Physical realizations of quantum operations

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(Received 29 May 2003; published 23 October 2003)

Quantum operations (QO's) describe any state change allowed in quantum mechanics, such as the evolution of an open system or the state change due to a measurement. We address the problem of which unitary transformations and which observables can be used to achieve a QO with generally different input and output Hilbert spaces. We classify all unitary extensions of a QO and give explicit realizations in terms of free-evolution direct-sum dilations and interacting tensor-product dilations. In terms of Hilbert space dimensionality the free-evolution dilations minimize the physical resources needed to realize the QO, and for this case we provide bounds for the dimension of the ancilla space versus the rank of the QO. The interacting dilations on the other hand, correspond to the customary ancilla-system interaction realization, and for these we derive a majorization relation which selects the allowed unitary interactions between system and ancilla.

DOI: 10.1103/PhysRevA.68.042113 PACS number(s): 03.65.Ta, 03.67.−a

I. INTRODUCTION

The recent progress in quantum information theory [1–3] offers the possibility of radically different information-processing methods that can achieve much higher performances than those obtained by classical means, in terms of security, capacity, and efficiency [4–7]. This urges a quantum system engineering approach for the production of the quantum tools needed for communication, processing, and storage of quantum information. A first step toward this goal is the search for a systematic method to implement in a controlled way any quantum state transformation.

The mathematical structure that describes the most general state change in quantum mechanics is the *quantum operation* (QO) of Kraus [2,8]. Such abstract theoretical evolution has a precise physical counterpart in its implementations as a unitary interaction between the system undergoing the QO and a part of the apparatus—the so-called *ancilla*—which after the interaction is read by means of a conventional quantum measurement. In this paper we address the problem of which unitary transformations and which observables can be used to achieve a given QO for a finite dimensional quantum system. We consider generally different input and output Hilbert spaces H and K, respectively, allowing the treatment of very general quantum machines, e.g., of the kind of quantum optimal cloners [9,10]. As will be clear from the physical implementations of the QO, schematically this corresponds to a general scenario in which the machine prepares a state in the Hilbert space H and couples it unitarily with a *preparation* ancilla in the Hilbert space R, which was previously set to a fixed state. The machine then transfers the joint system with Hilbert space $\mathsf{R}\otimes\mathsf{H}$ to a measuring section, which performs a measurement on another *measurement* ancilla, with space $\mathsf{L}\subset\mathsf{R}\otimes\mathsf{H}$. The output system will be in the Hilbert space K, where K is such that $\mathsf{L}\otimes\mathsf{K}=\mathsf{R}\otimes\mathsf{H}$. The result is a machine that performs a QO with input in H and output in K.

In the process of classification of all unitary extensions of a QO, we will give explicit realization schemes in terms of free-evolution direct-sum dilations and interacting tensor-product dilations, which in the following will be named briefly *free* and *interacting* dilations, respectively. The interacting dilations correspond to the ancilla-system interaction scenario just described above, whereas in the *free* dilations we only have the measurement ancilla, and the input space is embedded in a larger Hilbert space, where a kind of superselection rule forces the choice of the input state in a proper subspace before a free unitary evolution. In terms of Hilbert space dimensionality the free dilations minimize the physical resources needed to realize the QO, and for this case we will provide bounds for the dimension of the ancilla space versus the rank of the QO. For the interacting dilations, on the other hand, we will derive a majorization relation which allows us to preselect the admissible unitary interactions between system and ancilla, in relation to the ancilla preparation state and the measured observable.

The paper is organized as follows. After briefly recalling the notion of quantum operations in Sec. II, in Sec. III we introduce the Stinespring form for a QO and explicitly construct all possible unitary realizations, for both free and interacting dilations. We also address the problem of finding *unitary interacting power dilations* of a given QO, namely, interacting dilations that also provide the kth power of the map (i.e., with the map applied k times). In Sec. IV, we give the criterion to select the admissible unitary interactions for a QO in the form of a majorization relation. Section V finally closes the paper with a summary of the results.

II. QUANTUM OPERATIONS

In the following, by $\mathsf{T}(\mathsf{H})$ we denote the set of trace-class operators on the Hilbert space H (which can be simply re-

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garded as just the set of states on H). A *quantum operation* $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$ is a linear, trace not-increasing map that is also *completely positive* (CP), namely, that preserves the positivity of any input state of the system on H entangled with any other quantum system (mathematically, all trivial extensions $\mathcal{E}\otimes\mathcal{I}$ of the map must preserve the positivity of input states on the extended Hilbert space). The *input* and the *output* states are connected via the relation

$$\rho \mapsto \rho' = \frac{\mathcal{E}(\rho)}{\mathrm{Tr}[\mathcal{E}(\rho)]}, \tag{1}$$

where the trace $\mathrm{Tr}[\mathcal{E}(\rho)] \leqslant 1$ also represents the probability that the transformation in Eq. (1) occurs. An analog of the spectral theorem for positive operators in finite dimensions leads to the following canonical form of the QO $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$ [8]:

$$\mathcal{E}(\rho) = \sum_n E_n \rho E_n^\dagger, \tag{2}$$

where the bounded operators $E_n \in \mathsf{B}(\mathsf{H},\mathsf{K})$ from H to K are orthogonal, i.e., $\mathrm{Tr}[E_n^\dagger E_m]=0$ for $n\neq m$, and moreover they satisfy the condition

$$\sum_n E_n^\dagger E_n = K \leqslant I_{\mathsf{H}}. \tag{3}$$

In terms of the positive operator $K\in\mathsf{B}(\mathsf{H})$, the probability of occurrence of the QO can also be rewritten as $\mathrm{Tr}[K\rho]$. Notice that there are generally infinitely many noncanonical ways of writing the map $\mathcal{E}$ in the form of Eq. (2), with generally larger and nonorthogonal sets of elements $\{E_j'\}$ that satisfy Eq. (3). All such decompositions are usually called *Kraus forms* of the QO $\mathcal{E}$. In order to satisfy Eq. (3), the operators $\{E_j'\}$ of a noncanonical Kraus form are related to the canonical ones $\{E_i\}$ as $E_j' = \Sigma_i Y_{ji} E_i$ via an isometric matrix Y, i.e., a matrix with orthonormal columns. When the map is trace preserving, i.e., $\mathrm{Tr}[\mathcal{E}(\rho)]=1$—or equivalently $K=I_{\mathsf{H}}$—it occurs with unit probability, and is usually named *channel*.

