Quantum information encoded on Quantum Operations

Estimation, characterization, engineering of QO's

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Quantum Optics & Information Group

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Research group/collaborations

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C. Macchiavello (univ. researcher),

M. G. A. Paris (INFM researcher),

M. Sacchi (INFM postdoc),

O. Rudolph (ATESIT postdoc),

S. Virmani (EQUIP postdoc),

P. Lo Presti (phd student),

R. Mecozzi (graduated),

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Main focus on QO's instead of quantum states

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- Encoding on QO's: given a fixed input state ρ, the message m is encoded on it via $\rho \to \mathrm{E}_m(\rho)$. Anonymous $\rho \equiv$ encryption.

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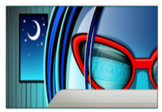
$\Rightarrow$ Without knowing $|\varphi_A\rangle$,  cannot tell m without significant error.

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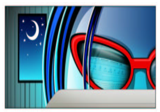
$\Rightarrow$ Without knowing $|\varphi_A\rangle$,  cannot tell m without significant error.

$\Rightarrow$ The function $f : m \rightarrow U_m^B$ can be regarded as a quantum one-way function with trapdoor information given by the knowledge of the actual input state $|\varphi_A\rangle$.

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  1.  prepares the Hilbert space H with the anonymous state $|\varphi\rangle \in$ H. He then sends H to  .
  2.  modulates the value b of the committed bit on a QO acting on the anonymous state $|\varphi\rangle$ and sends the output back to  .

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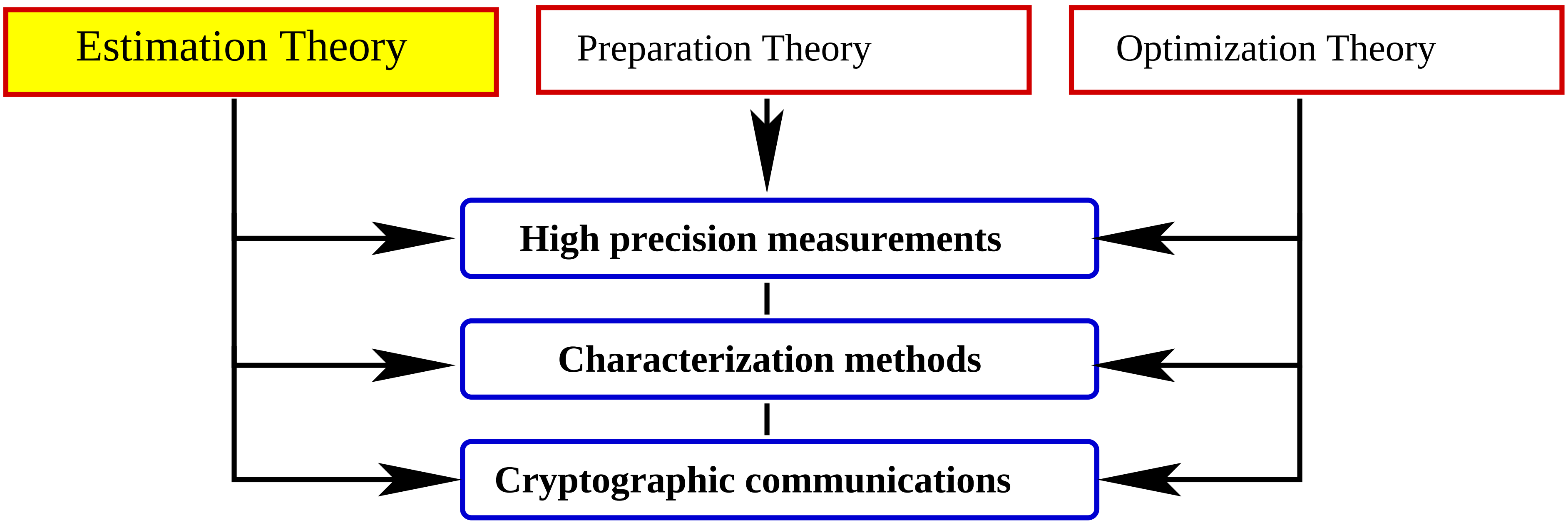

Discrimination among QO’s, POVM’s for estimating QO’s, tomographic characterization

Mayers, PRL **78** 3414 (1997); Lo and Chau, PRl **78** 3410 (1997),H. P. Yuen, quant-ph/0109055 –

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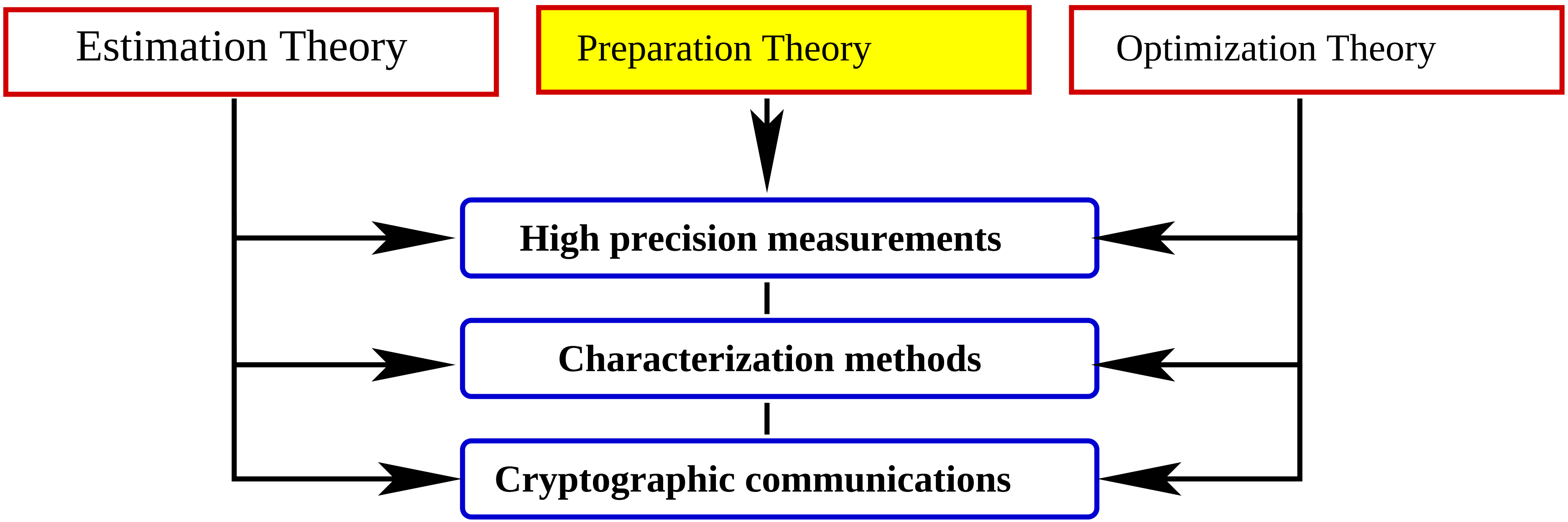

Which apparatuses for achieving a QO −−> Classification of unitary extensions of QO’s, ...

