Soliton equations, τ-functions and coherent states

G. M. D'ARIANO

Dipartimento di Fisica "A. Volta"
Universita' di Pavia, Italy

and

M. G. RASETTI

Dipartimento di Fisica
Politecnico di Torino, Italy

Introduction

It has been shown by M. Jimbo and T. Miwa [1] in a section of this book that the Bäcklund groups (i.e. the groups of transformations of solutions) for soliton equations are infinite dimensional Lie groups whose Lie algebras of infinitesimal generators are Kac-Moody algebras of infinite-order differential operators, called vertex operators. They have also shown that if one realizes the space of the complex polynomial algebra in terms of a Fock space of charged Fermions, writing the differential operators in terms of Fermi operators, the soliton equations become nothing but the defining differential equations of the group orbit of the highest weight vector in an infinite dimensional Fock space.

We will show that this algebraic treatment of soliton equations has a nice quantum mechanical interpretation: the solutions of the soliton equations can be viewed as quantum coherent states of an harmonic Fermi gas and the soliton dynamical evolution is thus mapped into a quantum hamiltonian evolution. The latter, which is coherence preserving, can be mapped back once more into a classical hamiltonian flow which corresponds to a succession of infinitesimal Bäcklund transformations.

Coherent states

One can define abstractly quantum coherent states associated to a Lie group $\mathcal{G}$ as follows [3, 4].

Let $\mathcal{U}$ a unitary irreducible representation of the Lie group $\mathcal{G}$ acting on the Hilbert space S of the states of the dynamical system. For every fixed "origin vector" $| \omega >\in S$ the manifold $\mathcal{M}_\omega$ of the coherent states is identified with the $\mathcal{G}$-orbit of the vector $| \omega >$ in S:

$$\mathcal{M}_\omega = U(\mathcal{G}) \mid \omega > \subset S \tag{1}$$

By definition one has:

$$\mathcal{M}_\omega \sim \mathcal{G}/K_\omega \tag{2}$$

i.e. the coherent states $| w >_\omega$ are labelled by points of the left coset space $w \in \mathcal{G}/K_\omega$, where K_ω is the stability subgroup of the vector $| \omega >$.

In general it is convenient to enlarge the unitary representation $\mathcal{U}$ to a holomorphic representation $\mathcal{T}$ of the complexified group $\mathcal{G}^c$.

If $\mathcal{M}_\omega$ is compact one has:

$$\mathcal{G}^c/K^c_\omega \sim \mathcal{G}/K_\omega \tag{3}$$

where K^c_ω is the stability subgroup of $| \omega >$ in $\mathcal{G}^c$. There follows that:

$$\mathcal{M}_\omega = \mathcal{T}(\mathcal{G}^c) \mid \omega > \subset S \tag{4}$$

Eq. (4) can also be interpreted as the definition of the coherent state manifold for the complex Lie group $\mathcal{G}^c$.

For a semisimple Lie group $\mathcal{G}^c$ the characterization of the stability subgroup is very simple if one chooses as origin vector the highest weight vector $| \lambda >$. In this case one has [5, 6]:

$$K^c_\lambda = e^{k_\lambda} \tag{5a}$$

$$k_\lambda = h \oplus g^+ \oplus o^-_\lambda \tag{5b}$$

$$o^-_\lambda = \text{span}\{x_{-\alpha} \in g_{-\alpha} \mid \alpha \in \Delta^+, (\alpha, \lambda) = 0\} \tag{5c}$$

where exp is the exponential map, h is the Cartan subalgebra of $g = \text{Lie}(\mathcal{G}^c)$, $g^+ = \bigoplus_{\alpha\in\Delta^+} g_\alpha$ is the positive root-space subalgebra and o^- is the subalgebra of negative root-spaces whose roots are orthogonal to the highest weight λ (Δ^+ is the positive roots sublattice). The stability subgroup K^c_λ is thus isomorphically identified with a parabolic subgroup of $\mathcal{G}^c$ as it contains the Borel subgroup $B = \exp(h \oplus g^+)$ and coincides with the latter only when λ belongs to the interior of the dominant Weyl chamber (i.e. $(\lambda, \alpha_i) > 0 \quad \forall \alpha_i \in \Pi \equiv$ simple roots set). The above construction gives a local chart in $\mathbb{C}^{\mathcal{N}}$ for M_λ:

$$| \varsigma >_\lambda = \exp\Big(\sum_{\alpha \in \Lambda_\lambda} \varsigma^\alpha x_{-\alpha} \Big) \, | \, \lambda > \tag{6a}$$

$$x_\alpha \in g_\alpha, \quad \Lambda_\lambda = \{ \alpha \in \Delta^+ \mid (\alpha, \lambda) > 0 \} \tag{6b}$$

$$\varsigma = \{ \varsigma^\alpha \} \in \mathbb{C}^{\mathcal{N}} \qquad \mathcal{N} = |\Lambda_\lambda| \tag{6c}$$

(In the following we shall drop the index λ of the ket $| \varsigma >_\lambda$ whenever not necessary).

