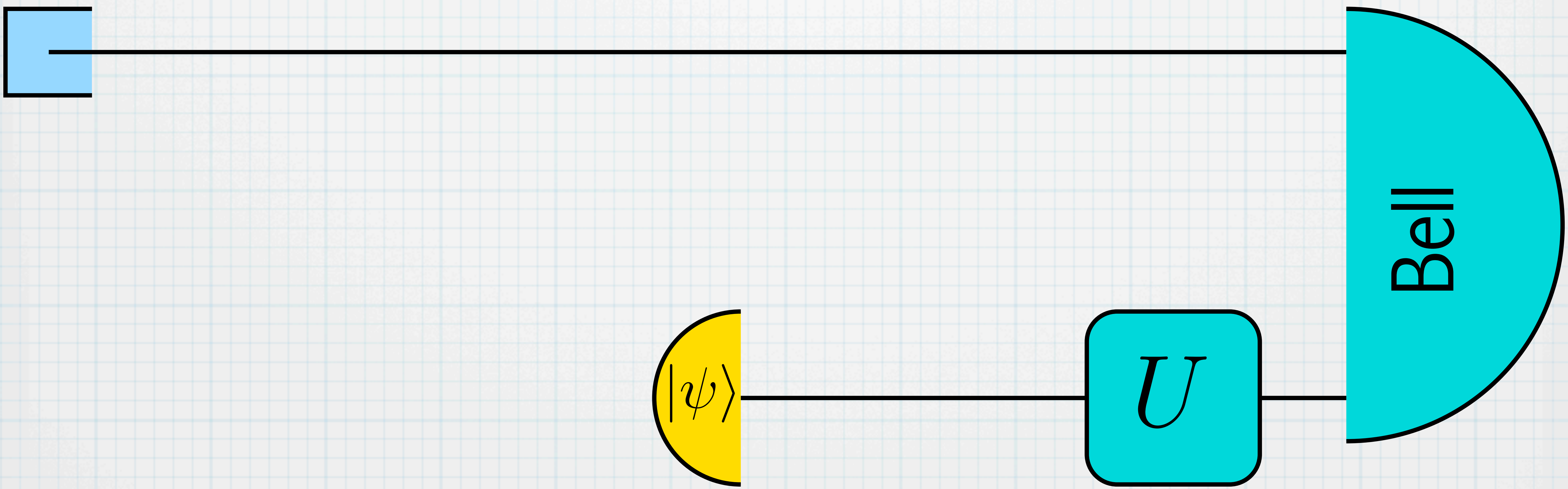
$\beta = [(d-1)(2+\sqrt{2}(d^2-1))]^{-1/2}$ channels