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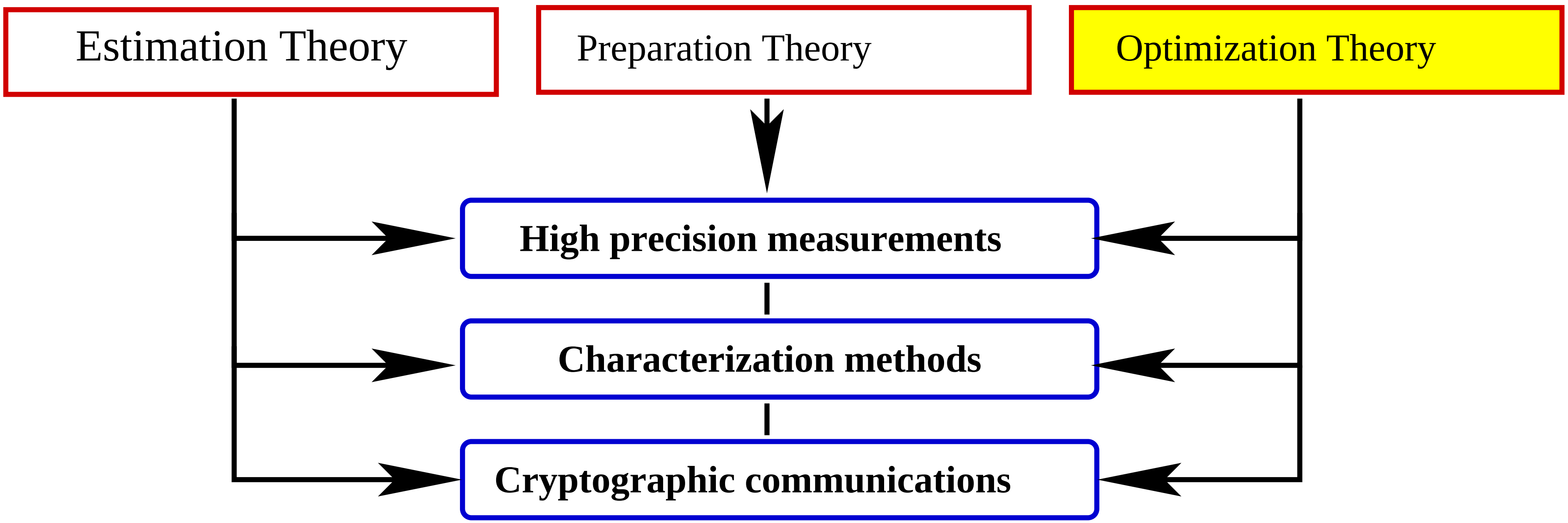

Which is the optimal QO to achieve a given purpose [in terms of a cost function]

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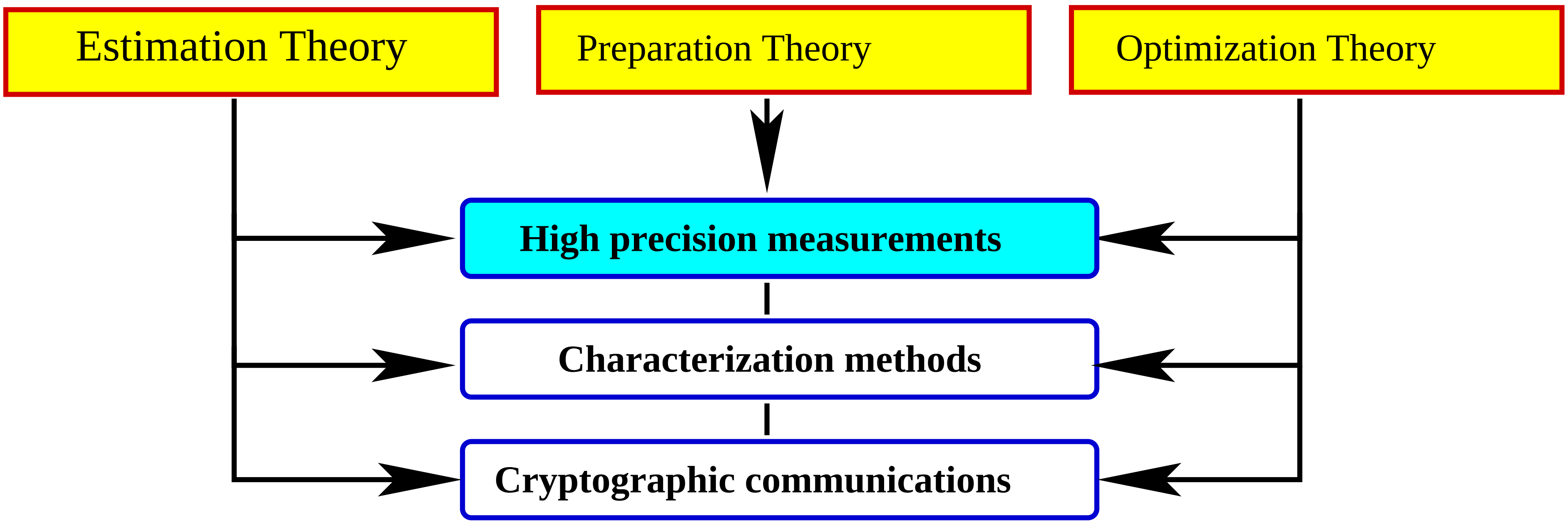

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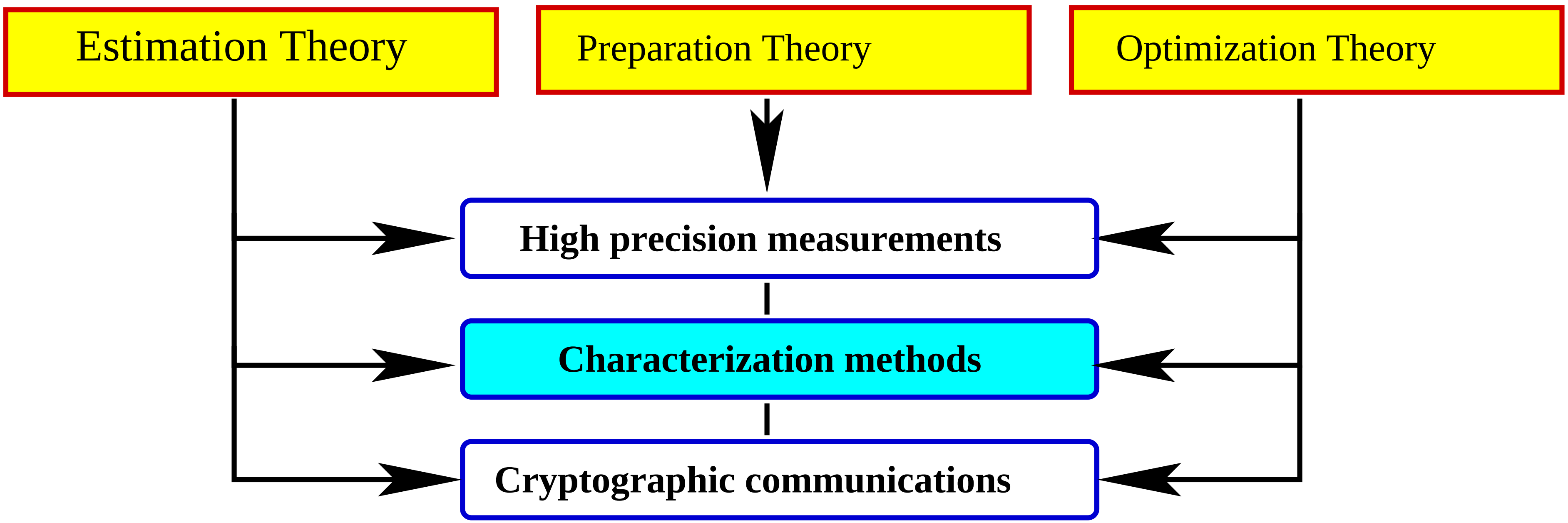