Thus M_λ is an almost complex manifold: indeed it is a Kaehler manifold with metric given by:

$$ds^2 = 2 \sum_{\alpha, \beta} g_{\alpha\overline{\beta}} d\varsigma^\alpha \, d\overline{\varsigma}^\beta \tag{7a}$$

$$g_{\alpha\overline{\beta}} = \frac{\partial^2 F}{\partial \varsigma^\alpha \partial \overline{\varsigma}^\beta} \tag{7b}$$

$$F(\varsigma, \overline{\varsigma}) = \ln < \varsigma \mid \varsigma > \tag{7c}$$

(The function F is positive definite as a consequence of Schwartz's inequality

$$< \varsigma \mid \varsigma > \geq | < \varsigma \mid \lambda > |^2 / < \lambda \mid \lambda > = 1).$$

The physical content of the above algebraic definition of coherent states lies mainly in their dynamical behaviour.

The quantum propagator between two (normalized) coherent states can be written as a path integral of the form [7]:

$$\begin{aligned} < \varsigma'', t'' \mid \varsigma', t' > &= < \varsigma'' \mid \exp\Big[-i\hat{H}(t'' - t')/\hbar \Big] \mid \varsigma' > = \\ &= \int \mathcal{D}[\varsigma(t)] \exp\left\{ \frac{i}{\hbar} S \right\} \end{aligned} \tag{8}$$

where the action functional is given by:

$$S[\varsigma(t)] = \int_{t'}^{t''} L dt = \int_{t'}^{t''} < \varsigma(t) \mid i\hbar \partial_t - \hat{H} \mid \varsigma(t) > dt \tag{9}$$

with the Lagrangian:

$$L = \frac{i\hbar}{2} \sum_{\alpha \in \Lambda_\lambda} \{\dot{\varsigma}^\alpha \partial_{\varsigma^\alpha} F(\varsigma, \bar{\varsigma}) - \dot{\bar{\varsigma}}^\alpha \partial_{\bar{\varsigma}^\alpha} F(\varsigma, \bar{\varsigma})\} - H(\varsigma, \bar{\varsigma}) \qquad (10)$$

$H(\varsigma, \bar{\varsigma})$ denotes the diagonal element $< \varsigma \mid \hat{H} \mid \varsigma >$ of the system Hamiltonian $\hat{H}$. Finally the measure in (8) is:

$$\mathcal{D}[\varsigma(t)] = \prod_{t' < t < t''} d\mu(\varsigma(t)) \qquad (11)$$

$$d\mu(\varsigma) = c_\lambda \det(g_{\alpha\bar{\beta}}) \Lambda_{\alpha \in \Lambda_\lambda} d\varsigma^\alpha \wedge d\bar{\varsigma}^\alpha$$
$$\int_{\mathcal{M}_\lambda} d\mu(\varsigma) = 1 \qquad (12)$$

where c_λ is a normalization constant.

The stationary phase approximation ($\delta S = 0$) of eq. (8) leads to the Euler-Lagrange equations for the trajectory $\varsigma(t)$ which can be put in the Hamilton's form:

$$i\hbar \sum_{\beta \in \Lambda_\lambda} g_{\alpha\bar{\beta}} \dot{\varsigma}^\beta = \frac{\partial H}{\partial \bar{\varsigma}^\alpha}$$
$$-i\hbar \sum_{\beta \in \Lambda_\lambda} g_{\alpha\bar{\beta}} \dot{\bar{\varsigma}}^\beta = \frac{\partial H}{\partial \varsigma^\beta} \qquad (13)$$