$\beta = 0$ unital channels

arXiv: 0806.1172

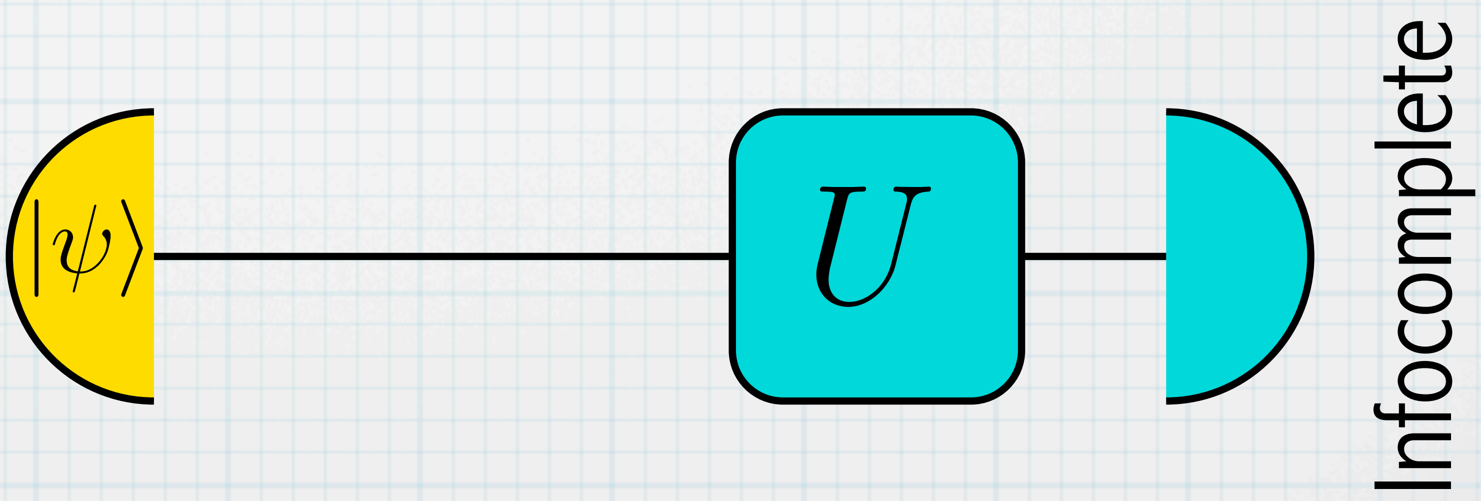
Optimal tomography

State tomography



Optimal tomography

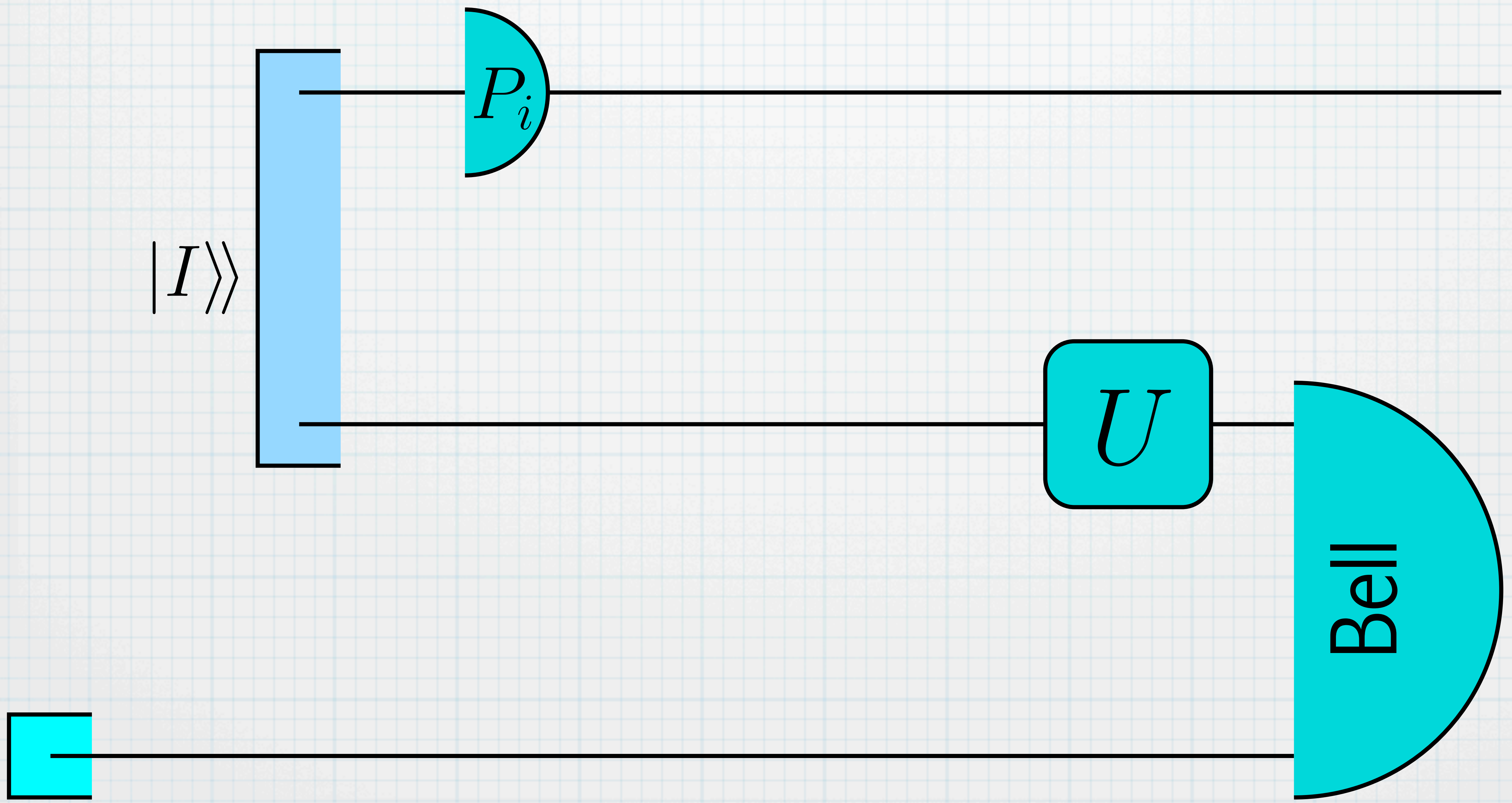
State tomography



arXiv: 0806.1172

Optimal tomography

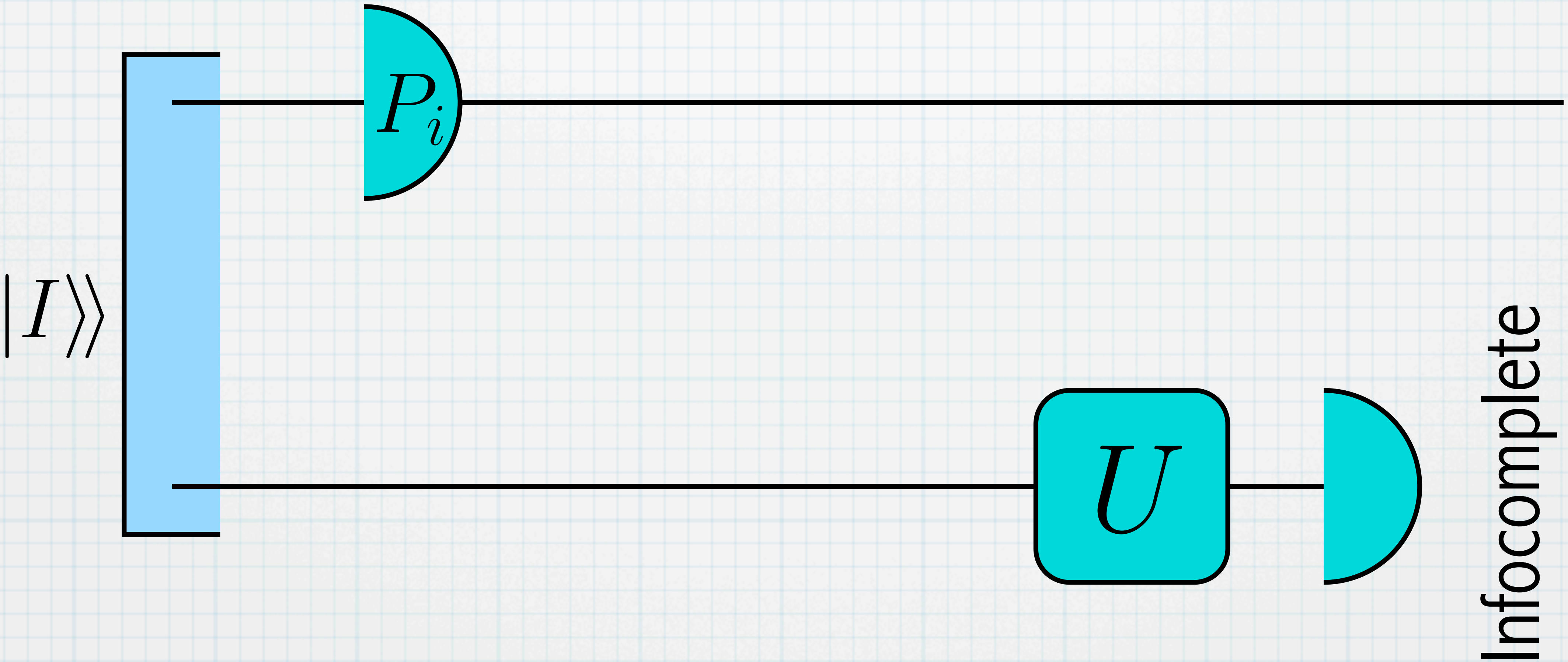
POVM tomography



arXiv: 0806.1172

Optimal tomography

POVM tomography