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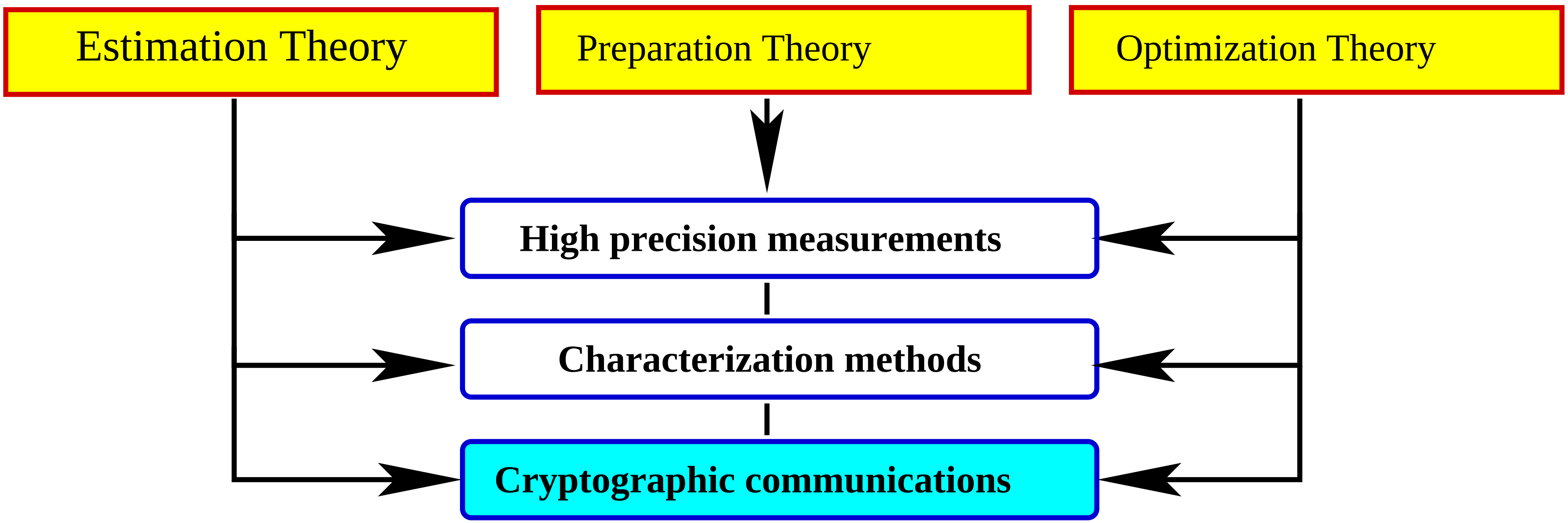

Quantum cryptography with anonymous states = encoding information on maps

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Main results on QO theory

1) Optimal discrimination between QO's (unitary)

2) Tomographic characterization of QO's using entangled input

3) Classification of all unitary extensions of QO's, extremal QO's and POVM's

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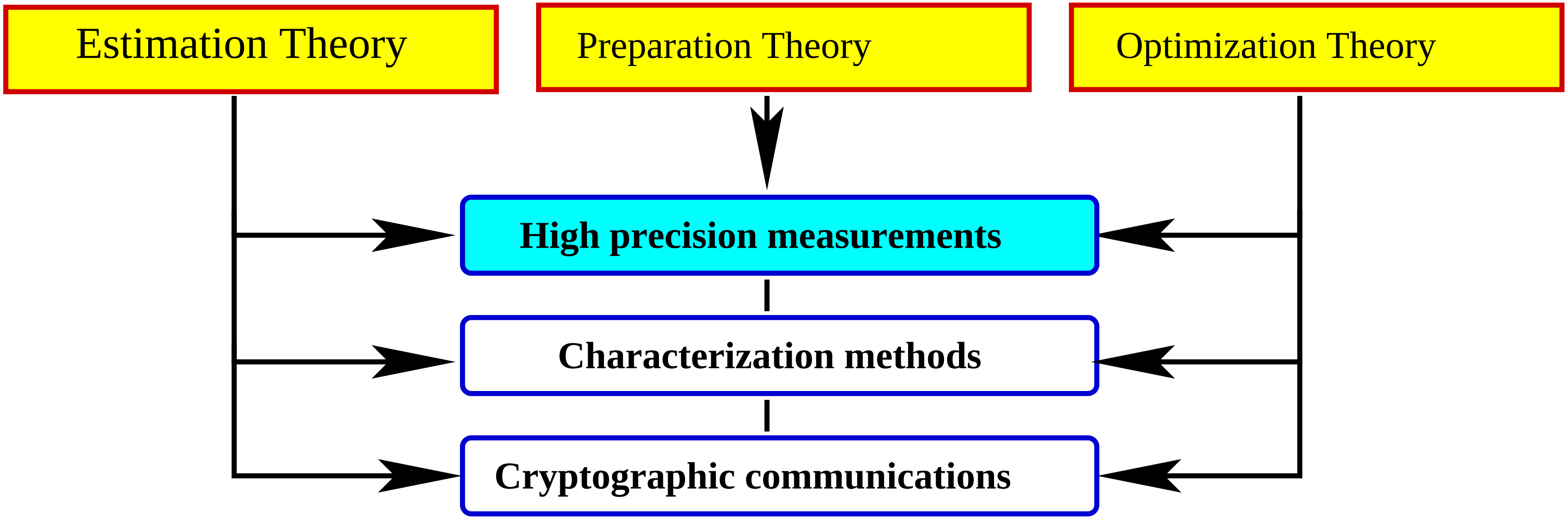

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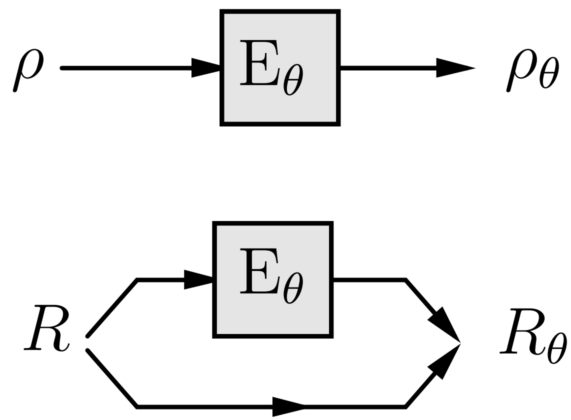

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  - Covariant discrimination: the Holevo bound is increased exactly by the amount of entanglement of the input state.

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3. One has the phenomenon of perfect discrimination between any two unitaries with a finite number N of copies of the QO (compare with *state* discrimination).

