acta physica slovaca vol. 47 No. 3/4, 281 – 290 July/August 1997

THE QUANTUM TOMOGRAPHIC ROULETTE WHEEL[1]

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Random-phase homodyne tomography of the field intensity is a concrete example of the Quantum Roulette of Helstrom. In this paper we give the explicit POM of such measurement and compare it with direct photodetection and heterodyne detection. Effects of nonunit quantum efficiency are also considered. Naimark extensions for the roulette POM are analyzed and its experimental realization is discussed.

I. Introduction

In recent years much attention has been devoted to state reconstruction techniques, namely to measurement schemes providing the elements of the density matrix in some representation [1-4]. The problem has a fundamental interest and also practical applications as, for example, in determining coherence properties of the field and in characterizing effective quantum interaction Hamiltonians of optical media. Nevertheless, in many practical situations the relevant information one aims to gain about the quantum state of a light beam regards the field intensity, namely the photon number. Actually, when looking at realistic situations, methods for precise measurements of photon intensity are especially welcome. Measurements on low-excited highly-nonclassical quantum states of radiation represent, in fact, a difficult task, due to limitations of currently available photodetectors. On one hand, it is very difficult to discriminate single photons in any desidered range of intensity. On the other hand, in the low photon-number region ($n < 10$) one can discriminate a single photon, however with low quantum efficiency. Therefore, alternative methods to detect intensity need to be considered, whereas their performances and their robustness to low quantum efficiency should be compared in the very-quantum regime, versus the limitations of customary photodetection.

A way to circumvent photodetection problems is that of amplifying the the output photocurrent by mixing the state under examination with a highly excited coherent

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[1] Presented at the Fifth Central-European Workshop on Quantum Optics, Prague, Czech Republic, April 25 - 28, 1997

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state. This is the basis of homodyne and heterodyne detection schemes, and, in turn homodyne detection is the basis of tomographic state-reconstruction. In this paper we address the measurements of the field intensity by random-phase homodyne tomography and compare it with heterodyne detection.

Random-phase homodyne tomography has been suggested [6] as an effective tool to measure photon intensity, and after has received some theoretical attention [5]. More recently experiments have nicely demonstrated its ability in determining photon statistics for highly nonclassical states of radiation [7]. Here, we derive the probability operator measure (POM) that describes the random-phase homodyning of the intensity, regarding this measurement as a concrete example of the quantum roulette of Helstrom [8]. This abstract framework is suitable to discuss different Naimark extensions, thus suggesting novel experimental realizations.

The paper is structured as follows. In Section II we describe the measurement of the field intensity by the quantum tomographic roulette. We derive its POM and discuss its precision. In Section III the tomographic roulette is compared with direct photodetection and heterodyne detection also in presence of nonunit quantum efficiency. In Section IV we analyze the possible Naimark extensions to the roulette POM, whereas Section V closes the paper with some concluding remarks.

II. Measuring Intensity by Quantum Roulette

The concept of *Quantum Roulette* has been introduced by Helstrom in his book [8] about twenty years ago. A Quantum Roulette is described by a POM of the form

$$\hat{\Pi}_m = \sum_{k=1}^{M} z_k \, \hat{E}_m^{(k)} \qquad m = 1, ..., N \; , \tag{1}$$

where $z_k \geq 0$, $\sum_{k=1}^M z_k = 1$) and the $\hat{E}_m^{(k)}$'s are families of orthogonal projectors corresponding to different observables labelled by k, say $\hat{O}^{(k)}$, in formula

$$\hat{E}_m^{(k)} \hat{E}_n^{(k)} = \delta_{mn} \hat{E}_n^{(k)} \qquad \sum_{m=1}^{N} E_m^{(k)} = \hat{1} \; . \tag{2}$$

