Number-phase squeezing through nonunitary scaling

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A nonunitary operator scaling mechanism is presented, which produces number-phase squeezing, useful at decreased signal power. It accounts for the performance of nonunitary schemes—better than unitary—in attaining squeezing in the number.

The potential application of nonclassical light in low-noise communications and high-precision measurements has focused the attention of many researchers on methods of production of such kind of light. Squeezed and amplitude-squeezed fields have recently been obtained in a variety of physical systems,[1] and newly improved generation schemes are presently under study.

Squeezed and amplitude-squeezed states are minimum-uncertainty states which can achieve an arbitrarily low noise level in one of the conjugated observables. In the squeezed states the conjugated pair corresponds to two quadrature phase components of the field, whereas in the amplitude (or number-phase) squeezing, the two observables are the number $\hat{n}$ and the phase of the field $\hat{\Phi}$.

Squeezed states may lead to substantial improvement of the sensitivity in high-precision interferometry. Nevertheless, they are not optimal for transmissions in which information is coded on the low-noise component of the field. In fact, reduction of the noise in a quadrature of the field requires an increased average number of photons, and the maximum signal-to-noise ratio turns out to be limited by the maximum power supported by the communication channel.

In order to achieve the maximum channel capacity, the information should be coded on the trasmitted photon number $\hat{n}$, where the noise can be reduced to zero without the requirement of an infinite number of photons. Actually, reduction of the photon-number noise can be attained to the detriment of the $\hat{n}$-conjugated variable $\hat{\Phi}$, whose enhanced noise does not consume energy at all.

Different schemes have been proposed to generate number-phase squeezing. They can be essentially divided into two main cathegories: (i) *entropy-preserving* schemes, based on unitary time evolution of the state; (ii) *entropy-varying* schemes, in which $\hat{n}$-noise reduction is obtained via nonunitary state reduction (i.e., quantum-nondemolition measurements or quantum measurements of the first kind).

In the entropy-preserving schemes the field evolves under the action of a nonlinear (Kerr) Hamiltonian.[2] The photon-number noise can be reduced well below the standard quantum limit, $\langle \Delta\hat{n}^2 \rangle < \langle \hat{n} \rangle$, but the Fano factor $F = \langle \Delta\hat{n}^2 \rangle / \langle \hat{n} \rangle$ still suffers restriction by the average power. (The minimum achievable noise is $\langle \Delta\hat{n}^2 \rangle_{\min} \simeq \langle \hat{n} \rangle^{1/3}$, much smaller than the optimum squeezed state value[3] $\langle \Delta\hat{n}^2 \rangle \simeq \langle \hat{n} \rangle^{2/3}$.)

The entropy-varying schemes are probably more suited for amplitude-squeezing than the entropy-preserving ones. As a matter of fact, the Fano factor is no more limited by the power and, in principle, an arbitrarily small $\langle \Delta\hat{n}^2 \rangle$ can be attained for every average number of photons.[4] The common procedure in the entropy-varying generation systems passes through two steps. The first establishes a quantum-mechanical correlation between the signal radiation field and an auxiliary *probe* degree of freedom, using a nonlinear interaction. The second is a nonunitary reduction of the signal state obtained by detecting some variable of the probe. For example, in the *high-Q micromaser Fock state generation*, the probe is an inverted two-level atom entering the cavity with a well-defined velocity;[5] the nonunitary reduction of the signal field is then obtained by means of nonselective measurements of the atomic variables. On the other hand, in other schemes the role of the probe is played by another electromagnetic wave: this is the case of the *quantum-nondemolition photon-number measurement* and the *parametrically amplified idler photon counting* schemes proposed in Ref. 4. In the former, the probe wave interacts with the signal mode, via Kerr effect, before being homodyne detected in a quadrature component. In the latter, the probe is the idler wave interacting with the signal in a second-order medium: the nonunitary reduction of the signal is then achieved by an idler photon-counting measurement.