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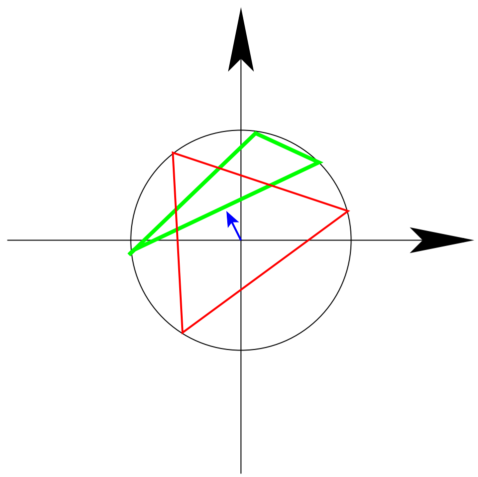

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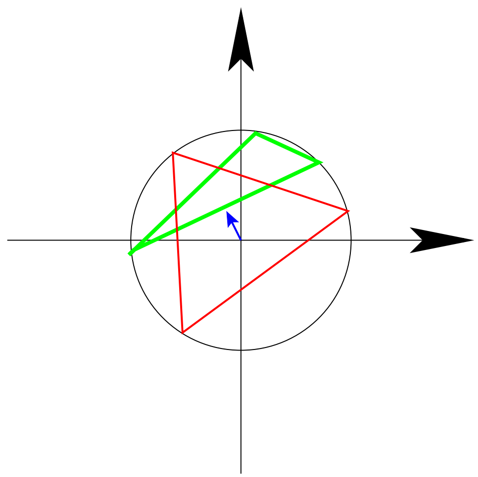

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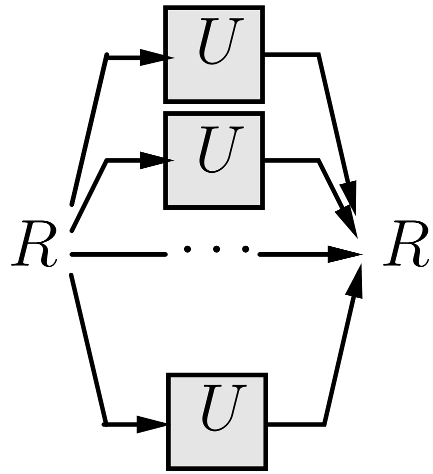

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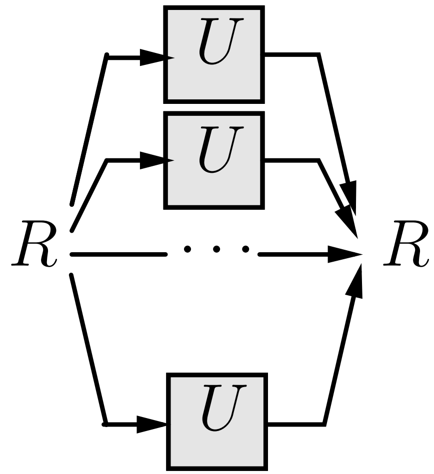

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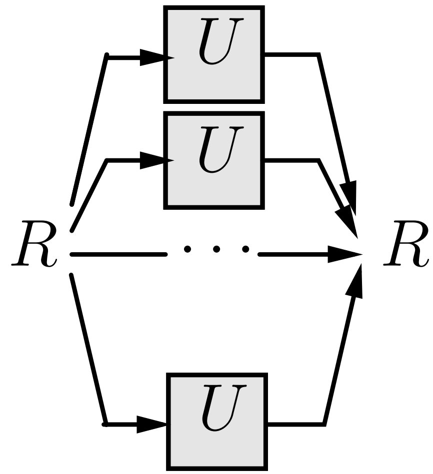

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- Conclusion: *the discrimination is always exact for sufficiently large* N! [see also Acín, quant-ph/0102064].

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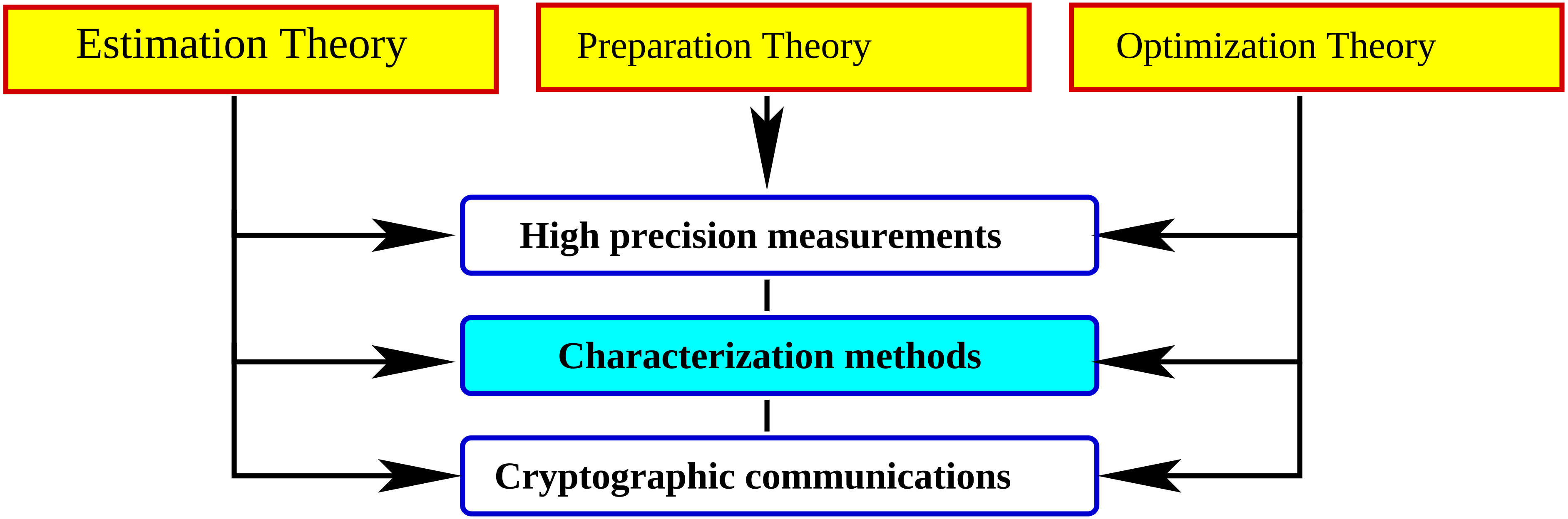

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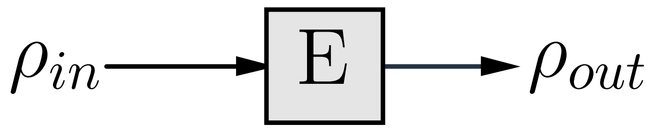

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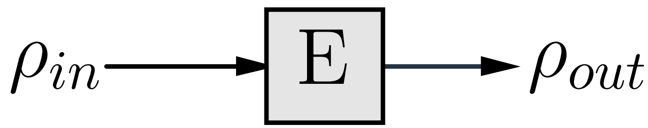

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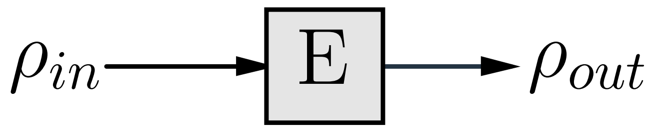

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G. M. D'Ariano and P. Lo Presti, Phys. Rev. Lett. **86** 4195 (2001) –

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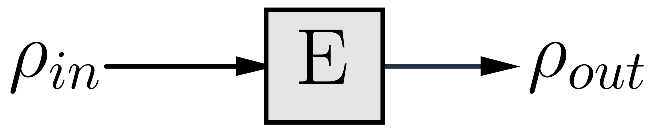

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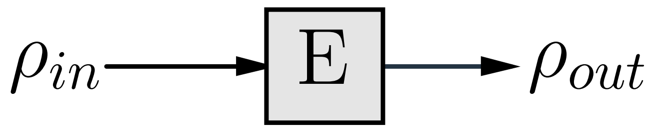

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- However, the availability of a basis of states in the lab is a very hard technological problem.