It was known since Kraus [8] that a trace-preserving QO admits a unitary realization on an extended Hilbert space. More generally, when we have a set of QO's that describe a general quantum measurement (also with a continuous spectrum and in infinite dimensions: the so-called *instruments*), Ozawa [11] proved the realizability in terms of an observable measurement over an ancilla after a unitary interaction with the quantum system. In the following we will derive explicitly all possible unitary dilations for a generic QO for finite dimension, and give ancillary realizations and bounds for the dimensions of the Hilbert spaces involved.

III. UNITARY DILATIONS OF A QUANTUM OPERATION

The Stinespring dilation [12,13] is a kind of "purification" of the QO. Originally, Stinespring's theorem was set for the *dual* version $\mathcal{E}^\tau$ of the QO, i.e., in the "Heisenberg picture"—instead of the "Schrödinger picture" of Eq. (1)—the two pictures being related as follows:

$$\mathrm{Tr}[\rho\mathcal{E}^\tau(O)] = \mathrm{Tr}[\mathcal{E}(\rho)O] \tag{4}$$

for every bounded operator $O\in\mathsf{B}(\mathsf{K})$. Analogously to Eqs. (2) and (3), one has

$$\mathcal{E}^\tau(O) = \sum_n E_n^\dagger O E_n \tag{5}$$

with

$$\mathcal{E}^\tau(I_{\mathsf{K}}) = K. \tag{6}$$

A variation of Stinespring's theorem can be restated by saying that for every QO $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$, there exists a Hilbert space L such that $\mathcal{E}$ can be obtained as follows:

$$\mathcal{E}^\tau(X) = E^\dagger (I_{\mathsf{L}}\otimes X) E, \tag{7}$$

where $E\in\mathsf{B}(\mathsf{H},\mathsf{L}\otimes\mathsf{K})$ is a contraction (i.e., E is an operator bounded as $\|E\|\leqslant 1$). In fact, consider any Kraus decomposition $\mathcal{E}=\Sigma_{i=1}^n E_i\cdot E_i^\dagger$ for $\mathcal{E}$, and let L be a Hilbert space with $\dim(\mathsf{L})\geqslant n$ and orthonormal basis $\{|l_i\rangle\}$. The operator

$$E = \sum_{i=1}^{\dim(\mathsf{L})} |l_i\rangle\otimes E_i \tag{8}$$

is a contraction, since $E^\dagger E = K \leqslant I_{\mathsf{H}}$ [if one considers $\dim(\mathsf{L})>n$, this means that extra null operators are appended to the Kraus decomposition]. Here and throughout the paper, for $A\in\mathsf{B}(\mathsf{H},\mathsf{K})$ and $|v\rangle\in\mathsf{L}$ the tensor notation $|v\rangle\otimes A$ will denote the linear operator from H to $\mathsf{L}\otimes\mathsf{K}$ defined as $(|v\rangle\otimes A)|\phi\rangle \doteq |v\rangle\otimes A|\phi\rangle$, for $|\phi\rangle\in\mathsf{H}$, whereas its adjoint $\langle v|\otimes A^\dagger$ is the linear operator from $\mathsf{L}\otimes\mathsf{K}$ to H given by $(\langle v|\otimes A^\dagger)|\varphi\rangle\otimes|\psi\rangle \doteq \langle v|\varphi\rangle A^\dagger|\psi\rangle$, for $|\psi\rangle\in\mathsf{K}$ and $|\varphi\rangle\in\mathsf{L}$. Using the Kronecker representation of the tensor product [15], the contraction E in Eq. (8) is easily represented by vertically joining the operators E_i. By substituting Eq. (8) into Eq. (7) one obtains Eq. (5), namely, the statement. On the other hand, the Schrödinger picture form of Eq. (5) is

$$\mathcal{E}(\rho) = \mathrm{Tr}_{\mathsf{L}}[E\rho E^\dagger]. \tag{9}$$

For a trace-preserving map the Stinespring contraction E is actually an isometry, since $E^\dagger E = I_{\mathsf{H}}$ [this case with isometric E is the original Stinespring theorem version of Eq. (7)].

It is possible to extend trace-decreasing maps to isometries also. For this purpose, first we prove the following lemma.

Lemma 1. For any given positive bounded operator $P\in\mathsf{B}(\mathsf{H})$ and for every Hilbert space K, there exists a set of bounded operator $A_i\in\mathsf{B}(\mathsf{H},\mathsf{K})$, $i=1,...,n$, such that

$$P = \sum_{i=1}^n A_i^\dagger A_i. \tag{10}$$

Proof. Let $P=\Sigma_{i=1}^{\mathrm{rank}(P)} |v_i\rangle\langle v_i|$ where $|v_i\rangle\in\mathsf{H}$ are the orthogonal eigenvectors of P, generally not normalized. One has two possibilities.

