SOLITON EQUATIONS AND COHERENT STATES

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The τ-functions, which represent the totality of solutions for hierarchies of equations in soliton theory, are identified with the coherent states of the infinite dimensional Lie algebra gl(∞). The associated quantum system can be realized by an infinite set of harmonically interacting fermionic modes. The soliton dynamical evolution is thus mapped into a quantum hamiltonian evolution, and the latter back into a classical hamiltonian flow corresponding to a succession of infinitesimal contact Bäcklund transformations.

In recent times a good deal of enphasis has been set on the deep connection between soliton theory and the representation of groups corresponding to infinite dimensional Lie algebras. In particular Sato, Jimbo, Miwa and their coworkers [1] have shown that there is a transitive action of an infinite dimensional (generalized Bäcklund) group on the manifold of solutions, whereby they were able to classify and solve in a very elegant unified manner all integrable soliton equations. A key concept in their scheme is that of τ-functions [2]. The latter represent the totality of (polynomial) solutions to the hierarchies of equations one encounters in soliton theory. Soliton equations in this approach are nothing but the defining differential equations of the group orbit of the highest weight vector in an infinite dimensional Fock space.

In the present note we identify, on the basis of the above observation, the space of τ-functions with the coherent state manifold of the infinite dimensional Lie algebra gl(∞). The latter has a realization in terms of bilinear products of fermion operators: it is therefore possible to map the soliton dynamics onto an equivalent quantum hamiltonian problem characterized by a harmonic interaction of infinitely many fermionic modes. It follows that, using the transformation properties of coherent states under the action of the Bäcklund group, one can map the stable [3] hamiltonian evolution of coherent states into a succession of infinitesimal contact transformations, namely again into a classical hamiltonian flow over the coherent state manifold.

For the sake of simplicity we refer, in the initial part of our discussion, to the KP (Kadomtsev–Petviashvili) hierarchy, which is the most basic one; even though the whole discussion could from the beginning be made in general terms. The KP equation reads [4]

$$3u_{yy} - (4u_t - 6uu_x - u_{xxx})_x = 0 \,. \qquad (1)$$

The whole hierarchy involves of course an infinity of variables that we denote $z = \{z_i\}$ (with $z_1 = x$, $z_2 = y$, $z_3 = t$). Hirota's bilinearization technique [2] allows writing (1) – upon setting $u = 2(\ln \tau)_{xx}$ – in the form

$$(D_x^4 + 3D_y^2 - 4D_xD_t)\,\tau \circ \tau = 0 \,, \qquad (2)$$

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where Hirota's bilinear differential operators are defined, for any polynomial P, by

$$P(D_x, D_y, D_t) f \circ g = P(\partial_x, \partial_y, \partial_t)[f(x+x', y+y', t+t') \times g(x-x', y-y', t-t')]|_{x'=y'=t'=0} . \quad (3)$$

Eq. (2) has a hidden internal symmetry, realized in terms of the action of the infinitesimal differential operators $Z_{ij}(z, \partial)$ given by the generating function

$$Z(p, q) = q\{\exp[\beta(z, p) - \beta(z, q)] \times \exp[-\beta(\tilde{\partial}, p^{-1}) + \beta(\tilde{\partial}, p^{-1})] - 1\}/(p-q) = \sum_{i,j \in Z} Z_{ij}(z, \partial) p^i q^{-j} , \quad (4)$$

where $\partial = \{\partial_n\}$, $\partial_n = \partial/\partial z_n$, $\tilde{\partial} = \{n^{-1}\partial_n\}$ and

$$\beta(z, p) = \sum_{n=1}^{\infty} z_n p^n .$$

The operators Z_{ij} are the infinitesimal Bäcklund transformation generators. The operators of infinite order out of which (4) is constructed,

$$X(k) = \exp[\beta(z, k) - \beta(\tilde{\partial}, k^{-1})] , \quad (5)$$

and its formal adjoint

$$\bar{X}(k) = \exp[-\beta(z, k) + \beta(\tilde{\partial}, k^{-1})] , \quad (6)$$

are referred to as vertex operators [5].

They realize a correspondence between the space of the polynomial algebra $C[z]$ and the Fock space $\mathcal{F}$ of charged fermions $\{\psi_i, \bar{\psi}_i\}$ by the Clifford algebra module isomorphism generated by the identification

$$\psi_i = \hat{X}_i , \quad \bar{\psi}_i = \hat{\bar{X}}_i , \quad (7)$$

where the $\hat{X}_i$ and $\hat{\bar{X}}_i$ are defined in the following way.