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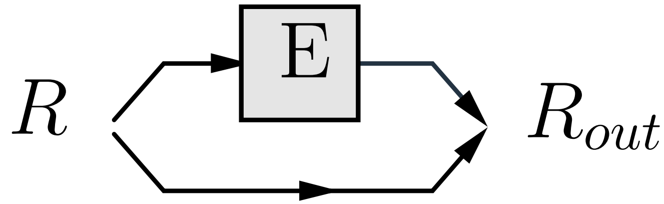

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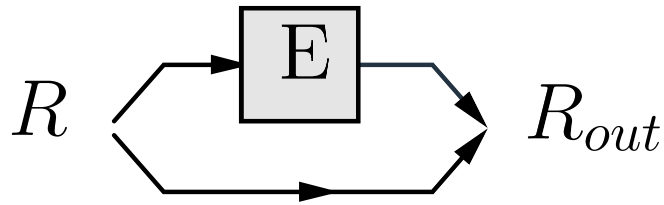

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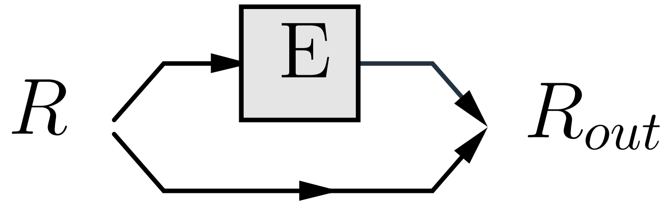

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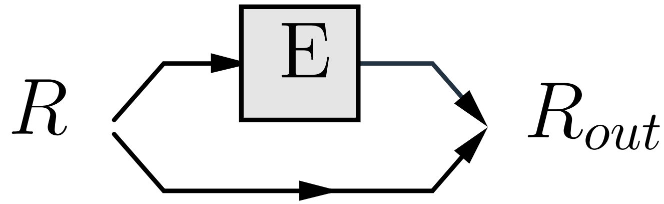

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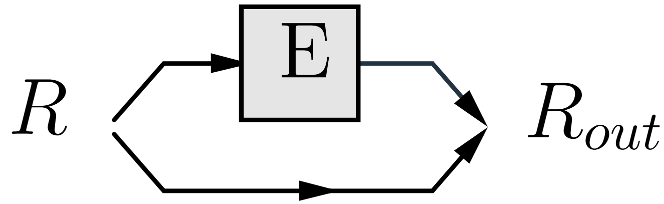

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- The method is very robust to noise [a state remains faithful under almost any kind of noise, e. g. depolarizing, etc].

G. M. D'Ariano and P. Lo Presti, unpublished –

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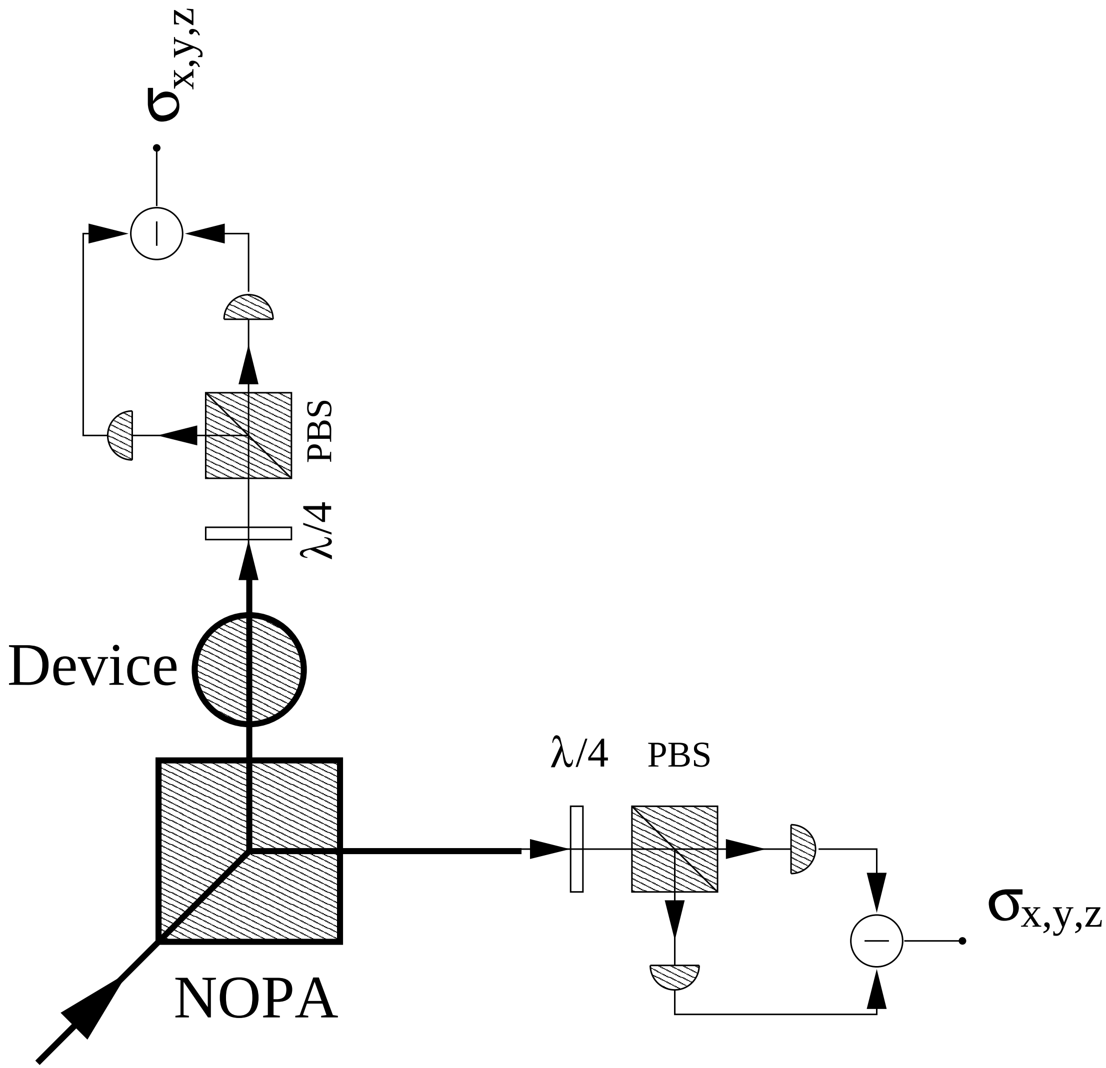