(a) $\dim(\mathsf{K})\geqslant\mathrm{rank}(P)$: the statement holds for $n=1$ with $P=A^\dagger A$ and $A=\Sigma_{i=1}^{\mathrm{rank}(P)}|k_i\rangle\langle v_i|$, with $\{|k_i\rangle\}$ any orthonormal set in K.

(b) $\dim(\mathsf{K})<\mathrm{rank}(P)$: then the result holds with $n=\mathrm{rank}(P)$ and $A_i=|\psi_i\rangle\langle v_i|$, with $\{|\psi_i\rangle\}$ any set of normalized vectors in K. ■

Notice that in case (b) of the proof, we can suitably choose the operators $\{A_i\}$ in order to minimize n as $n=\lceil r/k\rceil$, for $r\doteq\mathrm{rank}(P)$, $k\doteq\dim(\mathsf{K})$, and $\lceil x\rceil$ denoting the minimum integer greater than or equal to x. These are given by the operators

$$A_i=\sum_{j=1}^{k}|k_j\rangle\langle v_{(i-1)k+j}|,\quad i=1,\ldots,n\equiv\lceil r/k\rceil. \tag{11}$$

The lemma stated above can be used to prove the following theorem.

Theorem 1. A linear map $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$ is a QO if and only if its dual form can be written as

$$\mathcal{E}^\tau(X)=V^\dagger(\Sigma\otimes X)V \tag{12}$$

for a suitable ancillary Hilbert space L, where $V\in\mathsf{B}(\mathsf{H},\mathsf{L}\otimes\mathsf{K})$ is an isometry, and $\Sigma\in\mathsf{B}(\mathsf{L})$ is a nonvanishing orthogonal projector on a subspace of L. Furthermore, $\Sigma\equiv I_{\mathsf{L}}$ if and only if $\mathcal{E}$ is trace preserving.

Proof. Let us denote by $\{|\sigma_j\rangle\}_{j=1,\ldots,\mathrm{rank}(\Sigma)}\subset\mathsf{L}$ the eigenvectors of Σ having unit eigenvalue. Then the operators

$$E_j=(\langle\sigma_j|\otimes I_{\mathsf{K}})V,\quad j=1,\ldots,\mathrm{rank}(\Sigma) \tag{13}$$

provide a Kraus decomposition for the map $\mathcal{E}$, which then is a QO. This proves the sufficient condition.

For the necessary condition, consider a QO $\mathcal{E}$ where $\{E_i\}\subset\mathsf{B}(\mathsf{H},\mathsf{K})$ are the elements of any Kraus decomposition. From Lemma 1, there exists a set of operators $\{F_j\}\subset\mathsf{B}(\mathsf{H},\mathsf{K})$ such that $\Sigma_j F_j^\dagger F_j=I_{\mathsf{H}}-\Sigma_i E_i^\dagger E_i\geqslant 0$. Now, consider a set of orthonormal vectors $\{|e_i\rangle,|f_j\rangle\}$ in L, and define the orthogonal projector $\Sigma=\Sigma_i|e_i\rangle\langle e_i|$ and the isometry

$$V=\sum_i|e_i\rangle\otimes E_i+\sum_j|f_j\rangle\otimes F_j. \tag{14}$$

These operators will provide the desired dilation in Eq. (12).

To complete the proof we need to show that $\Sigma=I_{\mathsf{L}}$ if and only if $\mathcal{E}$ is trace preserving. If $\mathcal{E}$ is trace preserving, we do not need the operators $\{F_j\}$ and hence we can choose the space L to be spanned by the vectors $\{|e_i\rangle\}$ that form an orthonormal basis for L, namely, $\Sigma_i|e_i\rangle\langle e_i|=I_{\mathsf{L}}$. On the other hand, if $\Sigma=I_{\mathsf{L}}$, one has $\mathcal{E}^\tau(I_{\mathsf{K}})=V^\dagger V=I_{\mathsf{H}}$, namely, the map is trace preserving. ■

Theorem 1 allows us to derive a bound for the physical resources that one needs to obtain the dilation (12) of a QO (as we will see in Sec. III A, the unitary dilation of the isometry does not introduce any additional ancillary resource). It is clear that for trace-preserving maps one has $F_j=0$ for all j in the proof of the theorem. Notice also that since $V\in\mathsf{B}(\mathsf{H},\mathsf{L}\otimes\mathsf{K})$ is an isometry, one has $\dim(\mathsf{H})\leqslant\dim(\mathsf{L})\times\dim(\mathsf{K})$. The minimum dimension for L is obtained for the canonical Kraus decomposition and for the minimum cardinality of the complementary set of operators $\{F_j\}$ that is given by $\lceil\mathrm{rank}(I_{\mathsf{H}}-K)/\dim(\mathsf{K})\rceil$. Therefore, upon denoting by c the cardinality of the canonical Kraus decomposition, namely, the *rank* of the QO, one has

$$\left(c+\left\lceil\frac{\mathrm{rank}(I_{\mathsf{H}}-K)}{\dim(\mathsf{K})}\right\rceil\right)\times\dim(\mathsf{K})\geqslant\dim(\mathsf{H}) \tag{15}$$

for every map $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$. In fact, using Lemma 1, in Eq. (14) one has at least c elements E_i and at least $\lceil r/\dim(\mathsf{K})\rceil$ elements F_j, where r is the rank of $I_{\mathsf{H}}-K$. From Eq. (15) and the condition $\mathcal{E}^\tau(I_{\mathsf{K}})=K$, we have

$$\dim(\mathsf{L})\geqslant c+\left\lceil\frac{\mathrm{rank}[I_{\mathsf{H}}-\mathcal{E}^\tau(I_{\mathsf{K}})]}{\dim(\mathsf{K})}\right\rceil\geqslant\frac{\dim(\mathsf{H})}{\dim(\mathsf{K})}. \tag{16}$$

Equation (16) provides a bound on the resources that one needs to obtain an isometric dilation, without knowing *a priori* a Kraus decomposition for the map.