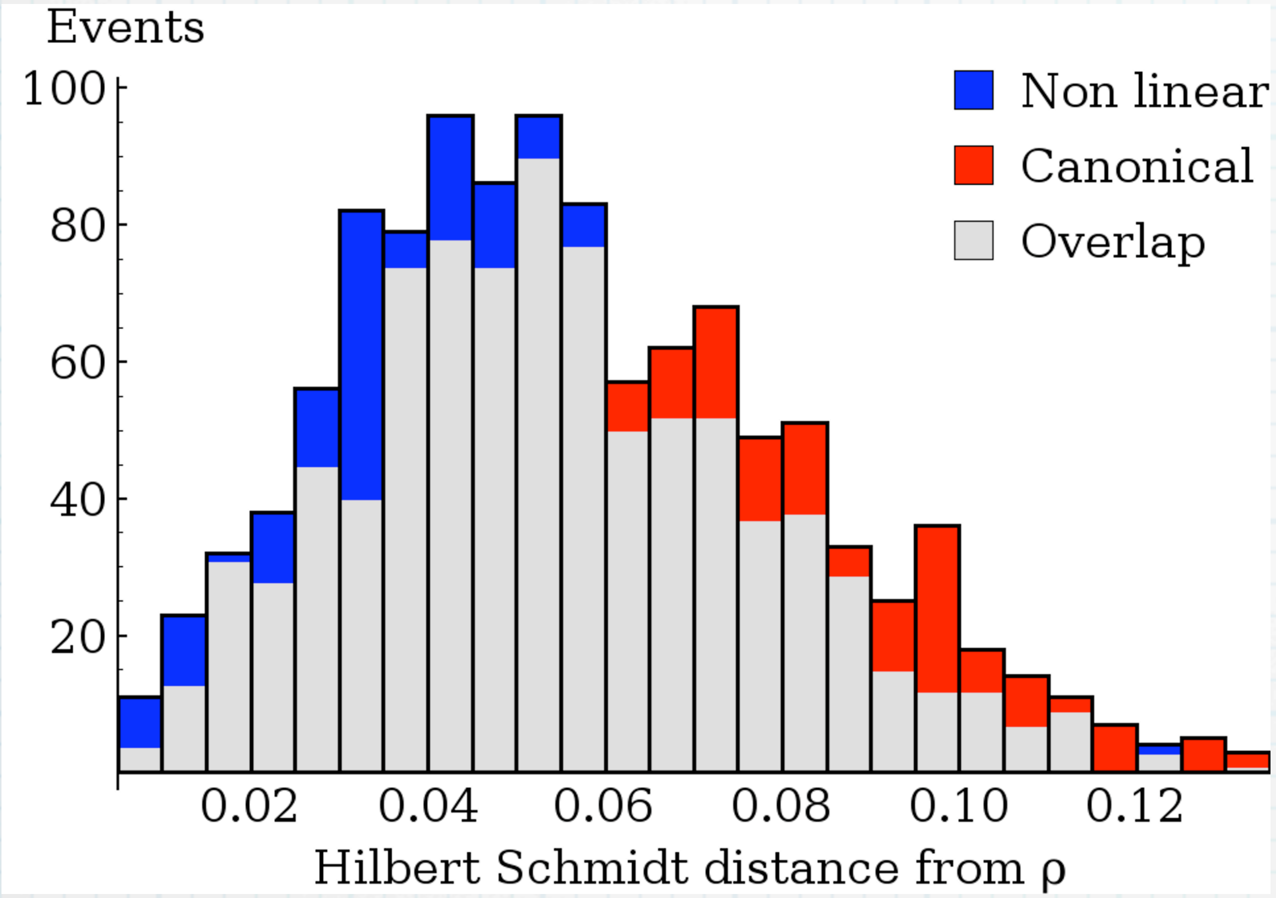
Adaptive Quantum Tomography

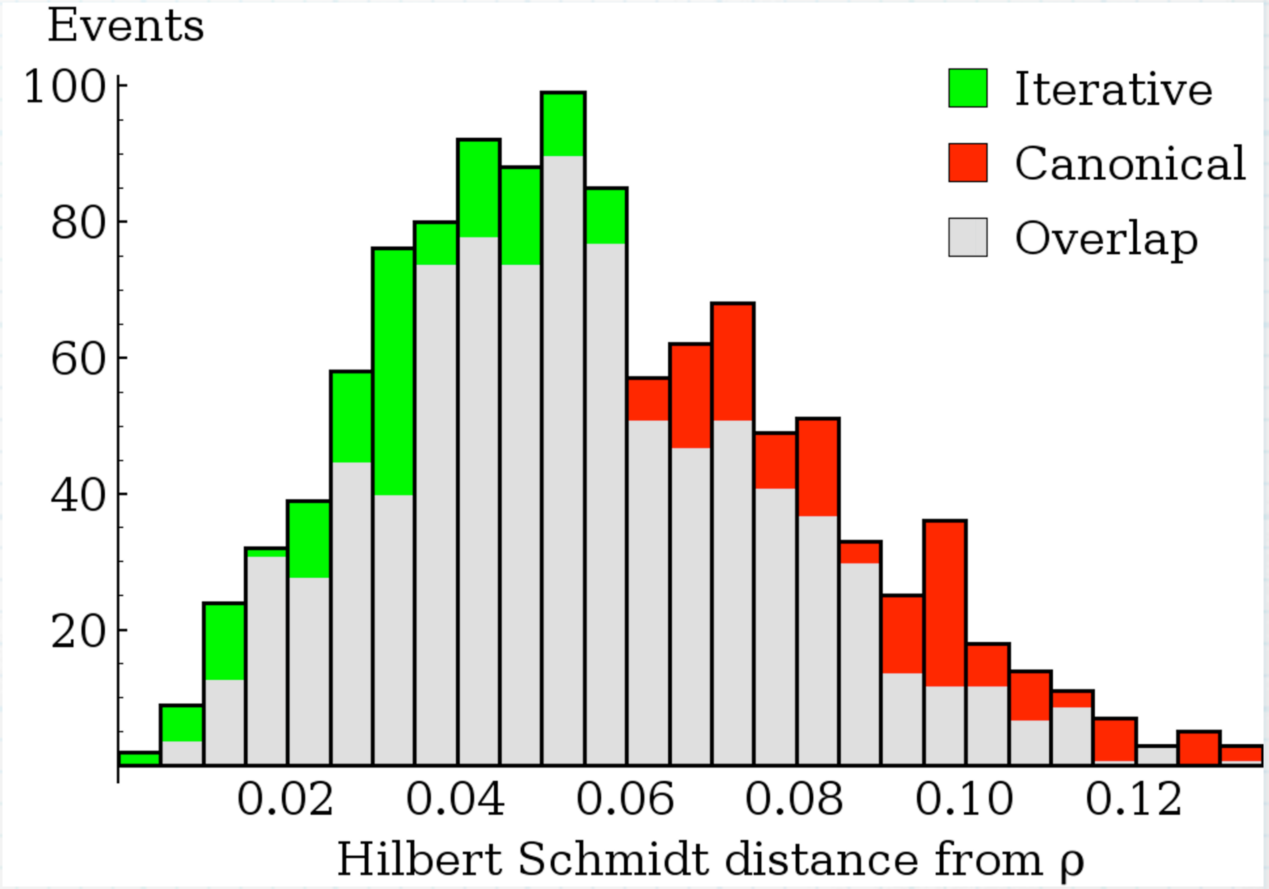
G. M. D'Ariano, D. F. Magnani, P. Perinotti arXiv:0807.5058

- Method 1 (Bayesian iterative procedure): Bayesian update of the prior distribution after the first state reconstruction, then iterate.
- Method 2 (Frequentistic approach) replace the theoretical probability distribution of the infocomplete in the optimal data-processing with the experimental frequencies.

arXiv:0807.5058

Adaptive Quantum Tomography





Histograms representing the number of experiments versus the Hilbert-Schmidt distance of the estimated state from the theoretical one. Right plot: the green bars correspond to the Bayesian processing, the red bars correspond to the plain processing without updating, the gray part is the overlap. Left plot: the blue bars corresponding to the frequentist processing method. Both plots show a well visible shift of the histograms corresponding to the new adaptive methods towards small errors compared to the plain processing without update.