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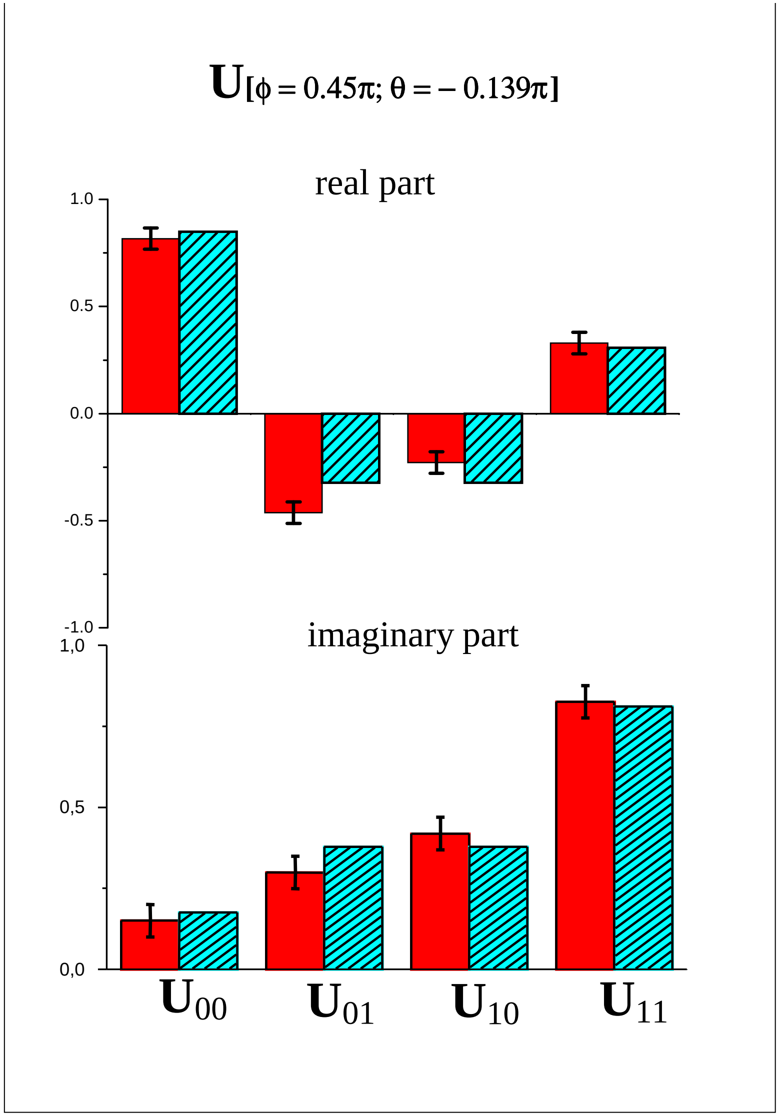
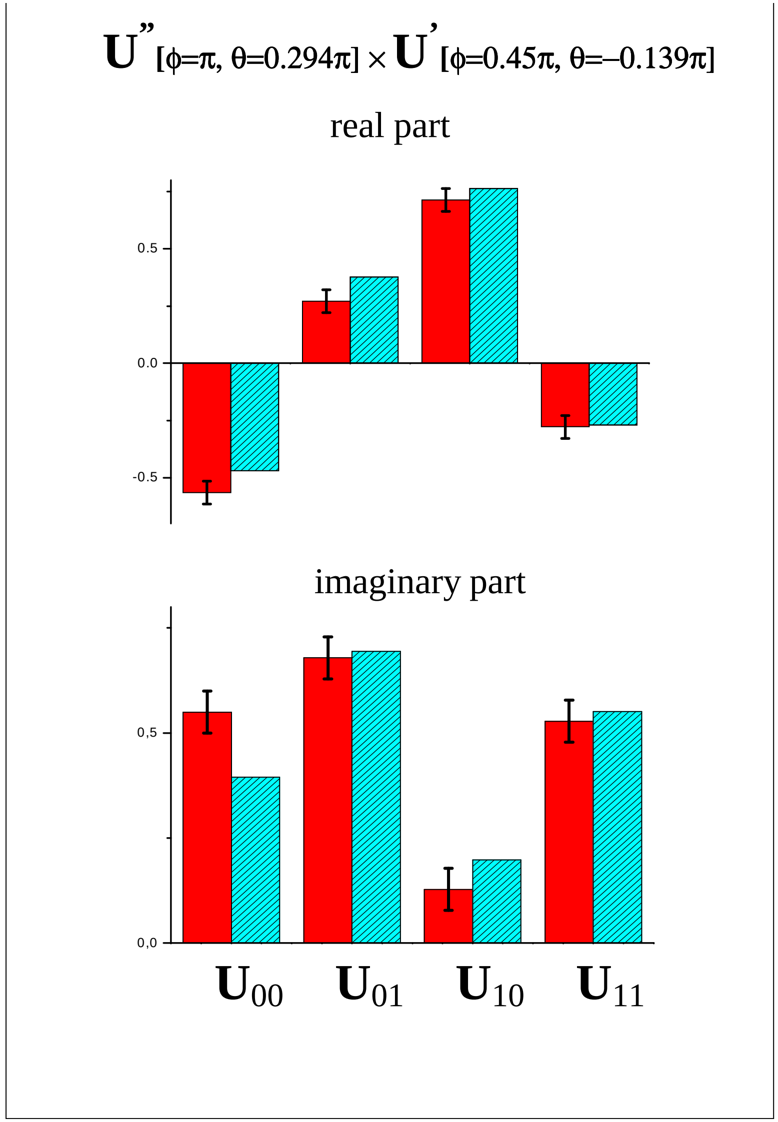
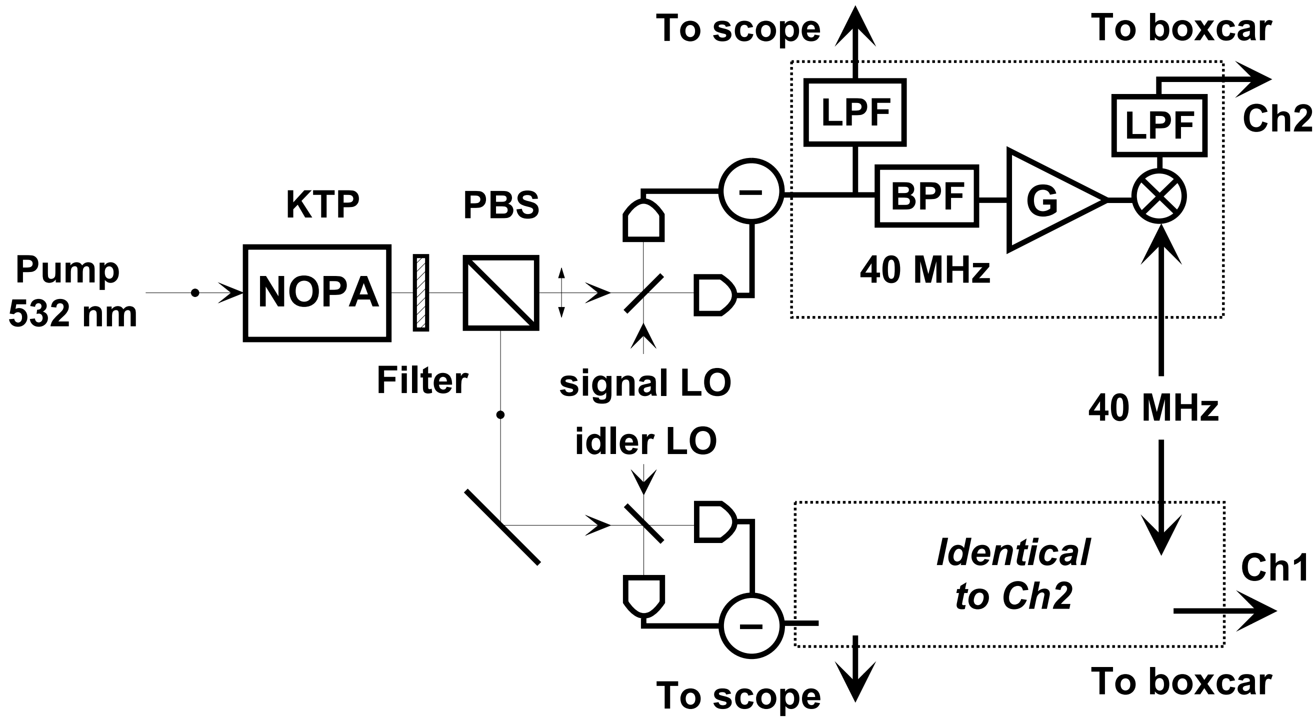