In Theorem 1 we have shown how to obtain a QO via an isometric embedding. In the following subsections, we explicitly derive the physical realizations for the QO for both the free and the interacting formulations.

A. Free dilations

We start by giving a proof of the well-known lemma of Gram-Schmidt unitary dilations [14].

Lemma 2. Every isometry $T\in\mathsf{B}(\mathsf{H}_{\mathrm{in}},\mathsf{H}_{\mathrm{out}})$ admits a unitary dilation $U\in\mathsf{B}(\mathsf{H}_{\mathrm{out}})$.

Proof. Introduce a Hilbert space $\mathsf{H}_{\mathrm{aux}}$ such that $\mathsf{H}_{\mathrm{out}}=\mathsf{H}_{\mathrm{in}}\oplus\mathsf{H}_{\mathrm{aux}}$. We consider the case $\dim(\mathsf{H}_{\mathrm{aux}})\geqslant 1$, otherwise $\mathsf{H}_{\mathrm{out}}\cong\mathsf{H}_{\mathrm{in}}$, and T is already unitary. For a given isometry $W\in\mathsf{B}(\mathsf{H}_{\mathrm{aux}},\mathsf{H}_{\mathrm{out}})$, define the operator $U\in\mathsf{B}(\mathsf{H}_{\mathrm{out}})$ as

$$U=T\cdot+\cdot W,$$

$$U|v_{\mathrm{out}}\rangle=(|v_{\mathrm{in}}\rangle\oplus|v_{\mathrm{aux}}\rangle)=T|v_{\mathrm{in}}\rangle+W|v_{\mathrm{aux}}\rangle. \tag{17}$$

In finite dimension, this can be obtained on a chosen basis just by joining *horizontally* the two matrices T and W so that, by construction, U is a square matrix, whence the symbol $\cdot+\cdot$. If the condition

$$T^\dagger W=0 \tag{18}$$

is satisfied, then the operator U is unitary on $\mathsf{H}_{\mathrm{out}}$. An operator $W\in\mathsf{B}(\mathsf{H}_{\mathrm{aux}},\mathsf{H}_{\mathrm{out}})$ that satisfies Eq. (18) has column vectors $[W(k)]$ for $k=1,\ldots,\dim(\mathsf{H}_{\mathrm{aux}})$ that make an orthonormal basis for $\mathsf{H}_{\mathrm{aux}}=\mathrm{Rng}(I_{\mathsf{H}_{\mathrm{out}}}-TT^\dagger)\subset\mathsf{H}_{\mathrm{out}}$. A set of vectors of this kind can always be obtained iteratively by the Gram-Schmidt procedure on $\mathsf{H}_{\mathrm{in}}\oplus\mathsf{H}_{\mathrm{aux}}$, with $\dim(\mathsf{H}_{\mathrm{aux}})>0$. ■

Using the previous lemma, one can obtain a unitary operator $U\in\mathsf{B}(\mathsf{L}\otimes\mathsf{K})$ from the isometry $V\in\mathsf{B}(\mathsf{H},\mathsf{L}\otimes\mathsf{K})$ by sticking V horizontally with an appropriate isometry $W\in\mathsf{B}(\mathsf{D},\mathsf{L}\otimes\mathsf{K})$, where D is a second auxiliary Hilbert space defined by the relation $\mathsf{H}\oplus\mathsf{D}=\mathsf{L}\otimes\mathsf{K}$. From U, conversely, it is possible to reconstruct V using the *dilation operator* D

$\in \mathsf{B}(\mathsf{H},\mathsf{L}\otimes\mathsf{K})$, which is the trivial isometry $D=I_{\mathsf{H}}\dot{+}\mathbf{0}_{\mathsf{H},\mathsf{D}}$ [in analogy with $\cdot+\cdot$, the symbol $\dot{+}$ means the *vertical* joining of two block matrices, whereas the symbol $\mathbf{0}_{\mathsf{H},\mathsf{D}}$ denotes the operator in $\mathsf{B}(\mathsf{H},\mathsf{D})$ corresponding to the rectangular matrix with all zero entries], such that

$$V=UD, \tag{19}$$

where D acts as follows:

$$D|v_{\mathsf{H}}\rangle = I_{\mathsf{H}}|v_{\mathsf{H}}\rangle \oplus \mathbf{0}_{\mathsf{H},\mathsf{D}}|v_{\mathsf{H}}\rangle = |v_H\rangle \oplus |0\rangle_{\mathsf{D}}, \tag{20}$$

with $|0\rangle_{\mathsf{D}}$ denoting the null vector in D.

In this way, we can reexpress Theorem 1 stating that a linear map $\mathcal{E}:\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$ is a QO if and only if its dual form can be written as

$$\mathcal{E}^{\tau}(X)=D^{\dagger}U^{\dagger}(\Sigma\otimes X)UD. \tag{21}$$

Therefore, any trace-decreasing QO can be interpreted in terms of a unitary interaction between the quantum system and an ancilla, followed by an orthogonal projection. The dilation operator D is needed just in order to reduce the output space of the unitary operator to the original output space of the map.