Each experimental event corresponds to the measurement of one of the observables $\hat{O}^{(k)}$, chosen at random according to the probability distribution z_k. At the time of Helstrom's proposal the Quantum Roulette was just a tool to illustrate an example of generalized measurements that do not correspond to selfadjoint operators. Nowadays, such measurement can be now performed in quantum optics labs. Let us consider the homodyne detection of a nearly single-mode radiation field. When the phase ϕ of the local oscillator is fixed, the field-quadrature $\hat{x}_\phi = \frac{1}{2}(a^\dagger e^{i\phi} + a e^{-i\phi})$ is detected and the measurement is described by the POM

$$d\hat{E}^{(\phi)}(x) = |x\rangle_{\phi\phi}\langle x| dx \; , \tag{3}$$

$|x\rangle_\phi$ denoting eigenstates of $\hat{x}_\phi$. The tomographic detection of the field corresponds to scan the local oscillator phase over $[0, \pi]$. When the phase ϕ of the local oscillator is unknown one deals with a Quantum Roulette, each experimental event being the measurement $d\hat{E}^{(\phi)}(x)$ with random ϕ. The experimental outcomes are distributed over the whole real axis according to the probability distribution $p(x) = \mathrm{Tr}\{\hat{\varrho} d\hat{\mu}(x)\}$ where $d\hat{\mu}(x)$ is the nonorthogonal POM

$$d\hat{\mu}(x) = dx \int_0^\pi \frac{d\phi}{\pi} |x\rangle_{\phi\phi}\langle x| \; . \tag{4}$$

Inserting the number state expansion of $|x\rangle_\phi$

$$|x\rangle_\phi = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2} \sum_{n=0}^{\infty} \frac{H_n(\sqrt{2}x)}{2^{n/2}\sqrt{n!}} e^{in\phi} |n\rangle \; , \tag{5}$$

in Eq. (4) we obtain the POM of the roulette

$$d\hat{\mu}(x) = dx \sqrt{\frac{2}{\pi}} e^{-2x^2} \sum_{n=0}^{\infty} \frac{H_n^2(\sqrt{2}x)}{2^n n!} |n\rangle\langle n| = dx \sqrt{\frac{2}{\pi}} e^{-2x^2} \frac{H^2_{a^\dagger a}(\sqrt{2}x)}{2^{a^\dagger a}(a^\dagger a)!} \; , \tag{6}$$

where $H_n(x)$ denote Hermite polynomials.
The POM $d\hat{\mu}(x)$ is an operator function of the number operator only and the outcome x will be an estimate–generally biased–of the field intensity. We now proceed in deriving an unbiased estimate. As it was shown by Richter [9] the expectation value of any normally ordered product $\langle a^{\dagger n} a^m \rangle$ can be obtained from tomographic data by averaging the kernel integral

$$\mathcal{R}[a^{\dagger n} a^m](x, \phi) = e^{i\phi(m-n)} \frac{H_{n+m}(\sqrt{2}x)}{2^{(n+m)/2} \binom{n+m}{m}} \; , \tag{7}$$

over the probability distribution $p(x, \phi)$. For the number operator $\hat{n}$ Eq. (7) defines the kernel

$$y \equiv \mathcal{R}[a^\dagger a](x) = 2x^2 - \frac{1}{2} \; , \tag{8}$$

which is a phase-independent quantity, hence is suitable for estimation by a Quantum Roulette. The quantity y traces the field intensity by averaging over the roulette outcomes distribution

$$\bar{y} = \int_{-\infty}^{\infty} dx \; \mathcal{R}[a^\dagger a](x) \; p(x) = \langle \hat{n} \rangle \; . \tag{9}$$

Indeed, Eq. (9) shows that th function $y(x)$ in Eq. (8) is the unbiased field intensity estimator for Quantum Roulette. The single outcomes $y \equiv \mathcal{R}[a^\dagger a](x)$ are random numbers distributed over the interval $[-1/2, \infty]$. It is clear that the determination in Eq. (9) is meaningful only when also a its statistical deviation is specified. The latter is given by

$$\overline{\Delta y} = \sqrt{\overline{y^2} - \bar{y}^2} \; , \tag{10}$$

where

$$\overline{y^2} = \int_{-\infty}^{\infty} dx\, \mathcal{R}^2[a^\dagger a](x)\, p(x)\;. \tag{11}$$