F. De Martini, G. M. D’Ariano, A. Mazzei, and M. Ricci, quant-ph/ – [17/32]

Tomography of a cv device

- Feasibility study for tomography of a displacer



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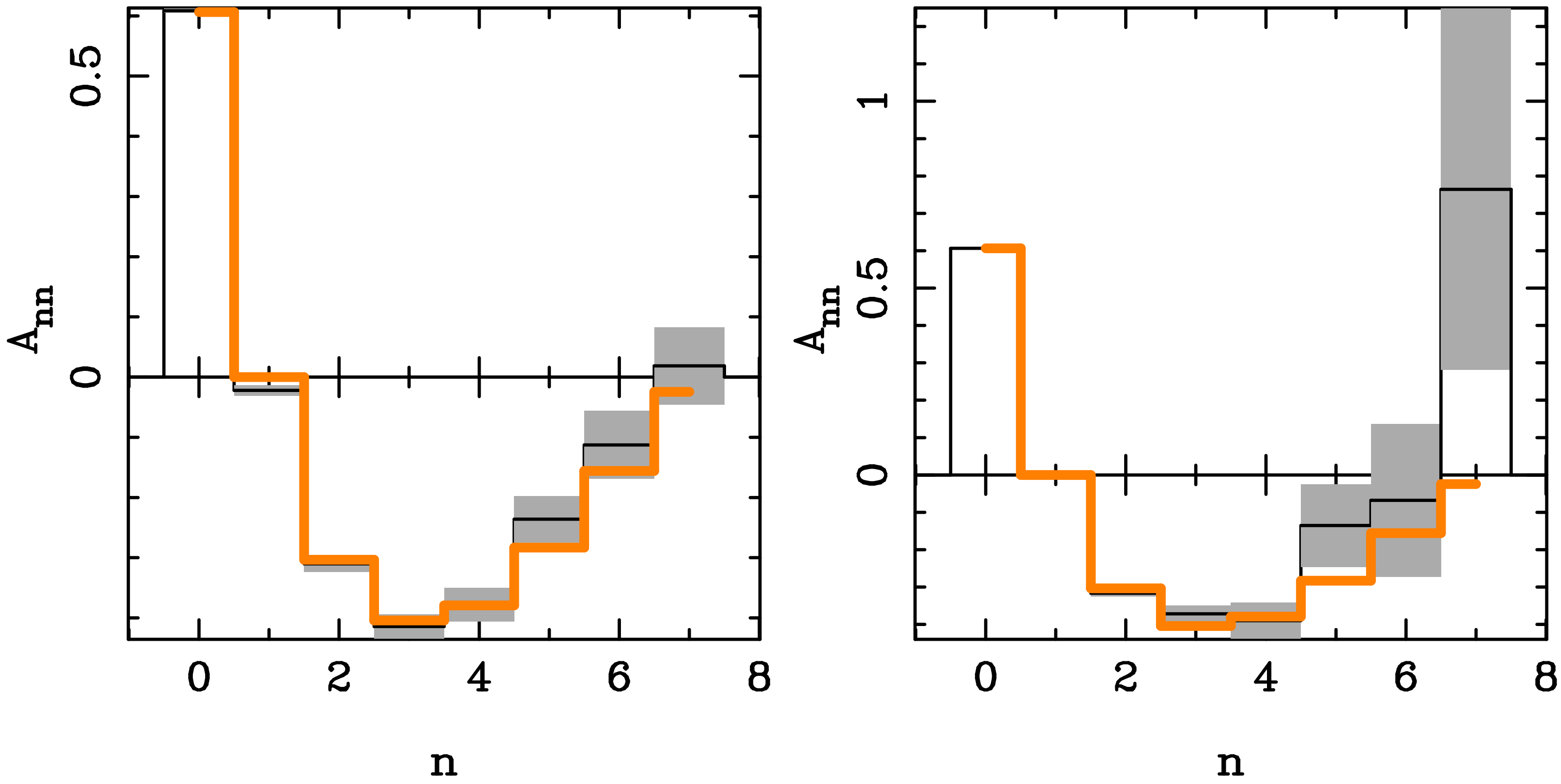

Left: $z = 1$, $\bar{n} = 5$, $\eta = 0.9$, and 150 blocks of 10^4 data have been used. Right: $z = 1$, $\bar{n} = 3$, $\eta = 0.7$, and 300 blocks of $2 \cdot 10^5$ data have been used.

D'Ariano and Lo Presti, PRL **86** 4195 (2001); Vasilyev, Choi, Kumar, and D'Ariano, PRL **84** 2354 (2000) –

Classification of QO extensions

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1) Optimal discrimination between QO's (unitary)

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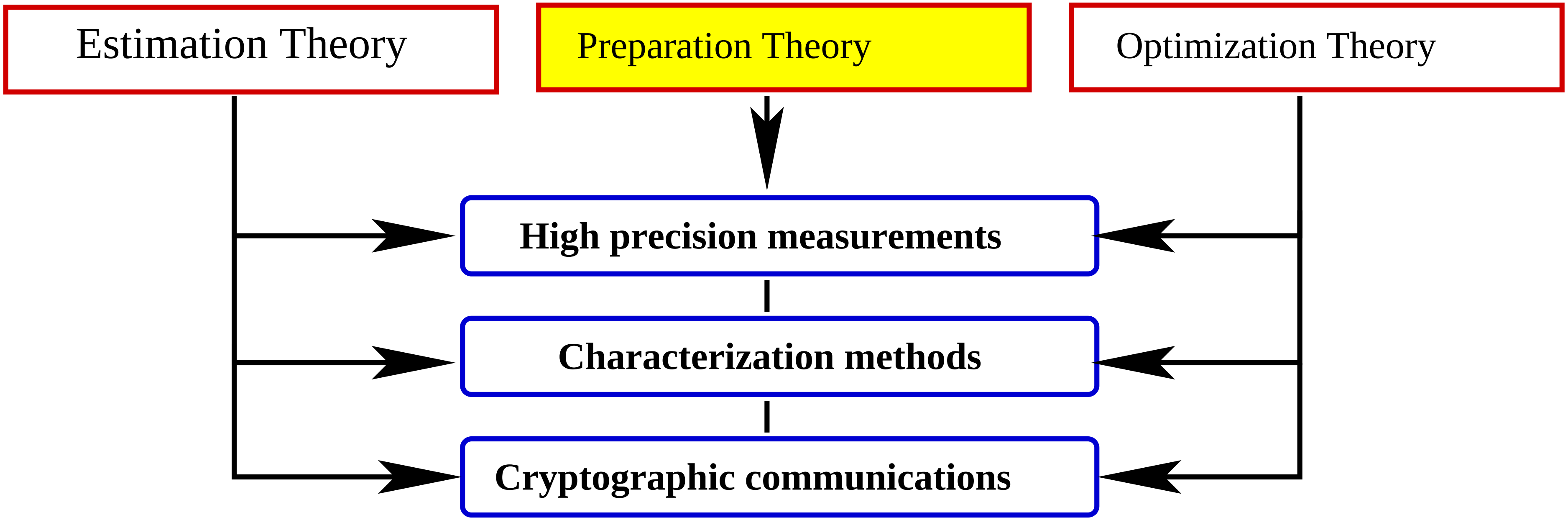