The Schrödinger form of Eq. (21) can be obtained as follows. From the duality relation in Eq. (4), one has

$$\begin{aligned}\mathrm{Tr}[\mathcal{E}^{\tau}(X)\rho] &= \mathrm{Tr}[D^{\dagger}U^{\dagger}(\Sigma\otimes X)UD\rho] \\ &= \mathrm{Tr}[(\Sigma\otimes X)UD\rho D^{\dagger}U^{\dagger}] \\ &= \mathrm{Tr}[X\,\mathrm{Tr}_{\mathsf{L}}[(\Sigma\otimes I_{\mathsf{K}})UD\rho D^{\dagger}U^{\dagger}]],\end{aligned} \tag{22}$$

whence

$$\begin{aligned}\mathcal{E}(\rho) &= \mathrm{Tr}_{\mathsf{L}}[(\Sigma\otimes I_{\mathsf{K}})UD\rho D^{\dagger}U^{\dagger}] \\ &= \mathrm{Tr}_{\mathsf{L}}[(\Sigma\otimes I_{\mathsf{K}})U(\rho\oplus\mathbf{0}_{\mathsf{D}})U^{\dagger}],\end{aligned} \tag{23}$$

where $\mathbf{0}_{\mathsf{D}}$ is the null operator on D. In Eq. (23) the term $U(\rho\oplus\mathbf{0}_{\mathsf{D}})U^{\dagger}$ represents a *free unitary evolution* of the system in the state $D\rho D^{\dagger}\equiv\rho\oplus\mathbf{0}_{\mathsf{D}}$, which is a positive block-diagonal operator in $\mathsf{T}(\mathsf{L}\otimes\mathsf{K})$ with unit trace (remember that $\mathsf{H}\oplus\mathsf{D}=\mathsf{L}\otimes\mathsf{K}$). Physically, such a trivial embedding of H in $\mathsf{H}\oplus\mathsf{D}$ can be regarded as kind of conservation law or superselection rule forbidding a subspace for the input states.

In conclusion of this subsection, we notice that the special case of V already unitary in Eq. (12) corresponds to no subspace D, and $D\equiv I_{\mathsf{H}}$ and $U\equiv V$. Then Eq. (23) becomes simply

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathsf{L}}[(\Sigma\otimes I_{\mathsf{K}})U\rho U^{\dagger}], \tag{24}$$

and one necessarily has $\dim(\mathsf{K})\leqslant\dim(\mathsf{H})$.

B. Interacting dilations

In the previous subsection we derived a general unitary realization for a given QO, in terms of a direct-sum dilation, using a measurement ancilla only, with the input space embedded in a larger Hilbert space, where a kind of superselection rule forces the choice of the input state in a proper subspace before a free unitary evolution on the extended space. We are now interested in the tensor-product types of realization schemes, in which the role of the dilation operator D (i.e., of the superselection rule) will be played by the tensor product of ρ with the state of a preparation ancilla, with the system interacting with such ancilla, and with a conventional observable measurement then performed on a different ancilla. This ancilla-system interaction scenario is more popular in the literature and is the one used in the extension theorems for instruments in Refs. [11]. It is obvious that composite schemes are also possible, with both direct-sum and tensor-product dilations.

The results of the previous subsection can be rewritten by choosing a dilation in terms of a Hilbert space L with dimension $\dim(\mathsf{L})\times\dim(\mathsf{K})=r\dim(\mathsf{H})$, for integer r. Then, upon introducing a second ancillary space R with $\dim(\mathsf{R})=r$, one has $\mathsf{L}\otimes\mathsf{K}\cong\mathsf{R}\otimes\mathsf{H}$, and we obtain the following theorem.

Theorem 2. A linear map $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$ is a QO if and only if its dual form can be written as follows:

$$\mathcal{E}^{\tau}(X)=\langle\phi_{\mathsf{R}}|U^{\dagger}(\Sigma\otimes X)U|\phi_{\mathsf{R}}\rangle, \tag{25}$$

where $X\in\mathsf{B}(\mathsf{K})$ is the input $\Sigma\in\mathsf{B}(\mathsf{L})$ is a nonvanishing orthogonal projector on a subspace of the ancillary space L, $U\in\mathsf{B}(\mathsf{L}\otimes\mathsf{K})$ is unitary, and $|\phi_{\mathsf{R}}\rangle\in\mathsf{R}$ is a fixed normalized vector. In the Schrödinger picture one has

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathsf{L}}[(\Sigma\otimes I_{\mathsf{K}})U(|\phi_{\mathsf{R}}\rangle\langle\phi_{\mathsf{R}}|\otimes\rho)U^{\dagger}], \tag{26}$$

where now the input is represented by the state $\rho\in\mathsf{T}(\mathsf{H})$. We refer to the spaces R and L as the preparation and measurement ancilla, respectively.

Proof. Notice that the notation $\langle\phi_{\mathsf{R}}|U^{\dagger}(\Sigma\otimes X)U|\phi_{\mathsf{R}}\rangle$ denotes a partial matrix element: in our tensor notation this corresponds to writing $(\langle\phi_{\mathsf{R}}|\otimes I_{\mathsf{H}})U^{\dagger}(\Sigma\otimes X)U(|\phi_{\mathsf{R}}\rangle\otimes I_{\mathsf{H}})$.