This Brief Report presents a simple argument which accounts for the better performance of nonunitary schemes — as opposed to unitary — in producing number-phase squeezing. A straightforward way to obtain $\hat{n}$-noise reduction and simultaneous $\hat{\Phi}$-noise amplification lies in the realization of a transformation of states which rescales the moments as follows:

$$\begin{aligned} \langle \hat{\Phi}^p \rangle &\to r^p \langle \hat{\Phi} \rangle , \\ \langle \hat{n}^p \rangle &\to r^{-p} \langle \hat{n} \rangle , \end{aligned} \qquad (1)$$

with $r > 1$. Such a transformation preserves the product of uncertainties $\langle \Delta\hat{n}^2 \rangle \langle \Delta\hat{\Phi}^2 \rangle$ and has the great advantage of rescaling the Fano factor

$$F = \langle \Delta\hat{n}^2\rangle / \langle \hat{n}\rangle \to r^{-1}F \tag{2}$$

by decreasing the average power

$$\langle \hat{n}\rangle \to r^{-1}\langle \hat{n}\rangle \ . \tag{3}$$

To avoid depletion of the radiation toward the vacuum, a driving unitary excitation of the state should be applied first, leaving F almost unchanged.

In the Heisenberg picture, the transformation corresponds to an operator scaling $\mathcal{S}_H$ resulting in a multiplication of the phase $\hat{\Phi}$ and a division of the number $\hat{n}$ by the same factor r:

$$\mathcal{S}_H(\hat{\Phi}) = r\hat{\Phi} \ , \tag{4}$$

$$\mathcal{S}_H(\hat{n}) = r^{-1}\hat{n} \ . \tag{5}$$

The general form for the operator scaling $\mathcal{S}_H$ can be inferred from the definition of the phase operator $\hat{\Phi}$:

$$\hat{E}_\pm = e^{\mp i\hat{\Phi}} \ , \tag{6}$$

$\hat{E}_\pm$ denoting the shift operators

$$\hat{E}_- = (a^\dagger a + 1)^{-1/2} a \ , \quad \hat{E}_+ = (\hat{E}_-)^\dagger \ , \tag{7}$$

$$\hat{E}_\pm |n\rangle = |n \pm 1\rangle \ . \tag{8}$$

(One should notice that, although $\hat{\Phi}$ is not Hermitian, it can be regarded as an almost Hermitian operator when applied on highly excited states; i.e., states approximately orthogonal to the vacuum $|0\rangle$. Correspondingly, in the limit of large mean numbers, one has the asymptotic commutation relation $[\hat{n}, \hat{\Phi}] \sim i$, which ensures that $\hat{n}$ and $\hat{\Phi}$ constitute a conjugated pair.)

From Eq. (6) one can see that the rescaling of the phase corresponds to transition from one-particle shift operators $\hat{E}_\pm$ to r-particle shift operators $(\hat{E}_\pm)^r$,

$$\mathcal{S}_H(\hat{E}_\pm) = (\hat{E}_\pm)^r \ , \tag{9}$$

$(\hat{E}_\pm)^r$ now acting on the Fock space as follows:

$$(\hat{E}_\pm)^r |n\rangle = |n \pm r\rangle \ . \tag{10}$$

From Eqs. (9) and (10) it turns out that the scaling $\mathcal{S}_H$ of a generic operator $\hat{O}$ has the form

$$\mathcal{S}_H(\hat{O}) = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda^\dagger \hat{O} \hat{S}_\lambda \ , \tag{11}$$

$$\hat{S}_\lambda = e^{i\phi_\lambda} \sum_{n=0}^{\infty} |n\rangle\langle nr + \lambda| \ . \tag{12}$$

[The phase factors are ineffective in the action (11) and will be dropped in the following.] $\hat{S}_\lambda$ are nonunitary operators satisfying the orthogonality and completeness relations

$$\hat{S}_\lambda \hat{S}_\mu^\dagger = \delta_{\lambda\mu} \hat{1} \ , \tag{13}$$

$$\sum_{\lambda=0}^{r-1} \hat{S}_\lambda^\dagger \hat{S}_\lambda = \hat{1} \ , \tag{14}$$

where $\hat{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|$. Although the scaling $\mathcal{S}_H$ is nonunitary, it preserves the operator products

$$\mathcal{S}_H(\hat{O}_1)\mathcal{S}_H(\hat{O}_2) = \mathcal{S}_H(\hat{O}_1\hat{O}_2) \ , \tag{15}$$

as it can be checked using the orthogonality conditions (13). Furthermore, $\mathcal{S}_H$ can be inverted in the following sense:

$$\mathcal{S}_H^{-1}(\mathcal{S}_H(\hat{O})) = \hat{O} \ , \tag{16}$$

$$\mathcal{S}_H^{-1}(\hat{O}) = r^{-1} \sum_{\lambda=0}^{r-1} \hat{S}_\lambda \hat{O} \hat{S}_\lambda^\dagger \ . \tag{17}$$

However, due to the nonunitarity of $\hat{S}_\lambda$, the inversion $\mathcal{S}_H^{-1}$ does not preserve the operator product in general.