arXiv:0807.5058

Adaptive Quantum Tomography

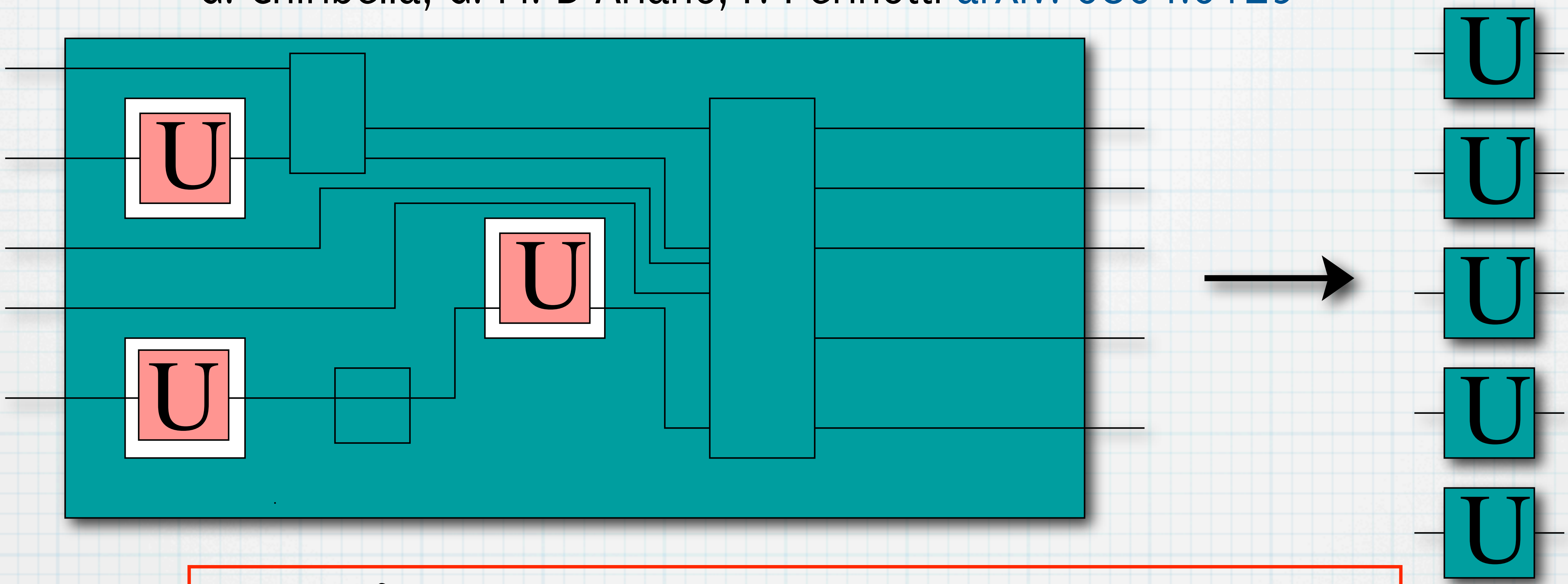
Procedure	$\langle H.S.dist.\rangle$	σ	$\Delta\left(\langle H.S.dist.\rangle\right)$	$\Delta(\sigma)$
Plain (no update)	0.06	0.03	-	-
Bayesian	0.05	0.02	-17%	-33.3%
Frequentist	0.05	0.02	-17%	-33.3%

Average Hilbert-Schmidt distance, variance σ of the histogram, and relative improvements compared to the plain un-updated procedure of the new data-processing strategies.

Other applications

Cloning of unitaries

G. Chiribella, G. M. D'Ariano, P. Perinotti arXiv: 0804.0129



$$F = \int \mathrm{d}U \, F(\mathscr{T}_U^{(N)}, \mathscr{T}_U^{\otimes N})$$ (channel fidelity)

1-to-2 cloning $$F = \frac{d + \sqrt{d^2 - 1}}{d^3} > F_{est} = \frac{6}{d^4} (d \neq 2)$$

for qubits: $$F \simeq 46.65\%, \; F_{est} = \frac{5}{16} \simeq 31\%$$

Quantum-algorithm learning

Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow

- Alice owns quantum circuit that performs a very valuable algorithm U that she wants to keep undisclosed.

- Bob needs to run Alice's algorithm on an input state that will be available tomorrow, but he can borrow the circuit from Alice only today for just a limited number of uses N, and with the circuit sealed.

Quantum algorithm learning

Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow

- The only thing that Bob can do today, with the circuit available, is to use it on a input state known to him.
- After that the only thing that remains available to Bob for tomorrow is the output state, which Bob can store on a quantum memory.
- Therefore, Bob needs a quantum device that is capable of "learning the quantum algorithm" from the output state, namely recovering U and then running it on a new unknown state.