Let us consider the unitary dilation in Eq. (21), and expand the space L such that

$$\dim(\mathsf{L}\otimes\mathsf{K})=\dim(\mathsf{L})\times\dim(\mathsf{K})=r\dim(\mathsf{H}) \tag{27}$$

for integer r. The Hilbert space D defined as $\mathsf{D}=(\mathsf{L}\otimes\mathsf{K})\ominus\mathsf{H}$, now has dimension $\dim(\mathsf{D})=\dim(\mathsf{L})\times\dim(\mathsf{K})-\dim(\mathsf{H})=(r-1)\dim(\mathsf{H})$. Let us introduce a Hilbert space R with dimension $\dim(\mathsf{R})=r$, so that

$$\dim(\mathsf{L}\otimes\mathsf{K})=\dim(\mathsf{R}\otimes\mathsf{K}), \tag{28}$$

whence

$$\mathsf{L}\otimes\mathsf{K}\cong\mathsf{R}\otimes\mathsf{H}. \tag{29}$$

Clearly, if the map has equal input and output spaces, the preparation ancilla and the measurement ancilla are isomorphic, i.e., $\mathsf{R}\cong\mathsf{L}$. On fixed orthonormal bases for K, L, H, and R, upon denoting by $|\phi_{\mathsf{R}}\rangle\in\mathsf{R}$ an element of the basis of R, we have the following identifications:

$$D\equiv|\phi_{\mathsf{R}}\rangle\otimes I_{\mathsf{H}} \tag{30}$$

and

$$\rho\oplus\mathbf{0}_{\mathsf{D}}\equiv|\phi_{\mathsf{R}}\rangle\langle\phi_{\mathsf{R}}|\otimes\rho. \tag{31}$$

The statement of the theorem is then obtained by rewriting Eqs. (21) and (23) with the use of Eqs. (30) and (31). ■

Alternative proof. The above proof is based on the direct-sum dilation of Eq. (21). An equivalent way to obtain the result in Theorem 2 is the following. From the Stinespring form in Eq. (12), let us introduce a Hilbert space R such that $\mathsf{L}\otimes\mathsf{K}\cong\mathsf{R}\otimes\mathsf{H}$. By a repeated use of the Gram-Schmidt procedure one obtains other isometries $W_i\in\mathsf{B}(\mathsf{H},\mathsf{L}\otimes\mathsf{K})$, for $i=2,...,r$, such that

$$V^\dagger W_i=0 \quad\text{and}\quad W_i^\dagger W_j=\delta_{ij}I_{\mathsf{H}}, \tag{32}$$

namely,

$$\mathsf{Rng}(V)\oplus\mathsf{Rng}(W_2)\oplus\cdots\oplus\mathsf{Rng}(W_r)=\mathsf{L}\otimes\mathsf{K}. \tag{33}$$

Let us consider the unitary operator

$$U=\langle r_1|\otimes V+\langle r_2|\otimes W_2+\cdots+\langle r_r|\otimes W_r, \tag{34}$$

where $\{|r_i\rangle\}\subset\mathsf{R}$ is an orthonormal basis for the space R. By taking $|r_1\rangle\equiv|\phi_{\mathsf{R}}\rangle$, one obtains the statement of the theorem. ■

This constructive proof was used in Ref. [16] to explicitly derive a unitary realization for the optimal transposition map.

The tensor-product form of the unitary dilation is generally more expensive in terms of resources (i.e., the dimension of the extended space) than the direct-sum form in Eq. (23); however, the physical realization of the tensor product could be more practical, since one just needs to prepare a fixed ancilla state, without the need of a superselection rule.

By a further enlargement of the ancilla space, the structure of the unitary interaction that realizes a given QO can be simplified. The following derivation generalizes the Halmos method [17] and has been used in [18] to provide unitary realizations of the ideal phase measurement.

From the Stinespring dilation (12), where we take L such that $\mathsf{L}\otimes\mathsf{K}\cong\mathsf{R}\otimes\mathsf{H}$, let us define the operators

$$\tilde{V}=V(\langle\phi_{\mathsf{R}}|\otimes I_{\mathsf{H}}) \quad\text{and}\quad \tilde{V}^\dagger=(|\phi_{\mathsf{R}}\rangle\otimes I_{\mathsf{H}})V^\dagger. \tag{35}$$

One can simply verify that both $\tilde{V}\tilde{V}^\dagger$ and $\tilde{V}^\dagger\tilde{V}$ are projectors, i.e., $(\tilde{V}\tilde{V}^\dagger)(\tilde{V}\tilde{V}^\dagger)=\tilde{V}\tilde{V}^\dagger$ and $(\tilde{V}^\dagger\tilde{V})(\tilde{V}^\dagger\tilde{V})=\tilde{V}^\dagger\tilde{V}$. Let us introduce a *third* ancilla space S and a linear operator W on S such that

$$W^2=W^{\dagger 2}=0 \quad\text{and}\quad WW^\dagger+W^\dagger W=I_{\mathsf{S}}. \tag{36}$$

These conditions imply that $WW^\dagger$ and $W^\dagger W$ are orthogonal projectors. We can now write the unitary operator $U\in\mathsf{B}(\mathsf{S}\otimes\mathsf{L}\otimes\mathsf{K})$ as follows:

$$U=WW^\dagger\otimes\tilde{V}-W^\dagger W\otimes\tilde{V}^\dagger+W^\dagger\otimes(I-\tilde{V}^\dagger\tilde{V})+W\otimes(I-\tilde{V}\tilde{V}^\dagger), \tag{37}$$

thus obtaining the map by the equation

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathsf{S},\mathsf{L}}[(I_{\mathsf{S}}\otimes\Sigma\otimes I_{\mathsf{K}})U(\sigma_{\mathsf{S}}\otimes|\phi_{\mathsf{R}}\rangle\langle\phi_{\mathsf{R}}|\otimes\rho)U^\dagger], \tag{38}$$

where $\sigma_{\mathsf{S}}=WW^\dagger/\mathrm{Tr}[WW^\dagger]$ is the fixed normalized state of the third ancilla. Notice that the space S and the operator W are arbitrary, provided that the constraints in Eq. (36) are satisfied. For $\dim(\mathsf{S})=2$ and $W=|0\rangle\langle 1|$ one recovers the Halmos unitary dilations [17].