When applied on the particle operators a and $a^\dagger$ the operator scaling gives the result:

$$\mathcal{S}_H(a) = \sum_{n=1}^{\infty} |n-1\rangle \sqrt{[n/r]} \langle n| \ , \qquad \mathcal{S}_H(a^\dagger) = [\mathcal{S}_H(a)]^\dagger \ , \tag{18}$$

where $[x]$ denotes the maximum integer $\le x$. Equations (18) show that the scaled particle operators $\mathcal{S}_H(a)$ and $\mathcal{S}_H(a^\dagger)$ are nothing but the r-boson operators $b_{(r)}$ and $b_{(r)}^\dagger$ introduced in Ref. 6:

$$\mathcal{S}_H(a) = b_{(r)} = \left(\frac{(1 + [\hat{n}/r])\hat{n}!}{(\hat{n}+r)!} \right)^{1/2} a^r, \qquad \mathcal{S}_H(a^\dagger) = b_{(r)}^\dagger \ . \tag{19}$$

In Eq. (19) $b_{(r)}$ and $b_{(r)}^\dagger$ annihilate and create r photons simultaneously and satisfy the commutation relations $[b_{(r)}, b_{(r)}^\dagger] = 1$, $[\hat{n}, b_{(r)}] = -r b_{(r)}$. From Eq. (15) it follows that the scaling $\mathcal{S}_H$ of a generic operator $\hat{O} = F(a, a^\dagger)$ (Hermitian analytic function of a and $a^\dagger$) can simply be obtained substituting a and $a^\dagger$ with $b_{(r)}$ and $b_{(r)}^\dagger$; i.e., $\mathcal{S}_H(\hat{O}) = F(b_{(r)}, b_{(r)}^\dagger)$. Therefore, the present operator scaling corresponds to the construction of the r-photon observables of Ref. 7.

Regarding the scaling of the number operator $\hat{n}$, from the defining equations (11) and (12) one obtains

$$\mathcal{S}_H(\hat{n}) = [\hat{n}/r] \equiv b_{(r)}^\dagger b_{(r)} \ . \tag{20}$$

It follows that the number operator satisfies the rescaling (5) only asymptotically in the limit of large mean numbers $\langle \hat{n}\rangle \gg r$,

$$\mathcal{S}_H(\hat{n}) \simeq r^{-1}\hat{n} \ . \tag{21}$$

In the Schrödinger picture the nonunitary evolution $\mathcal{S}_S$ is defined by the identity

$$\mathrm{Tr}[\rho \mathcal{S}_H(\hat{O})] = \mathrm{Tr}[\mathcal{S}_S(\rho)\hat{O}] \ , \tag{22}$$

where ρ denotes a general density matrix state. Using the invariance of trace under cyclic permutations one obtains

$$\mathcal{S}_S(\rho) = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda \rho \hat{S}_\lambda^\dagger \ . \tag{23}$$

The state evolution (23) is proportional to $\mathcal{S}_H^{-1}$ and, as a consequence, does not satisfy Eq. (15). However, $\mathcal{S}_S$ is well defined on the density matrices ρ, because they do not constitute an operator algebra. Due to the completeness relation (14) $\mathcal{S}_S$ preserves the normalization of the state ρ, despite the fact that it is not unitary. It resembles the reduced evolutions of the quantum unstable systems,[8] as a consequence of the semigroup property of the corresponding Heisenberg evolution $\mathcal{S}_H$.

