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COHERENT STATES AS SOLITON SOLUTIONS

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The totality of solutions for a hierarchy of equations in soliton theory is identified with the set of coherent states associated with a quantum system of infinitely many harmonically interacting fermions.

It has long been known that the classical dynamics associated with a quantum magnetic chain, exhibits both wave and soliton solutions [1]. In present paper we discuss a possible fundamental structure whereby such a behavior can be generated [2]. In order to make the discussion more definite, we shall focus our attention on the specific case in which the semi-classical approximation to the quantum dynamical problem turns out to be exact. The latter requirement is equivalent to determining the coherent states of the system in such a way that the Hamiltonian preserves coherence. In turn this can be obtained by individuating first the dynamical algebra $\mathfrak{g}$ of the system, and constructing then the coherent states as generalized coherent states of such an algebra. The largest dynamical algebra one can recognize in this scheme is $gl(\infty)$, whose coherent states are indeed the soliton solutions of the whole Kadomtsev–Petviashvili (KP) hierarchy. In Hirota's [3] bilinear differential form the KP equation writes, in terms of the so called τ-functions,

$$\left(D_x^4 + 3D_y^2 - 4D_x D_t\right)\tau \circ \tau = 0, \tag{1}$$

where, for any polynomial P

$$\begin{aligned} P(D_x, D_y, D_t) f \circ g &= P(\partial_{x'}, \partial_{y'}, \partial_{t'}) \\ &\times [f(x+x', y+y', t+t') \\ &\times g(x-x', y-y', t-t')]|_{x'=y'=t'=0}. \end{aligned} \tag{2}$$

The customary form of eq. (1) in terms of the variable $u = 2(\ln \tau)_{xx}$ has the form

$$3u_{yy} - (4u_t - 6uu_x - u_{xxx})_x = 0. \tag{3}$$

So far the problem of recognizing the dynamical algebra of a given dynamical system has no general solution and has to be handled case by case with ad hoc procedures. A well-known example is the method adopted by Lieb, Mattis and Schultz [4] who were able to identify the algebra of the XY spin-1/2 chain in a transverse field as su(2). It is interesting to notice that in order to do so, they had to resort to mapping the original model onto a system of harmonically coupled fermions. $gl(\infty)$ is the most general algebra spanned by bilinear fermion operator forms. It is therefore a suitable dynamical algebra for generic bilinear fermion Hamiltonians.

The generalized coherent states associated to the Lie group $\mathscr{G}$ of $\mathfrak{g}$ that we consider [5] constitute an overcomplete set of quantum states, labelled by a point in a Kählerian manifold $\mathscr{M}$, which can be identified as the classical phase space of the system. If $\mathscr{U}$ denotes a unitary irreducible representation of $\mathscr{G}$ acting on the Hilbert state space $\mathscr{S}$ of the dynamical system, and $|\omega\rangle$ is a fixed vector in $\mathscr{S}$; $\mathscr{M}$ is defined as

$$\mathscr{M} \sim \mathscr{G}/\mathscr{K}_\omega, \tag{4}$$

where $\mathscr{K}_\omega$ is the subgroup of $\mathscr{G}$ which leaves $|\omega\rangle$ invariant (up to a phase), and the coherent states

$$\{|\zeta\rangle\} = \mathscr{U}(\mathscr{G})|\omega\rangle \tag{5}$$

are the points of $\mathscr{M}$ (orbit of $|\omega\rangle$ under $\mathscr{G}$). The states have the following features:

i) The fact that $\mathscr{G}$ is a dynamical group implies that the time evolution of a state initially represented by a point on $\mathscr{M}$ is a path entirely in $\mathscr{M}$ [6].

ii) The quantum Schrödinger evolution path in $\mathscr{M}$, defined by the propagator [7]

$$\begin{aligned} \langle \zeta'' t'' | \zeta' t' \rangle &= \langle \zeta'' | \exp[-i\hat{H}(t''-t')/\hbar] | \zeta' \rangle \\ &= \int \mathscr{D}[\zeta(t)]\, e^{iS\hbar}, \end{aligned} \tag{6}$$

where the action functional is given by

$$S[\zeta(t)] = \int_{t'}^{t''} L\,dt = \int_{t'}^{t''} \langle \zeta(t) | i\hbar\partial_t - \hat{\mathscr{H}} | \zeta(t) \rangle dt, \tag{7}$$

$\hat{\mathscr{H}}$ denoting the system Hamiltonian, coincides with the classical flow on $\mathscr{M}$ determined by the Euler–Lagrange equations and the lagrangian L defined in eq. (7)

iii) A set of local charts of canonical coordinates of the orbit can be constructed, which allows the description of the above time evolution as an infinite sequence of infinitesimal contact (Bäcklund) transformation.

We shall examine now the coherent states representatives for $gl(\infty)$, and show their relation to the Hirota's solution of the KP hierarchy.