C. Power interacting dilations

We have shown in Theorem 2 how to obtain a unitary interaction U that realizes a given QO $\mathcal{E}$: $\mathsf{T}(\mathsf{H})\to\mathsf{T}(\mathsf{K})$, as in Eq. (26). Consider now a trace-preserving QO with $\mathsf{H}\equiv\mathsf{K}$, namely, a customary *channel*. The equivalence of the input and output spaces implies, in the interacting scheme, coincidence also between the preparation and the measurement ancillas, namely, $\mathsf{R}\equiv\mathsf{L}$ [19]. The map $\mathcal{E}$ can now be applied recursively, and we study the properties of its *powers*

$$\rho\mapsto\mathcal{E}(\rho)\mapsto\mathcal{E}(\mathcal{E}(\rho))\doteq\mathcal{E}^2(\rho)\mapsto\cdots\mapsto\mathcal{E}^n(\rho). \tag{39}$$

Of course, the unitary realization given in Eq. (26) does not satisfy the composition law for powers of the map, namely,

$$\mathcal{E}^n(\rho)\neq\mathrm{Tr}_{\mathsf{R}}[U^n(|\phi_{\mathsf{R}}\rangle\langle\phi_{\mathsf{R}}|\otimes\rho)(U^\dagger)^n]. \tag{40}$$

In fact, the unitary dilation needs a *fresh resource*, i.e., a *disentangled input ancilla*, whereas generally it returns an *entangled* output. For this reason, powers of U do not correspond to powers of $\mathcal{E}$.

Here we address the problem of finding *unitary power interacting dilations* for a given map. Using the unitary U and the ancilla state $|\phi_{\mathsf{R}}\rangle$ of Eq. (26), let us define the n-copy ancilla state $\sigma=|\phi_{\mathsf{R}}\rangle\langle\phi_{\mathsf{R}}|^{\otimes n}$ and the unitary operator on $\mathsf{R}^{\otimes n}\otimes\mathsf{H}$,

$$W=\left(\prod_{i=1}^{n-1}E_{i,n}\otimes I_{\mathsf{H}}\right)(I_{1,2,\ldots,n-1}\otimes U), \tag{41}$$

where the product of swap operators $E_{i,n}|\psi\rangle|\phi\rangle=|\phi\rangle|\psi\rangle$ for $|\psi\rangle\in\mathsf{R}_i$ and $|\phi\rangle\in\mathsf{R}_n$ performs a cyclic permutation of the ancilla spaces R_i. One has

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathsf{R}^{\otimes n}}[W(\sigma\otimes\rho)W^\dagger]. \tag{42}$$

It is now easy to check that the unitary realization in Eq. (42) satisfies the composition law for k powers up to n,

$$\mathcal{E}^k(\rho)=\mathrm{Tr}_{\mathsf{R}^{\otimes n}}[W^k(\sigma\otimes\rho)(W^\dagger)^k], \quad k=1,...,n. \tag{43}$$

In fact, the permutation operator selects one *fresh ancilla* at *every step* of the interaction, leaving the others unchanged.

IV. MAJORIZATION SELECTION OF UNITARY DILATIONS

We give now a criterion to select the unitary dilations of a QO in terms of a *majorization* relation. We recall that for two vectors $x,y\in\mathbb{R}^n$ we say that x is *majorized* by y, i.e., $x\prec y$, if and only if [20,21]

$$\sum_{j=1}^{k} x_j^{\downarrow}\leqslant\sum_{j=1}^{k} y_j^{\downarrow}, \quad 1\leqslant k<n, \tag{44}$$

and

$$\sum_{j=1}^{n} x_j^{\downarrow}=\sum_{j=1}^{n} y_j^{\downarrow}, \tag{45}$$

where $v^{\downarrow}$ denotes the vector obtained from v by rearranging its entries in nonincreasing order.

In Ref. [22] Nielsen proved the following theorem that characterizes the ensembles corresponding to a given density operator ρ by means of a majorization relation. Let $\rho\in\mathsf{T}(\mathsf{H})$ and (p_i) be a probability vector. There exist normalized vectors $|\psi_i\rangle\in\mathsf{H}$ such that

$$\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{46}$$

if and only if

$$(p_i)\prec(\lambda_\rho), \tag{47}$$

where (λ_ρ) is the vector of eigenvalues of ρ.

We now apply the Nielsen theorem in order to select the unitary dilations of a given QO, by exploiting the isomorphism [23] between CP maps from $\mathsf{T}(\mathsf{H})$ to $\mathsf{T}(\mathsf{K})$ and positive operators on $\mathsf{K}\otimes\mathsf{H}$. This correspondence is defined by the relations [10,24]

$$R_{\mathcal{E}}=\mathcal{E}\otimes\mathcal{I}(|I\rangle\rangle\langle\langle I|),$$

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathsf{H}}[(I_{\mathsf{K}}\otimes\rho^T)R_{\mathcal{E}}], \tag{48}$$

where $|I\rangle\rangle\in\mathsf{H}\otimes\mathsf{H}=\Sigma_n|n\rangle\otimes|n\rangle$ is the maximally entangled unnormalized vector, T denotes the transposition on the basis $\{|n\rangle\}$, $\mathcal{I}$: $\mathsf{T}(\mathsf{H})\rightarrow\mathsf{T}(\mathsf{H})$ is the identity map, and we used the notation [25]

$$|A\rangle\rangle=\sum_{n,m} A_{nm}|n\rangle\otimes|m\rangle \tag{49}$$

for bipartite pure states. In terms of the positive operator $R_{\mathcal{E}}$, the identity (3) becomes $\mathrm{Tr}_{\mathsf{K}}[R_{\mathcal{E}}]=K$.

Denoting by $\|A\|_2=\sqrt{\mathrm{Tr}[A^\dagger A]}$ the Hilbert-Schmidt norm of the operator A, we have the following theorem.