The explicit expression of the statistical deviation is given by

$$\overline{\Delta y}^2 = \langle \widehat{\Delta n^2} \rangle + \frac{1}{2}\left[\langle \widehat{n^2} \rangle + \langle \hat{n} \rangle + 1 \right]\;, \tag{12}$$

$\langle \widehat{\Delta n^2} \rangle$ being the intrinsic photon number fluctuations. We can also specify the whole probability distribution $p(y)$. In fact, from the Radon-Nikodym derivative of the roulette POM in Eq. (6) we arrive at the roulette POM for field intensity

$$d\hat{\mu}(y) = \frac{dy}{\sqrt{\pi}} \frac{e^{-(y+1/2)}}{\sqrt{y+1/2}} \frac{H^2_{a^\dagger a}(\sqrt{y+1/2})}{2^{a^\dagger a}(a^\dagger a)!}\;. \tag{13}$$

III. Quantum Roulette versus Photodetection and Heterodyning

As it emerges from Eq. (12) the Quantum Roulette measurement of the field intensity is noisy, as compared with ideal photodetection. Here we analyze its performances in the realistic case of nonunit quantum efficiency. When dealing with $\eta < 1$ the overall output noise $\langle \widehat{\Delta n^2} \rangle_\eta$ is larger than the intrinsic quantum fluctuations $\langle \widehat{\Delta n^2} \rangle$ which represents the minimum attainable noise in a measurement of the intensity. For non unit quantum efficiency, the roulette POM becomes a Gaussian convolution of the POM (6)

$$d\hat{\mu}_\eta(y) = \int_{-\infty}^{\infty} \frac{dy'}{\sqrt{2\pi\sigma^2_\eta}}\, d\hat{\mu}(y') \exp\left\{ -\frac{(y-y')^2}{2\sigma^2_\eta} \right\} \quad \sigma^2_\eta = \frac{1-\eta}{4\eta}\;, \tag{14}$$

whereas the unbiased estimator is now given by [10]

$$y_\eta \equiv \mathcal{R}_\eta[a^\dagger a](x) = 2x^2 - \frac{1}{2\eta}\;. \tag{15}$$

Inserting Eqs. (14) and (15) in Eq. (10) one has

$$\overline{\Delta y_\eta}^2 = \langle \widehat{\Delta n^2} \rangle + \frac{1}{2}\langle \widehat{n^2} \rangle + \langle \hat{n} \rangle \left(\frac{2}{\eta} - \frac{3}{2} \right) + \frac{1}{2\eta^2}\;. \tag{16}$$

This noise has to be compared with the rms variance of direct detection for nonunit quantum efficiency, which is given by

$$\langle \widehat{\Delta n^2} \rangle_\eta = \langle \widehat{\Delta n^2} \rangle + \langle \hat{n} \rangle \left(\frac{1}{\eta} - 1 \right)\;. \tag{17}$$

The difference between $\overline{\Delta y_\eta}^2$ and $\langle \widehat{\Delta n^2} \rangle_\eta$ defines the noise $N_R[\hat{n}]$ added by Quantum Roulette with respect to direct detection

$$N_R[\hat{n}] = \frac{1}{2}\left[\langle \widehat{n^2} \rangle + \langle \hat{n} \rangle \left(\frac{2}{\eta} - 1 \right) + \frac{1}{\eta^2} \right]\;. \tag{18}$$

$N_R[\hat{n}]$ is always positive, and roulette determination is more noisy than direct photodetection even for nonunit quantum efficiency.
Let us now consider heterodyne detection, namely the joint measurement of two commuting photocurrents, which, in turn, trace a pair of conjugated field quadratures. Each experimental event corresponds to a point in the complex plane of the field amplitude and these outcomes are distributed according to the generalized Wigner function $W_s(\alpha,\bar{\alpha})$ with ordering parameter s related to the quantum efficiency as $s = 1 - 2\eta^{-1}$. Starting from the relation