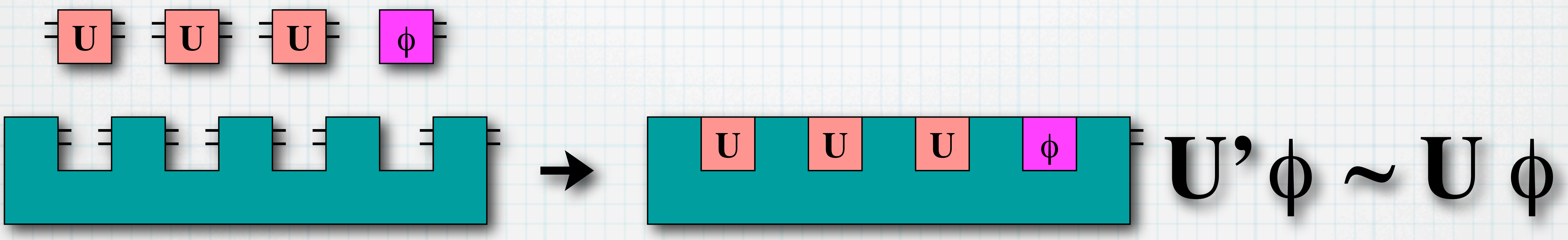
Quantum algorithm learning

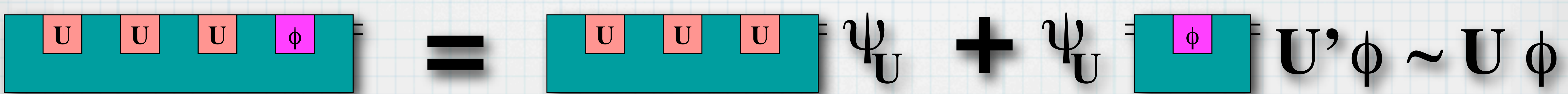
Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow

- In principle:
  - Exact storing of quantum states is possible (quantum memory is a technological problem)
  - Perfect storing of undisclosed unitaries over a quantum state is impossible (Nielsen-Chuang no-programming theorem)

Quantum algorithm learning

Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow





Cloning versus learning

$$(d \neq 2) F_{est} = \frac{6}{d^4} < F_{learn} = \frac{1}{d^2} < F_{clon} = \frac{d + \sqrt{d^2 - 1}}{d^3}$$

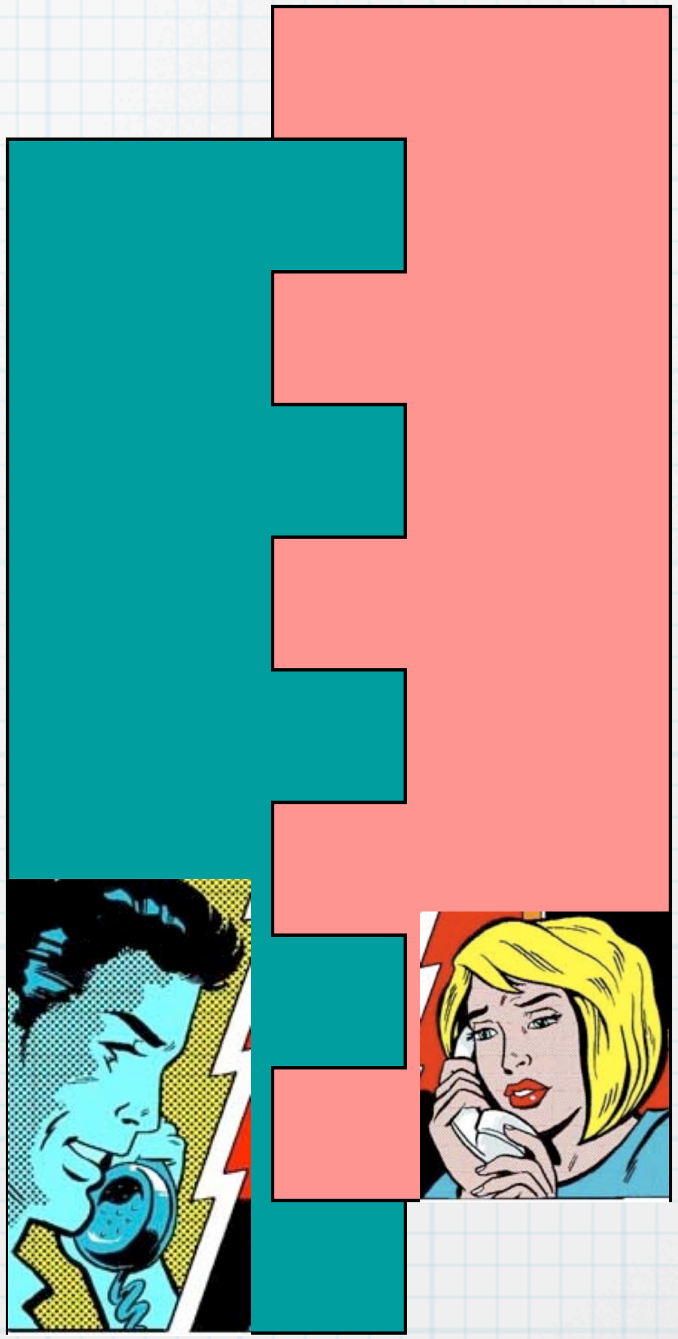
- Optimal cloning of U outperforms optimal cloning of states to which U is applied locally (the learning gives the optimal recovering of U from the state).
- Applying U to a state = "degrading" U irreversibly ".