Thus $\mathcal{M}_\lambda$ is interpreted as a curved canonical phase space for the system with metric given by (7 b). Also eq. (9) shows that the coherence preserving Schrödinger evolution of the quantum state coincides with the classical lagrangian flow on the phase space. Infact, if one computes the variation of the action one gets, after an integration by parts:

$$\delta S = \int_{t'}^{t''} \left\{ \left[\delta < \varsigma(t) \mid \right] (i\hbar \partial t - \hat{H}) \mid \varsigma(t) > + < \varsigma(t) \mid (-i\hbar \overleftarrow{\partial}_t - \hat{H}) \left[\delta \mid \varsigma(t) > \right] \right\} dt \qquad (14)$$

where:

$$\delta < \varsigma(t) \mid = \sum_{k=1}^{dim\, \mathcal{T}} \delta f_k(\varsigma(t)) < k \mid \qquad f_k(\varsigma) = < \varsigma \mid k > \qquad (15)$$

$\mid k >$ being a complete orthonormal set of vectors in $\mathcal{S}$.

It is clear from eq. (14) that if the time dependent coherent state $|\varsigma(t)\rangle$ satisfies the Schrödinger equation, the first variation of the action is zero. In other words, if the Schrödinger evolution, starting on the manifold $\mathcal{M}_\lambda$, remains on the manifold for every time t, the trajectory of the representative point on $\mathcal{M}_\lambda$ coincides with the classical Euler-Lagrange trajectory. Furthermore it is possible to characterize in algebraic terms the coherence preserving hamiltonian [5, 6]: it should be an element of the algebra $g = \mathrm{Lie}(\mathcal{G})$, if the latter is semisimple, whereas it can belong to a Levi extension (by a semisimple algebra) of g if this is solvable.

In summary, the given definition of coherent states permits to interpret also in the generalized case the coherent state manifold as the canonical phase space and moreover it implies that the quantum coherence-preserving time evolution coincides with the classical lagrangian one.

Summary of main results of τ-function theory

For the sake of simplicity we refer to the KP (Kadomtsev-Petviashvili) [8, 9] hierarchy, which is also the most basic.

The KP equation reads:

$$3u_{yy} - (4u_t - 6uu_x - u_{xxx})_x = 0 \tag{16}$$

The whole hierarchy involves of course an infinity of variables that we denote $z = \{z_i\}$ (with $z_1 = x$, $z_2 = y$, $z_3 = t$). Hirota's bilinearization technique [10] – upon setting $u = 2(\ln\tau)_{xx}$ – allows writing (16) in the form:

$$(D_x^4 + 3D_y^2 - 4D_xD_t)\tau\cdot\tau = 0 \tag{17}$$

where Hirota's bilinear differential operators are defined, for any polynomial P by:

$$P(D_x, D_y, D_t)f\cdot g = P(\partial_x, \partial_y, \partial_t)\cdot \left.\cdot\left[f(x+x', y+y', t+t')g(x-x', y-y', t-t')\right]\right|_{x'=y'=t'=0} \tag{18}$$

The Lie algebra of the infinitesimal Bäcklund transformation generators for eq. (7)is given by [1]:

$$\mathcal{A} = \mathrm{span}\{Z_{ij}(z,\partial)\} \oplus \mathbb{C} \tag{19}$$

where the generating function of the differential operators $Z_{ij}(z,\partial)$ writes:

$$Z(p,q) = \frac{q}{p-q}\left\{\exp\left[\sum_{n=1}^{\infty}(p^n - q^n)z_n\right]\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}(p^{-n} - q^{-n})\frac{\partial}{\partial z_n}\right]\right\}$$
$$Z(p,q) = \sum_{i,j\in\mathbf{Z}} Z_{ij}(z,\partial)p^i q^{-j} \tag{20}$$

where $\partial = \{\partial/\partial z_n\}$. For example the transformation:

$$\tau(z) \mapsto e^{aZ(p,q)}\tau(z) \qquad a \in \mathbb{C} \tag{21}$$

is a Bäcklund transformation for the bilinear KP hierarchy which maps a solution of eq. (17) to another solution.

The Lie algebra $\mathcal{A}$ is isomorphic to $\mathcal{GL}(\infty)$, i.e. the Lie algebra of the infinite dimensional sector diagonal matrices (that is, there exists an integer N such that the matrix elements $a_{ij} = 0$ for $|i-j| > N$).