$$\overline{|\alpha|^2} = \int_{\mathbf{C}} d^2\alpha\, \alpha\alpha^*\, W_s(\alpha,\bar{\alpha}) = \langle a^\dagger a\rangle + \frac{1}{\eta}\,, \tag{19}$$

we are led to consider the shifted square modulus $I_\eta = |\alpha|^2 - 1/\eta$ as the heterodyne unbiased estimator for the field intensity. For unit quantum efficiency $\eta = 1$ heterodyne detection measures the Husimi Q-function $\langle\alpha|\hat{\varrho}|\alpha\rangle$, and thus the POM for the field intensity $d\hat{\mu}(I)$ is the marginal one of the Arthurs-Kelly coherent-state POM $d\hat{\mu}(\alpha) = \pi^{-1}|\alpha\rangle\langle\alpha|$, we have

$$d\hat{\mu}(I) = dI \sum_{k=0}^{\infty} e^{-(I+1)}\, \frac{(I+1)^k}{k!}\, |k\rangle\langle k| = dI\, e^{-(I+1)}\, \frac{(I+1)^{a^\dagger a}}{(a^\dagger a)!}\,. \tag{20}$$

For the determination $\overline{I_\eta} = \langle\hat{n}\rangle$ we need to specify the statistical deviation

$$\overline{\Delta I_\eta}^2 = \overline{I_\eta^2} - \bar{I_\eta}^2\,. \tag{21}$$

By using Eq. (19) and the following relation

$$\overline{|\alpha|^4} = \int_{\mathbf{C}} d^2\alpha\, \alpha^2\alpha^{*2}\, W_s(\alpha,\bar{\alpha}) = \langle\widehat{n^2}\rangle + \left(\frac{2}{\eta} - 1\right)\langle\hat{n}\rangle + \frac{1}{\eta^2}\,, \tag{22}$$

we arrive at the result

$$\overline{\Delta I_\eta}^2 = \langle\widehat{\Delta n^2}\rangle + \left(\frac{2}{\eta} - 1\right)\langle\hat{n}\rangle + \frac{1}{\eta^2}\,, \tag{23}$$

which represents the precision of heterodyne detection in measuring the field intensity. From Eqs. (17) and (23) we obtain the noise added by heterodyne detection with respect to direct detection

$$N_H[\hat{n}] = \frac{1}{\eta}\left[\langle\hat{n}\rangle + \frac{1}{\eta}\right]\,. \tag{24}$$

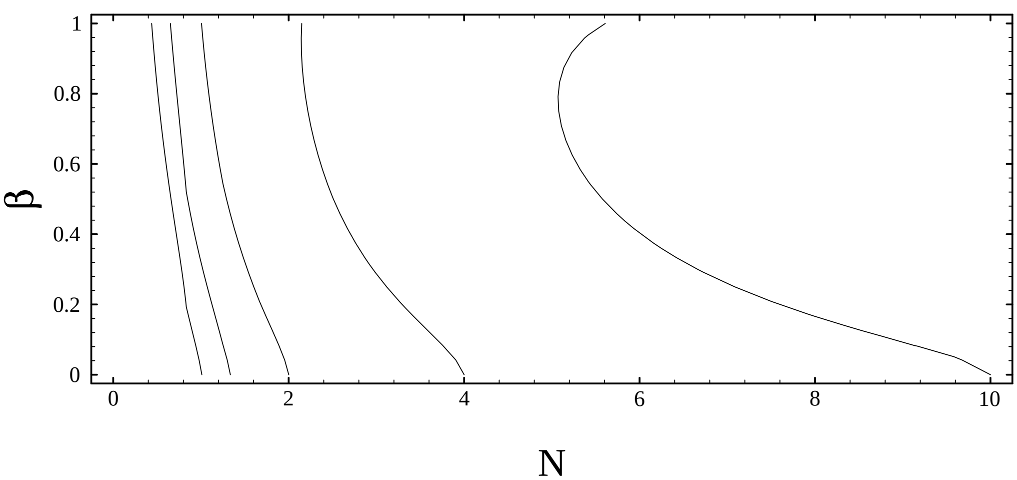
$N_H[\hat{n}]$ is always a positive quantity, thus also heterodyne detection is more noisy than direct detection for any value of the quantum efficiency.
A direct comparison between Quantum Roulette and heterodyne detection can be obtained by considering the difference $\Delta_{RH}[\hat{n}] = \overline{\Delta y_\eta}^2 - \overline{\Delta I_\eta}^2$. From Eqs. (16) and (23)