Upon setting

$$X(k) = \sum_{i \in Z} X_i(z, \partial) k^i , \quad \bar{X}(k) = \sum_{i \in Z} \bar{X}_i(z, \partial) k^{-i} ,$$

consider copies $\{V_r\}$ of $C[z]$. Then, for $f_r(z) \in V_r$,

$$\hat{X}_i : \quad V_r \to V_{r+1} , \quad f_r(z) \to \hat{X}_{i-r}(z, \partial) f_r(z) , \quad (8)$$

$$\hat{\bar{X}}_i : \quad V_r \to V_{r-1} , \quad f_r(z) \to \hat{\bar{X}}_{i-r+1}(z, \partial) f_r(z) . \quad (9)$$

Consider now the Lie algebra $gl(\infty)$ defined as the vector space

$$\mathcal{A} = gl(\infty) = \Big(\sum_{i,j \in Z} a_{ij} :\psi_i \bar{\psi}_j:\Big) \oplus C , \quad (10)$$

where : : denotes the usual normal ordered product defined according to Wick's theorem, a_{ij} are the elements of a sector diagonal matrix A (i.e. there exists an integer N such that $a_{ij} = 0$ for $|i - j| > N$), and C is the center, spanned by the identity $\mathbb{I}$ and the nontrivial element

$$H_0 = \sum_{i \in Z} :\psi_i \bar{\psi}_i: . \quad (11)$$

There is an eigenvalue decomposition of the Fock space $\mathcal{F}$ into charged fermion subspaces $\mathcal{F}_r$, $\mathcal{F} = \oplus_r \mathcal{F}_r$, where each $\mathcal{F}_r$ is invariant under the central element H_0. Eqs. (8), (9) define then an isomorphism between V_r and $\mathcal{F}_r$. $gl(\infty)$ has a Heisenberg subalgebra $\mathcal{E}$ spanned by $\mathbb{I}$ and

$$H_n = \sum_{i \in Z} :\psi_i \bar{\psi}_{i+n}: , \quad n \neq 0 , \quad (12)$$

with

$$[H_n, H_m] = n\delta_{n,-m} \mathbb{I} . \quad (13)$$

It was shown in ref. [1] that the space of τ-functions is given by the group orbit of the highest weight vector $|r\rangle \in \mathcal{F}_r$ under the action of $G = \exp(\mathcal{A})$. The correspondence between the functional space and the group orbit $\mathcal{M}$ is realized in that scheme by the equation

$$\tau = \tau_r(z; g) = \langle r|\exp[H(z)] g|r\rangle , \quad g \in G , \quad (14)$$

where

$$H(z) = \sum_{n=1}^{\infty} z_n H_n . \quad (15)$$

Upon recalling that $\mathcal{M}$ is indeed the homogeneous factor space quotient of G by the stability subgroup K leaving $|r\rangle$ fixed, the above structure has then a natural interpretation from the point of view of generalized coherent states $\{|\zeta\rangle\}$ for an arbitrary Lie group, say $\mathcal{G}$ [6]. These are defined as an overcomplete set of quantum states, labelled by a point in a suitable homogeneous space of $\mathcal{G}$; and their mathematical construction proceeds as follows.

Let the Hilbert space of states $\mathcal{S}$ of the dynamical system be transitive under $\mathcal{G}$. Consider then the cyclic vector $|\omega\rangle \in \mathcal{S}$ and denote by $\mathcal{K}$ the stability subgroup of $\mathcal{G}$ for $|\omega\rangle$, namely the subgroup which leaves the state corresponding to $|\omega\rangle$ invariant.

Let finally $T(g), g \in \mathcal{G}$ be the holomorphic representation of $\mathcal{G}$ (suitably complexified). The coherent state $|\zeta\rangle$ is defined by

$$|\zeta\rangle = |\zeta_g\rangle = \pi^{-1}(g)\, T(g)\, |\omega\rangle\,, \tag{16}$$

where $\pi(g)$ is a holomorphic character for all g's in the coset labelled by the point $\zeta \in \mathcal{G}/\mathcal{K}$. The manifold $\mathcal{G}/\mathcal{K}$ is locally isomorphic with $\mathrm{C}^{\mathcal{N}}$, for some (possibly infinite) integer $\mathcal{N}$. Hence ζ can be written in a local chart as $\zeta = \{\zeta_i; i = 1, \ldots, \mathcal{N}\}$. Out of the continuous set of coherent states $\{|\zeta\rangle\}$, the quantum propagator can be constructed by a path integral of the form [7]

$$\langle \zeta'', t'' | \zeta', t' \rangle = \int \mathcal{D}\,[\zeta(t)]\, \exp[(\mathrm{i}/\hbar) S]\,, \tag{17}$$

where the action functional is given by

$$S[\zeta(t)] = \int_{t'}^{t''} L\, \mathrm{d}t$$

$$= \int_{t'}^{t''} \langle \zeta(t) | \mathrm{i}\hbar\, \partial_t - \mathcal{H} | \zeta(t) \rangle\, \mathrm{d}t\,, \tag{18}$$

with the lagrangian

$$L = \tfrac{1}{2}\mathrm{i}\hbar \sum_{i=1}^{\mathcal{N}} (\dot{\zeta}_i \partial_{\zeta_i} \ln \langle \zeta | \zeta \rangle - \dot{\bar{\zeta}}_i \partial_{\bar{\zeta}_i} \ln \langle \zeta | \zeta \rangle)$$