Quantum protocols

N-party quantum protocols are described by N interlaced combs

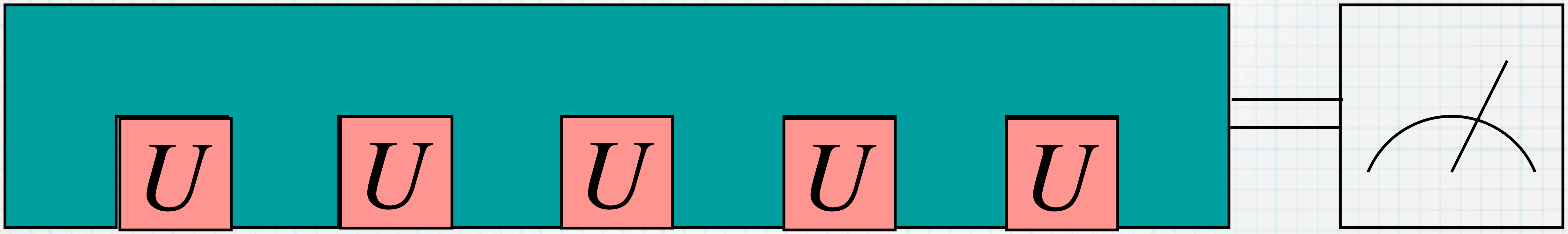
Comb= single-party strategy

For quantum protocols and causal networks:
see G. Gutoski and J. Watrous, “Toward a General Theory of Quantum Games, quant-ph/0611234v2



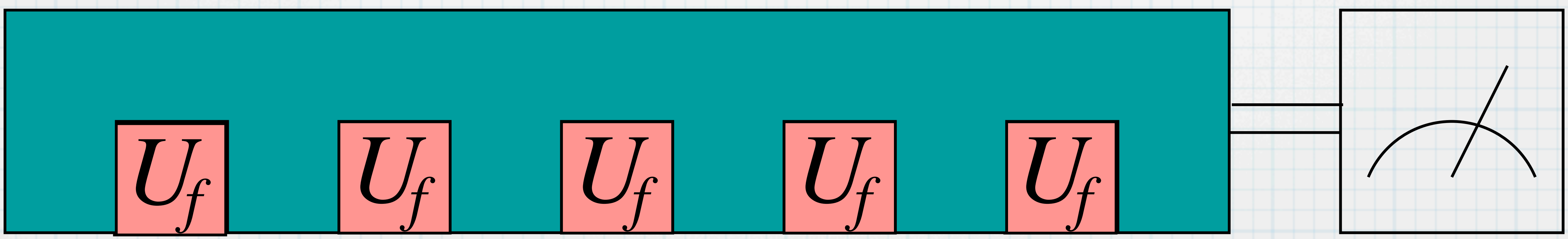
Quantum bit committment

Optimal algorithms



Discrimination of equivalence classes of unitaries (oracles)

= generalization of Deutsch-Jozsa problem



Systematic method to determine the optimal algorithm

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Conclusions

PRL101 060401(2008)
arXiv:0804.0180
0803.3237
0806.1172
0807.5058

- New Quantum Estimation Theory, with multiple copies, and optimization of the setup → optimization of quantum circuits architecture, engineering high-precision operations
  - Quantum circuit board = quantum comb = supermap
  - Comb algebra (link-product)
  - Convex optimization method
- Applications:
  - estimation/discrimination of unitaries and memory channels
  - optimal process tomography
  - cloning of unitary transformations, quantum-algorithm learning, optimal quantum algorithms, quantum protocols