The above description of the KP τ-function can be cast into an algebraic language. Consider the "vertex operators" [11] out of which (20) is constructed:

$$X(k) = \exp\left(\sum_{n=1}^{\infty} z_n k^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial}{\partial z_n}k^{-n}\right) \tag{22}$$

and its formal adjoint:

$$\overline{X}(k) = \exp\left(-\sum_{n=1}^{\infty} z_n k^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial}{\partial z_n}k^{-n}\right) \tag{23}$$

The above operators realize a correspondence between the space of the polynomial algebra $\mathbb{C}\,[z]$ and the Fock space $\mathcal{F}$ of charged Fermions $\{\psi_i, \overline{\psi}_i\}$, $i \in \mathbf{Z}$, by the Clifford algebra module isomorphism generated by the identification:

$$\psi_i = \hat{X}_i\,; \quad \overline{\psi}_i = \hat{\overline{X}}_i \tag{24}$$

where the $\hat{X}_i$'s and $\hat{\overline{X}}_i$'s are defined in the following way. Upon setting:

$$X(k) = \sum_{i\in\mathbf{Z}} X_i(z,\partial)k^i$$
$$\overline{X}(k) = \sum_{i\in\mathbf{Z}} \overline{X}_i(z,\partial)k^{-i} \tag{25}$$

consider copies $\{V_l\}$ of $\mathbb{C}[z]$. Then for $f_l(z) \in V_l$,

$$\begin{aligned} \hat{X}_i : V_l &\longrightarrow V_{l+1}; \quad f_l(z) \longrightarrow X_{i-l}(z,\partial) f_l(z) \\ \hat{\overline{X}}_i : V_l &\longrightarrow V_{l-1}; \quad f_l(z) \longrightarrow X_{i-l+1}(z,\partial) f_l(z) \end{aligned} \tag{26}$$

The Lie algebra (19) is now realized as follows:

$$\mathcal{A} = \mathrm{span}\{:\psi_i \overline{\psi}_j:\} \oplus C \tag{27}$$

where $:\;:$ denotes the usual normal ordered product defined according to Wick's theorem, the linear span is done in terms of $\mathcal{GL}(\infty)$ matrices and C is the center, spanned by the identity $\mathbf{I}$ and the non-trivial element

$$H_0 = \sum_{i \in \mathbf{Z}} :\psi_i \overline{\psi}_i: \tag{28}$$

It follows that the representation of $\mathcal{A}$ on the Fock space is reducible and the Fock space is decomposed into eigenspaces of H_0, the "charged subspaces":

$$\mathcal{F} = \oplus_n \mathcal{F}_n \tag{29}$$

Eqs. (26) define then an isomorphism between V_n and $\mathcal{F}_n$. Furthermore $\mathcal{A}$ has an Heisemberg subalgebra $\mathcal{E}$:

$$\mathcal{E} = \mathrm{span}\{H_n, \mathbf{I}; n \in \mathbf{Z} - \{0\}\} \tag{30}$$

$$H_n = \sum_{i \in \mathbf{Z}} :\psi_i \overline{\psi}_{i+n}: \quad ; \qquad n \neq 0 \tag{31}$$

with commutation relations:

$$[H_n, H_m] = n\delta_{n,-m}\mathbf{I} \tag{32}$$

The existence of such a subalgebra enables to construct an explicit realization of the isomorphism between the vector spaces V_n and $\mathcal{F}_n$ and between the algebras of operators acting on them. The isomorphism is realized by the following map:

$$\begin{aligned} \mathcal{F} = \oplus_n \mathcal{F}_n &\longrightarrow V = \oplus_n V_n \\ a \mid 0 > &\longrightarrow \oplus_n < n \mid e^{H(z)} a \mid 0 > \end{aligned} \tag{33}$$

$$H(z) = \sum_{n=1}^{\infty} z_n H_n \tag{34}$$

where a is an arbitrary operator of the Clifford algebra, $|\,0>$ is the vacuum vector, $|\,n>\in \mathcal{F}_n$ ($|\,n>=\psi_{n-1}\ldots\psi_0\,|\,0>$, $n>0$; $|\,n>=\overline{\psi}_n\cdots\overline{\psi}_{-1}\,|\,0>$, $n<0$) is the vector of charge n selected as highest weight vectors for the irreducible components of the representation of $\mathcal{A}$ on $\mathcal{F}$. We can now use the above isomorphism to write the space of τ-functions for the KP hierarchy. As the Bäcklund group is transitive on the space of solutions and $\tau=1$ is a solution, the τ-functions manifold can be identified with the Bäcklund group orbit on the constant function $\tau=1$. The orbit can be written in algebraic terms using the isomorphism (33) for a fixed copy V_n of the polynomial algebra as follows:

$$\tau_n(z;\gamma)=<n\,|\,e^{H(z)}\gamma\,|\,n> \qquad \gamma\in\exp(\mathcal{A}) \tag{35}$$