one has

$$\Delta_{RH}[\hat{n}] = \langle\widehat{\Delta y^2_\eta}\rangle - \langle\widehat{\Delta I^2_\eta}\rangle = \frac{1}{2}\left[\langle\widehat{n^2}\rangle - \langle\hat{n}\rangle - \frac{1}{\eta^2}\right]$$
$$= \sum_{n=0}^{\infty}\frac{1}{2}\left[n^2 - n - \frac{1}{\eta^2}\right]\varrho_{nn}\ , \tag{25}$$

ϱ_{nn} being the diagonal elements of the signal density matrix. $\Delta_{RH}[\hat{n}]$ has no definite sign, when changing the states of radiation. Therefore, it is matter of convenience to choose between the two kinds of detection scheme, depending on the state under examination. For any value of the quantum efficiency η the quantity $[n^2 - n - \frac{1}{\eta^2}]$ becomes positive for n larger than the threshold value

$$n_T = \frac{1 + \sqrt{1 + 4/\eta^2}}{2}\ . \tag{26}$$



Comparison between the Quantum Roulette and heterodyne detection noises in the determination of the field-intensity. The lines where $\Delta_{RH}[\hat{n}] = 0$ for squeezed states are plotted for different values of the quantum efficiency (from left to right: $\eta = 1.0, 0.75, 0.5, 0.25$ and 0.1) as a function the total mean photon number N and the squeezing photon fraction β. The region on the left of each curve corresponds to states for which the Quantum Roulette is more convenient than heterodyne detection. This region becomes larger for decreasing η.

This means that the Quantum Roulette is more precise than heterodyne detection for low excited states, as the higher-n terms get a lower weight ϱ_{nn}. Moreover, the lower is the quantum efficiency the larger is the threshold value, and hence the region where the Quantum Roulette is convenient. This means that the Quantum Roulette is very robust to low quantum efficiency. For coherent states the Quantum Roulette becomes convenient with respect to heterodyne detection when the field intensity is lower than the value $\langle \hat{n} \rangle = \eta^{-1}$. For squeezed states (we consider the case of zero signal and zero squeezing phases) Eq. (25) can be rewritten as

$$\Delta_{RH}[\hat{n}] = N^2 + 2\beta N(1+\beta N) + (1-\beta)N(1+2\beta N + 2\sqrt{\beta N(1+\beta N)}) - N - \frac{1}{\eta^2} \,,$$

where N denotes the total mean photon number of the state and β the squeezing photon fraction, namely the ratio between the (mean) photons engaged in squeezing and the total one (for $\beta = 0$ we have a coherent state and for $\beta = 1$ we have squeezed vacuum). In Fig. the lines where $\Delta_{RH}[\hat{n}]$ is zero are plotted for different values of the quantum efficiency. The region on the left of each curve corresponds to states for which the Quantum Roulette is more convenient than heterodyne detection.

IV. Naimark Extensions

Naimark theorem [11] assures that every POM is a partial trace of a customary orthogonal projection-valued measure on a larger Hilbert $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_P$ space which, itself, represents the original system interacting with appropriate probe mode(s). This extension is not unique, corresponding to the different possible physical implementation of a given measurement.