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$gl(\infty)$ is the Lie algebra of infinite dimensional sector-diagonal matrices $\{a_{ij}\}$ ($a_{ij}=0$ for $|i-j|>N$ for some N) [8]. It has a realization in terms of products of fermion operators ψ_i, $\bar{\psi}_i$, $i\in\mathbb{Z}$ of the form $:\psi_i\bar{\psi}_j:$, where $::$ denotes the customary normal ordered product. Together with the identity $\mathbb{I}$, the element

$$H_0=\sum_{i\in\mathbb{Z}} :\psi_i\bar{\psi}_i: \tag{8}$$

generates the center of $\mathfrak{g}$. Moreover $\mathfrak{g}$ has a Heisenberg subalgebra which is spanned by $\mathbb{I}$ and the elements of the form

$$H_n=\sum_{i\in\mathbb{Z}} :\psi_i\bar{\psi}_{i+n}: \tag{9}$$

whose commutation relations read

$$[H_n,H_m]=n\delta_{n,-m}\mathbb{I}. \tag{10}$$

An irreducible representation for $\mathfrak{g}$ can be constructed over the Fock space $\mathscr{F}$, which results to be decomposed into an infinite set of eigenspaces of the central element H_0,

$$\mathscr{F}=\bigoplus_{n\in\mathbb{Z}}\mathscr{F}_n. \tag{11}$$

A possible choice – which will turn out to be particularly convenient for the construction of the coherent states – is to identify the fixed vector $|\omega\rangle$ with the highest weight vector $|n\rangle\in\mathscr{F}_n$. The latter has the form:

$$\begin{aligned}|n\rangle&=\psi_{n-1}\cdots\psi_0|\rangle \quad \text{for } n>0,\\ |n\rangle&=\bar{\psi}_n\cdots\bar{\psi}_{-1}|0\rangle \quad \text{for } n<0,\end{aligned} \tag{12}$$

where $|0\rangle$ is the vacuum. The stability subalgebra of $|n\rangle$ is spanned by elements of the form [9] $\psi_i\bar{\psi}_j$, $i<j$; $\psi_i\bar{\psi}_j$, $i>j$, $n>i$ or $n<j$; $\psi_{i-1}\bar{\psi}_{i-1}-\psi_i\bar{\psi}_i$ $(i,\ j\in\mathbb{Z})$ and is strictly related to the Kac–Moody algebra. In order to exhibit manifestly the connection with the soliton equations, one has to resort to the differential realization of the Fermi operators $\{\psi_i,\ \bar{\psi}_i\}$. Upon denoting by $z=\{z_i\in\mathbb{R},\ i\in\mathbb{Z}^{(+)}\}$ the independent variables of such a realization, the latter is obtained by the following procedure. Let

$$\begin{aligned}X(k|z)&=\exp\left(\sum_{n=1}^{\infty}z_nk^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial}{\partial z_n}k^{-n}\right)\\ &=\sum_{i\in\mathbb{Z}}X_i(z,\partial)k^i\end{aligned} \tag{13}$$

be the vertex operator, and $\bar{X}(k|-z)=X(k|z)=\sum_{i\in z}\bar{X}_ik^{-i}$ its formal adjoint, considered as generating functions for the differential operators $X_i(z,\ \partial)$, $\bar{X}_i(z,\ \partial)$ respectively. Let V_r be a space of formal polynomials in infinitely many variables $\{z_i\}$ with complex coefficients, $r\in\mathbb{Z}$ being an index labelling different copies of it.

One can check that the identification

$$\psi_i=\hat{X}_i,\ \bar{\psi}_i=\hat{\bar{X}}_i, \tag{14}$$

where

$$\begin{aligned}\hat{X}_i&: V_r\to V_{r+1};\quad f_r(z)\to X_{i-r}(z,\partial)f_r(z),\\ \hat{\bar{X}}_i&: V_r\to V_{r-1};\quad f_r(z)\to X_{i-r+1}(z,\partial)f_r(z),\end{aligned} \tag{15}$$

$f_r(z)\in V_r$, realizes a Clifford algebra module isomorphism. It was shown by Sato and coworkers [10] that the infinitesimal generators of the Bäcklund transformations for the KP hierarchy in Hirota's form, are bilinear forms in the operators $\hat{X}_i$, $\hat{\bar{X}}_i$, $i\in\mathbb{Z}$ and span the whole algebra $gl(\infty)$. There follows that the solutions of the hierarchy, transform as

$$\tau(Z)\to e^Z\tau(Z);\ Z\in gl(\infty). \tag{16}$$

Since $\tau=1$ is obviously a solution of eq. (1) and moreover, due to the identification (15) it corresponds to the maximal weight vector in Fock space $\mathscr{F}$ and since the Bäcklund group acts transitively on the solutions, the whole manifold of solutions of the hierarchy can be thought of as the group orbit of the maximal weight vector. Keeping the definition (5) in mind, one can conclude that the coherent state manifold for the dynamical algebra of a general system of harmonically coupled Fermi oscillators, is the complete set of soliton solutions of the KP hierarchy.

[1] J.A. Krumhansl and J.R. Schrieffer, Phys. Rev. B11 (1975) 3535.
[2] G. D'Ariano and M. Rasetti, Phys. Lett. 107A (1985) 291.
[3] R. Hirota, in: Solitons, eds. K. Bullough and P.J. Caudrey (Springer-Verlag, Berlin, 1980).
[4] T. Schultz, D. Mattis and E. Lieb, Rev. Mod. Phys. 36 (1964) 856.
[5] M. Rasetti, Intern. J. Theor. Phys. 13 (1973) 425. A.M. Perelomov, Commun. Math. Phys. 26 (1972) 222.
[6] G. D'Ariano, M. Rasetti and M. Vadacchino, J. Phys. 18A (1985) 1295.
[7] H. Kuratsuji and T. Suzuki, Progr. Theor. Phys. Suppl. 74&75 (1983) 209.
[8] V.G. Kac, Infinite Dimensional Lie Algebras (Birkhäuser, Boston, 1983).
[9] G. D'Ariano and M. Rasetti, in: Integrable Systems in Statistical Mechanics, eds. G. D'Ariano, A. Montorsi and M. Rasetti (World Scientific, Singapore, 1985).
[10] E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Proc. RIMS Symp. on Non-Linear Integrable systems: Classical Theory, Quantum Theory, eds. M. Jimbo and T. Miure World Scientific, Singapore, (1983) p. 39.