τ-function theory in the coherent states language

Comparing eq. (35) with the definition (4), it appears clearly that the τ-functions are nothing but coherent states representatives associated to the Lie group $\mathcal{G}=\exp(\mathcal{A})$:

$$\begin{gathered}\tau_n(z;\gamma)\equiv\Psi_n(\varsigma,z)={}_n\!<\Psi\,|\,\varsigma>_n \qquad \varsigma\in\mathbb{C}^\infty\\ {}_n\!<\Psi\,|=<n\,|\,e^{H(z)}\\ |\,\varsigma>_n:\quad \text{coherent state for}\quad \mathcal{G}=\exp(\mathcal{A})\end{gathered} \tag{36}$$

The characterization of the stability subgroup of $|\,n>$ is very similar to that for a semisimple Lie group. The subalgebras h, $g^{\pm}$ of $\mathcal{A}=\mathcal{GL}(\infty)$ are given, in the charged Fermion representation, by [1]:

$$\begin{aligned} g^+ &=\text{gen}\{e_i=\psi_{i-1}\overline{\psi}_i\}\\ g^- &=\text{gen}\{f_i=\psi_i\overline{\psi}_{i-1}\}\\ h &=\text{span}\{h_i=\psi_{i-1}\overline{\psi}_{i-1}-\psi_i\overline{\psi}_i\} \qquad i\in\mathbf{Z}\end{aligned} \tag{37}$$

where gen{ } denotes the algebra generated via commutations by the operators in the curly brackets. The vector $|\,n>$ is an highest weight vector for the irreducible component of the representation of $\mathcal{A}$ on the invariant subspace $\mathcal{F}_n$:

$$e_i\,|\,n>=0, \qquad h_i\,|\,n>=\delta_{in}\,|\,n> \qquad \forall i\in\mathbf{Z} \tag{38}$$

The root system and the root spaces of $\mathcal{A}$, are given by [12]:

$$\Delta^+ = \{\alpha_i + \alpha_{i+1} \cdots + \alpha_j; \quad i \leq j \in \mathbf{Z}\} \qquad g_{\alpha_i} = \mathrm{span}\{e_i\} \quad (39)$$

and the stability subalgebra of $| n >$ is given by (compare (5b), (5c)):

$$k_n = h \bigoplus g^+ \bigoplus o_n^- \quad (40)$$

$$o_n = \mathrm{span}\{x_{-\alpha} \in g_{-\alpha} \mid \alpha \in \Delta^+ - \Lambda_n\} \quad (41)$$

$$\Lambda_n = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j; \; i \leq j \in \mathbf{Z}; i \leq n \leq j\} \quad (42)$$

The most general coherence preserving Hamiltonian can be written as an hermitian element of $\mathcal{A}$ as follows:

$$\hat{H} = \sum_{i,j \in \mathbf{Z}} h_{ij} {:}\psi_i \overline{\psi}_j{:} \qquad h_{ij} = h^*_{ji} \in \mathbb{C} \quad (43)$$

which is the hamiltonian of an harmonic Fermi gas. The time evolution of the quantum dynamical system can be viewed as an infinite sequence of local infinitesimal contact Bäcklund transformations, in that the element $\gamma = \gamma(t) \in \mathcal{G} = \exp(\mathcal{A})$ representing the time evolution operator can be written

$$\gamma = e^{Z_1} \dots e^{Z_k}; \quad Z_1, \dots, Z_k \in \mathcal{A} \quad (44)$$

(The Z_i are locally nilpotent, i.e. for any $| v > \in \mathcal{F}$, one can find a sufficiently large integer M such that $Z_i^M \mid v >= 0$).

This procedure maps the original non-linear classical system into an equivalent quantum system.

On the other hand, under the action of the same Hamiltonian the representative point $\varsigma(t)$ of the coherent state over the phase space $\mathcal{M}_n$ evolves in time according to the lagrangian dynamics of a system of (infinitely many) canonical degrees of freedom, thus defining an hamiltonian flow – once more classical – on $\mathcal{M}_n$ itself.

In conclusion, the identification of the τ-function manifold with that of quantum coherent states, permits the mapping of the non linear soliton evolution into a quantum Schrödinger evolution and simultaneously into a classical hamiltonian flow on the coherent states manifold interpreted as a canonical phase space.

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