Theorem 3. Let $\mathcal{E}$ be a QO from $\mathsf{T}(\mathsf{H})$ to $\mathsf{T}(\mathsf{K})$ with canonical Kraus decomposition given by $\mathcal{E}(\rho)=\Sigma_{j=1}^{c}E_j\rho E_j^\dagger$ with $\mathrm{Tr}[E_i^\dagger E_j]=\|E_i\|_2^2\delta_{ij}$. Then all the possible unitary interacting dilations for $\mathcal{E}$ obtained by Theorem 2 must satisfy the majorization constraint

$$(\|\langle\sigma_i|U|\phi_{\mathsf{R}}\rangle\|_2^2)\prec(\|E_i\|_2^2), \tag{50}$$

where $\{|\sigma_i\rangle\}\subset\mathsf{L}$ form an orthonormal basis for $\mathsf{Rng}(\Sigma)$ (see Theorem 2).

Proof. When representing the CP map $\mathcal{E}$ with the positive operator $R_{\mathcal{E}}$ as in Eq. (48), a Kraus decomposition $\mathcal{E}(\rho)=\Sigma_iE_i'\rho E_i'^\dagger$ for $\mathcal{E}$ can be regarded as the "ensemble" realization $R_{\mathcal{E}}=\Sigma_i|E_i'\rangle\rangle\langle\langle E_i'|$ for the "density operator" $R_{\mathcal{E}}$. Hence, different Kraus decompositions $\{E_1',\ldots,E_m'\}$ for $\mathcal{E}$ correspond to different ensembles, with probability vector $(\|E_i'\|_2^2)$ given by the Hilbert-Schmidt norms of the operators E_i'. On the other hand, the probability vector $(\|E_i\|_2^2)$ of the canonical Kraus decomposition corresponds to the vector of eigenvalues of $R_{\mathcal{E}}$, whence Eq. (47) in the present context becomes

$$(\|E_i'\|_2^2)\prec(\|E_i\|_2^2), \tag{51}$$

and Eq. (3) guarantees that the two vectors have the same length. Then the statement of the theorem follows from the identification $E_i'=(\langle\sigma_i|\otimes I_{\mathsf{K}})U(|\phi_{\mathsf{R}}\rangle\otimes I_{\mathsf{H}})$. ■

The above theorem provides another bound on the dimension of the ancilla space L. Since in Eq. (50) one has $i=1,\ldots,\mathrm{rank}(\Sigma)$, and $j=1,\ldots,c=\mathrm{rank}(R_{\mathcal{E}})$ (c is the cardinality of the canonical Kraus decomposition), then

$$\dim(\mathsf{L})\geqslant\mathrm{rank}(\Sigma)\geqslant c. \tag{52}$$

This bound can be compared with the tighter one in Eq. (16).

Equation (50) can also be used to introduce a partial ordering [26] between all possible unitary interacting dilations (26) for the same QO $\mathcal{E}$. In fact, Eq. (50) states that the unitary interactions from a canonical Kraus decomposition majorize in the sense of Eq. (50) all those derived from a generic Kraus decomposition. In other words, the more the Kraus decomposition $\{E_i'\}$ is "mixed," i.e., is an isometric combination of the canonical one $E_i'=\Sigma_{j=1}^{c}Y_{ij}E_j$ for $Y^\dagger Y=I_c$, the more the unitary interaction constructed with the $\{E_i'\}$ will be "flat" in the Hilbert-Schmidt norms of its partial matrix elements $\|\langle\sigma_i|U|\phi_{\mathsf{R}}\rangle\|_2^2$. This means that the partial ordering would also reflect a minimization of the ancillary resource in terms of its Hilbert space dimension.

V. CONCLUSIONS

Given a QO, generally trace nonincreasing and with different input and output spaces, we have seen how to obtain its unitary realizations in terms of both free and interacting dilations. These different forms of dilation require different amounts of resources in order to achieve the unitary interaction, and the minimum resource in terms of the Hilbert space dimension is obtained with the free dilation, where the input state is embedded in a larger Hilbert space and a kind of superselection rule forces the choice of the input state in a proper subspace before the free unitary evolution. For this case we derived bounds for the physical resources needed to achieve a QO in terms of the dimension of the measurement ancilla space. The interacting dilations, on the other hand, correspond to the customary realization in terms of an ancilla-system interaction. Then we have seen how the construction can be generalized in order to include also unitary power dilations of a given QO, namely, unitary interacting realizations that also provide the kth power of the map. Finally, we have seen how all possible interactions can be preselected by means of a majorization inequality, involving the unitary operator, the ancilla preparation state, and the measured observable.

ACKNOWLEDGMENTS

This work has been sponsored by INFM through Project No. PRA-2002-CLON, and by EEC and MIUR through the cosponsored ATESIT Project No. IST-2000-29681 and Cofinanziamento 2002. G.M.D. also acknowledges partial support from Department of Defense Multidisciplinary University Research Initiative (MURI) program administered by the Army Research Office under Grant No. DAAD19-00-1-0177.

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$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B & \cdots \\ a_{21}B & a_{22}B & \cdots & a_{2n}B & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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[19] Notice that the isomorphism $\mathsf{R}\equiv\mathsf{L}$ of the two ancillary Hilbert spaces does not mean that the two ancillas are physically equivalent, but only that there exist unitary realizations with the same physical ancilla. In fact, the ancillary spaces can be embedded in different ways in the overall tensor product, different embeddings can be included in the operator of the unitary interaction with the system. These kinds of ambiguity are due to the implicit identification between physical quantum system and its Hilbert space of states within a multipartite (i.e., tensor product) composite system. The misidentification is resolved by more properly identifying the quantum system with its algebra of observables.

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