The nonpreservation of the operator product implies also the nonpreservation of the Neumann-Shannon entropy

$$S(\rho) = -\mathrm{Tr}\rho \, \log \rho \ . \tag{24}$$

One then concludes that a transformation $\mathcal{S}_H$ rescaling the moments as in Eq. (1) does not conserve the entropy (24) in general. For example, starting with a pure state $\rho = |\omega\rangle\langle\omega|$, the mixed state is obtained,

$$\mathcal{S}_S(|\omega\rangle\langle\omega|) = \sum_{\lambda=0}^{r-1} |\Omega_\lambda\rangle\langle\Omega_\lambda| \ ,$$
$$|\Omega_\lambda\rangle = \hat{S}_\lambda|\omega\rangle = \sum_{n=0}^{\infty} |n\rangle\langle nr+\lambda|\omega\rangle \ . \tag{25}$$

From Eqs. (25) one can see that the number eigenstates are left pure and the vacuum is invariant. Furthermore, the entropy remains zero for an r-photon state[9] $|\omega\rangle_{(r,\mu)} = \hat{S}_\mu^\dagger|\omega\rangle$, whereas for an s-photon state ($s \neq r$) a fractional s/r-mixed state is obtained.[7]

The experimental realization of the scaling evolution $\mathcal{S}_S$ is probably not a simple task. As $\mathcal{S}_S$ does not conserve the entropy, an open quantum system is needed,[10] the operators (12) being eventually implemented by means of ideal photon amplifiers.[11] However, no other state transformation is available which rescales the moments as in Eq. (1) whichever input state is considered. At present, the experimental realizations of phase-number squeezing need a very careful tuning of the input state (see, for example, Ref. 12) making the process very sensitive to the destroying effect of damping.[13] Here, on the contrary, the input states are only restricted by the inequality $\langle\hat{n}\rangle \gg r$, which, besides satisfying Eq. (21), ensures a good definition of the phase operator itself. The average number $\langle\hat{n}\rangle$ does not limit the minimum achievable noise $\langle\Delta\hat{n}^2\rangle$, which can be reduced in many low-r scaling steps, each preceded by a driving excitation of the state.

As regards the possibility of implementing the rescaling (4) and (5) by means of a unitary transformation, one should notice that an exact transformation is not allowed by pure algebraic constraints. As a matter of fact, the mathematical constructed number-phase minimum-uncertainty states of Jackiw[14] cannot be obtained, for example, from a coherent state through a unitary evolution.[15] However, a crudely approximated scaling may be achieved by an operator of the form

$$U = \exp\left(\frac{i}{2}\log r(\hat{S}\hat{n} + \hat{n}\hat{S})\right) \ , \tag{26}$$

$\hat{S}$ denoting the sine operator

$$\hat{S} = \frac{1}{2i}(\hat{E}_- - \hat{E}_+)$$

($\hat{S} \sim \hat{\Phi}$ for $\langle\hat{\Phi}\rangle = 0$ and $\langle\Delta\hat{\Phi}^2\rangle \ll 1$). The operator U resembles the usual squeezing operator,[16] where the roles of $\hat{S}$ and $\hat{n}$ are played by two quadrature phase components of the field. Numerical and asymptotical evaluations[17] show that the validity of the approximate commutation $[\hat{n}, \hat{S}] \sim i$ now restrict more dramatically the set of states where the unitary transformation (26) attains the scalings (1). In addition, no physical scheme is available to realize the operator (26). At present, the unitary production of amplitude-squeezing is obtained via self-modulation processes, which are usually represented by an operator of the form[15]

$$U = \exp(\xi a^\dagger - \xi^* a)\exp[-i\chi\hat{n}(\hat{n}-1)] \ . \tag{27}$$

The operator (27) simulates the action of the operator (26) on coherent states, for a suitable choice of the parameters ξ and χ as a function of the input state.

The nonunitary scaling, as well as the unitary, is essentially simulated in the experimental realizations of the amplitude-squeezing based on entropy-varying schemes. Here the role of the dummy variable λ in Eq. (23) is played by some quantum number of the auxiliary probe field. There is no obvious strict comparison between the theoretical scaling (23) and the actual realization of amplitude-squeezing; however, some similarities can be recognized *a posteriori*.

Let us consider, for example, the case of the *high-Q micromaser Fock state generation*, where a monoenergetic low-density beam of two-level atoms is injected inside a lossless single-mode cavity. The injection rate is sufficiently low so that the atoms enter the cavity one at a time, but still high enough that a large number of atoms pass through the resonator before damping becomes important. Furthermore, exact resonance between the field frequency and the atomic transition frequency is assumed.