Which apparatuses for achieving a QO −−> Classification of unitary extensions of QO’s, ...

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- $(\mathsf{H} \oplus \mathsf{D}) \otimes \mathsf{A} \simeq \mathsf{K} \otimes \mathsf{F}, \qquad \left(\mathrm{rank}(\mathrm{E}) + \left\lfloor \frac{\mathrm{rank}(I_{\mathsf{H}} - \mathrm{E}^\tau(I_{\mathsf{K}}))}{\dim(\mathsf{K})} \right\rfloor\right) \dim(\mathsf{K}) \geq \dim(\mathsf{H})$

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where $\Sigma_{\mathsf{F}} = \sum_i |\sigma_i\rangle\langle\sigma_i|_{\mathsf{F}}$, and

$$\dim \mathsf{F} \geq \mathrm{rank}(\Sigma_{\mathsf{F}}) \geq \mathrm{rank}(\mathrm{E}).$$

Extremal QO's and POVM's

D'Ariano and Buscemi (unp.); D'Ariano, Lo Presti and Mecozzi (unp.); Parthasaraty, Inf. Dim. Anal. **2** 557 (1999)

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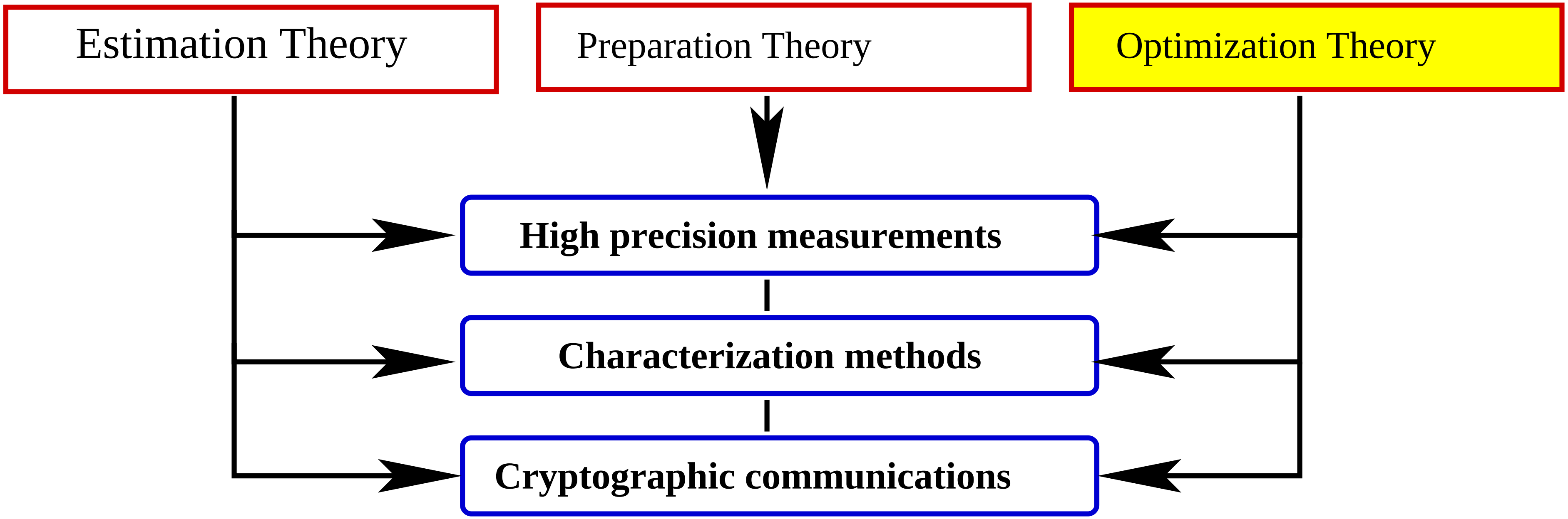

Which is the optimal QO to achieve a given purpose [in terms of a cost function]

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  - *Theorem:* A POVM $\{P_e\}_{e\in\mathsf{E}}$ with spectral resolution $P_e = \sum_i |v_i^{(e)}\rangle\langle v_i^{(e)}|$ is extremal if and only if the operators

$$|v_i^{(e)}\rangle\langle v_j^{(e)}|,\quad \text{for all events } e\in\mathsf{E},\ \text{and all } i,j$$

are linearly independent.

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G. M. D'Ariano, QCM&C 2002, Boston (preprint available)

The Quantum Bit Commitment

- Commitment:  provides  with a piece of evidence that she has chosen a bit $b = 0, 1$ which she commits to him.
- Opening: Later  will open the commitment, revealing b to , and proving that it is indeed the committed bit with the evidence in Bob's possession, i. e.  will check the commited bit.

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- Both parties are supposed to possess unlimited technology, and the protocol is said *unconditionally secure* if neither Alice nor Bob can cheat with significant probability of success as a consequence of physical laws.

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- Since  has unlimited technology, she can always achieve the map *knowingly*, i. e. she has the option of achieving each QO as a *perfect pure measurement*.

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$\Rightarrow$ For simplicity, we focus attention on non aborting protocols.

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- On the other side,  can try to discriminate between the two mixtures of QO's by launching his own EPR attach at the very beginning of the commitment, by entangling the anonymous state with a system in his possession.

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- Alice EPR-cheating transformation: unitary V on P $\otimes$ F: corresponds to change the Kraus decomposition from $\{E_J^{(0)}\} \to \{E_J^{(0)}(V)\}$

Bounds for cheating probabilities

$$P_c^A(V,\varphi) \geq \sqrt{1 - \sum_J \left\| E_J^{(0)}(V) - E_J^{(1)} \right\|^2},$$

$$2P_c^B - 1 \leq \left\| \mathrm{M}^{(1)} - \mathrm{M}^{(0)} \right\|_{cb} \leq \sqrt{\sum_J \left\| E_J^{(0)}(V) - E_J^{(1)} \right\|^2}.$$

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- However, it has not been proved that there is a V such that

$$\sum_J \left\| E_J^{(0)}(V) - E_J^{(1)} \right\|^2 \leq \omega\left(\left\| \mathrm{M}^{(1)} - \mathrm{M}^{(0)} \right\|_{cb} \right),$$

with $\omega(\varepsilon)$ vanishing with ε.

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