$$- \mathcal{H}(\zeta, \bar{\zeta})\,. \tag{19}$$

$\mathcal{H}(\zeta, \bar{\zeta})$ denotes the diagonal element $\langle \zeta | \mathcal{H} | \zeta \rangle$ of the system hamiltonian $\mathcal{H}$. The stationary phase approximation of (17) leads to the Euler–Lagrange equations for the trajectory $\zeta(t)$, thus allowing one to interpret $\Pi = \mathcal{G}/\mathcal{K}$ as the canonical phase space of the system. Also, eq. (18) shows that the coherence preserving Schrödinger evolution of the quantum state coincides with the classical lagrangian flow on the phase space. It appears from the definition (16) that the manifold Π is but the group orbit of $|\omega\rangle$ under the action of $\mathcal{G}$. Upon identifying $\mathcal{S}$ with $\mathcal{F}_r$, $\mathcal{G}$ with G, $\mathcal{K}$ with K, and selecting as cyclic invariant vector $|\omega\rangle$ the highest weight vector $|r\rangle \in \mathcal{F}_r$, eqs. (14) and (16) show that the τ-functions are but the coherent state representatives corresponding to the holomorphic section of the line bundle associated with the (complexified) principal fibre bundle $\mathrm{K} \to \mathrm{G} \to \mathrm{G/K} \to \mathcal{M}$ by the character π, induced by the element $H(z) \in \mathcal{A}$.

In the above scheme the stability subgroup K is thus isomorphically identified with a parabolic subgroup P of G [3]. The minimal parabolic subgroup is the Borel subgroup B. Once the highest weight vector has been selected, B is the exponentiation of the subalgebra $\mathrm{b} = \mathrm{n}_+ \oplus \mathrm{h}$, where n_+ and h are factors of the Cartan decomposition $\mathrm{gl}(\infty) = \mathrm{n}_- \oplus \mathrm{h} \oplus \mathrm{n}_+ \oplus \mathrm{C}$. The subalgebras n_-, n_+ and h (the latter the Cartan subalgebra of $\mathrm{gl}(\infty)$) are spanned, in the charged fermion representation, by

$$e_i = \psi_{i-1}\bar{\psi}_i\,, \quad f_i = \psi_i \bar{\psi}_{i-1}\,,$$

$$h_i = \psi_{i-1}\bar{\psi}_{i-1} - \psi_i \bar{\psi}_i\,, \quad i \in \mathrm{Z}\,, \tag{20}$$

respectively. The generators (20) have the same formal defining relations as the generators of an infinite rank Kac–Moody algebra [8].

It was shown in ref. [3] that in order to preserve coherence, a hamiltonian should be an element of the algebra of $\mathcal{G}$, if the latter is semisimple. Since in the present context this is the case, the most general coherence preserving hamiltonian can be written in the form

$$\mathcal{H} = \sum_{i,j \in \mathrm{Z}} E_{ij} : \psi_i \bar{\psi}_j :\,, \quad E_{ij} = E^*_{ji} \in \mathrm{C}\,. \tag{21}$$

The time evolution of the quantum dynamical system can then be viewed as an infinite sequence of local infinitesimal contact Bäcklund transformations, in that the element $g = g(t) \in \mathcal{G}$ representing the time evolution operator can be written

$$g = \mathrm{e}^{Z_1} \ldots \mathrm{e}^{Z_k}\,, \quad Z_1, \ldots, Z_k \in \mathrm{gl}(\infty)\,, \tag{22}$$

with the Z_i locally nilpotent (i.e. such that for any $|v\rangle \in \mathcal{F}$, one can find a sufficiently large integer M such $Z_i^M |v\rangle = 0$). This procedure maps the original non-linear classical system into an equivalent quantum system. On the other hand, under the action of this same hamiltonian the representative point $\zeta(t)$ of the coherent state over the phase space Π evolves in time according to the lagrangian dynamics of a system of (infinitely many) canonical degrees of freedom, thus defining an hamiltonian flow – once more classical – on Π itself. Indeed the coherent state representative dynamics is given by $\tau(z;g(t)) = \tau(z, \zeta(t))$.

In conclusion eqs. (17)–(19) provide the construction of local charts of canonical coordinates, which for coherence preserving hamiltonians are mapped into each other by the infinitesimal Bäcklund transformations associated with the elements (22) [‡1]. The whole procedure thus gives a generalization to the case of infinite dimensional dynamical algebras of the construction of canonical coordinates for orbits of coadjoint representation possessing polarization recently given by Kamalin and Perelomov [9]. It is worth signalling that when the algebra $\mathscr{g}$ of $\mathscr{G}$ is not semisimple, the hamiltonian can belong to a suitable extension of $\mathscr{g}$ [3]. In such a case, domains of coherence stability can be defined on $\mathcal{M}$ by constructing a partition of $\mathcal{M}$ itself in cells by Morse functions realized in terms of the coherent states representatives [10]. Work is in progress along these lines.

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‡1 More generally, by construction of the coherent states, the orbits of G intersect at single points each fibre of the fibre bundle over G/P determined by the Lie algebra associated with P. On the other hand P contains the kernel of the symplectic two-form over $\mathcal{M}$ defined by the commutations of gl(∞). Thus the construction of P (and of the associated coherent states) naturally defines a locally canonical dynamical system over Π.

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