At the initial time $t = 0$, when the first atom enters the cavity, the state of the system is described by the density matrix $\rho_0\rho_a$, ρ_0 and ρ_a being the density matrices of the field and of the atom. When the atom exits the cavity after the time of flight τ, the field density matrix reduces to

$$\rho_1 = \mathrm{Tr}_a[U(\tau)\rho_0\rho_a U^\dagger(\tau)] \ , \tag{28}$$

where Tr_a denotes the trace over the atomic variables (the state of the atom is not measured as it exits the cavity[10]) and $U(\tau)$ represents the evolution operator in the Dirac picture $U(\tau) = \exp(-iV\tau)$, the interacting Hamiltonian V being

$$V = \kappa(aJ_+ + a^\dagger J_-) . \quad (29)$$

Here κ denotes the atom-field dipole coupling constant, $J_3, J_\pm$ the usual Pauli spin matrices or the spin-J angular momentum operators if the model is generalized to the case of $(2J+1)$-level atoms ($[J_+, J_-] = 2J_3$, $[J_3, J_\pm] = \pm J_\pm$). In the general case ($J > \frac{1}{2}$) the Hamiltonian (29) equivalently describes radiation interacting with bunches of $N = 2J$ two-level atoms having mean interatomic distance much smaller than the wavelength of the cavity field (but still well separated, so that their wave functions do not overlap).[18] For negligible dissipation, the field density matrix in the interaction representation does not evolve before the next atom (or bunch) will enter the cavity. It follows that after N atoms pass through the cavity, the field density matrix can be obtained recursively as follows:

$$\rho_N = \mathrm{Tr}_a[U(\tau)\rho_{N-1}\rho_a U^\dagger(\tau)] . \quad (30)$$

In the experimentally interesting case, in which each atom enters the cavity in the upper state and there are no photons in the cavity at $t = 0$, one has

$$\rho_N = \sum_{\lambda=0}^{2NJ} \rho_N(\lambda)|\lambda\rangle\langle\lambda| , \quad (31)$$

where the probabilities $\rho_N(\lambda)$ can be obtained recursively as follows:

$$\rho_N(\lambda) = \sum_{\nu=0}^{2J} a_\nu(\lambda-\nu)\rho_{N-1}(\lambda-\nu) ,$$
$$a_\nu(n) = |{}_a\langle 2J-\nu|\langle n+\nu \mid U(\tau)|n\rangle|2J\rangle_a|^2 , \quad (32)$$

$|n\rangle|\sigma\rangle_a$ denoting the basis vectors [$\hat{n}|n\rangle = n|n\rangle$, $n \geq 0$; $J_3|\sigma\rangle_a = (\sigma - J)|\sigma\rangle_a$, $0 \leq \sigma \leq 2J$]. From Eq. (31) one can see that the evolution of ρ_N simulates the same variation of entropy attained by the scaling $\mathcal{S}_H$ acting on the pure state $|\omega\rangle\langle\omega|$, where

$$|\omega\rangle \doteq \sum_{n=0}^{r-1} \sqrt{\rho_N(n)}|n(r+1)\rangle . \quad (33)$$

The role of the scaling factor r is played by the total spin multiplicity

$$r = 2NJ + 1 . \quad (34)$$

One can see that Eq. (33) represents a $(r+1)$-photon state, whereas the density matrix (31) is a fractional $(1+r)/r$ -mixed state.

The above correspondence can be further carried out when a pure Fock state is reached in the limit $N \to \infty$. This is the case, for example, of the *trapping states*,[12] where $J = \frac{1}{2}$ and the time of flight τ is tuned to have vanishing probability of photon-emission $a_1(n_0) = \sin^2(\sqrt{n_0+1}\kappa\tau)$ when n_0 photons are present in the cavity. An asymptotic analysis[17] in the neighborhood of the trapping state $|n_0\rangle\langle n_0|$ shows that $\langle\hat{n}\rangle \sim n_0 - \alpha N^{-1}$ [$\alpha = 4(n_0+1)^2\pi^{-2}$], whereas $\langle\Delta\hat{n}^2\rangle \sim n_0\alpha N^{-1}$. It follows that the Fano factor rescales according to the rule

$$F \sim \alpha N^{-1} . \quad (35)$$

The asymptotic scaling (35) can also simply be checked numerically. It further confirms the identification (34) of the scaling factor r, as one can see by comparing Eq. (35) with Eq. (2) and noting that $2NJ + 1 \simeq N$ for $J = \frac{1}{2}$ and $N \to \infty$.

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