For the abstract Quantum Roulette of Helstrom, defined in Eq. (1), there exists a standard recipe to obtain a Naimark extension. This is found by considering the following projectors [8]

$$\hat{E}_m = \sum_{k=1}^{M} \hat{E}_m^{(k)} \otimes |\omega_k\rangle\langle\omega_k| \,, \tag{27}$$

where $|\omega_k\rangle$ denotes an orthogonal set of states in the extension (probe) $\mathcal{H}_P$ Hilbert space. In fact, by preparing the probe in the state

$$|\psi_P\rangle = \sum_{k=1}^{M} z_k^{1/2} \, |\omega_k\rangle \,, \tag{28}$$

we have

$$\mathrm{Tr}_P \left\{ \hat{1}_S \otimes |\psi_P\rangle\langle\psi_P| \, \hat{E}_m \right\} = \hat{\Pi}_m \,. \tag{29}$$

The POM in Eq. (4) is the continuous version of the quantum roulette with ϕ playing the role of the label k. The Helstrom recipe to achieve a Naimark extension requires an orthogonal POM for the phase ϕ in $\mathcal{H}_P$ yielding a resolution of identity in $[-\pi, \pi]$. As it describes a phase variable, such a POM has to satisfy the additional requirement of covariance, namely

$$d\hat{\mu}(\phi) = \frac{d\phi}{2\pi} \, e^{-i\hat{S}\phi} \, \hat{P} \, e^{i\hat{S}\phi} \,, \tag{30}$$

$\hat{S}$ being the phase-shift generator in the probe Hilbert space, and $\hat{P}$ a suitable positive operator.
The "minimal" implementation for the roulette POM would be a two-mode system, with the additional mode playing the role of the probe. However, such an implementation cannot be fully quantum, because no single-mode orthogonal set the phase variable exists. By using the canonical (Susskind-Glogower) phase POM $d\hat{\mu}(\phi) = (2\pi)^{-1}d\phi\sum_{nm}\exp\left[i(n-m)\phi\right]|n\rangle\langle m|$ we have an approximated extension described by the two-mode (non-orthogonal) POM

$$d\hat{M}(x) = dx\int\frac{d\phi}{2\pi}\,|x\rangle_{\phi\phi}\langle x|\,\otimes\,\sum_{nm}\exp\left[i(n-m)\phi\right]\,|n\rangle\langle m|\,, \tag{31}$$

which corresponds to the measured photocurrent

$$\hat{X} = \int x\,d\hat{M}(x) = a^\dagger\hat{e}_- + a\hat{e}_+\,, \tag{32}$$

$\hat{e}_\pm$ being the raising and lowering operators on $\mathcal{H}_P$

$$\hat{e}_+ = b^\dagger\,\frac{1}{\sqrt{b^\dagger b+1}} \qquad \hat{e}_- = \frac{1}{\sqrt{b^\dagger b+1}}\,b\,. \tag{33}$$

An exact two-mode implementation can be achieved in the semiclassical limit of highly excited probe mode. The basic idea is that coherent states $|z\rangle$ provide exact phase states for high amplitude $|z|\to\infty$. In this limit we can neglect the ordering in (33) and writing

$$a^\dagger\hat{e}_- + a\hat{e}_+ \simeq \frac{a^\dagger b + ab^\dagger}{\sqrt{b^\dagger b+1}} \simeq \frac{a^\dagger b - +ab^\dagger}{\sqrt{\langle b^\dagger b\rangle}}\,, \tag{34}$$

which coincides with the homodyne photocurrent. The prescription (28) for the probe preparation becomes

$$\hat{\varrho}_P = \lim_{|z|\to\infty}\int\frac{d\phi}{2\pi}\,\left||z|e^{i\phi}\rangle\langle|z|e^{i\phi}\right|\,. \tag{35}$$

Eqs. (34) and (35) unambiguously identify random-phase homodyne detection as the minimal (semiclassical) implementation of the Quantum Roulette POM of Eq. (4).
A fully quantum extension requires more than one mode as the probe. An example is provided by the heterodyne phase eigenstate, which are defined on a two-mode Hilbert space [12].

V. Conclusions

The Quantum Roulette is a concrete example of the Quantum Roulette of Helstrom. In this paper we compared intensity measurement by the Quantum Roulette with that from direct photodetection and from heterodyne detection. Direct photodetection, though only in principle, remains the most precise detection scheme, also in the case of nonunit quantum efficiency. On the other hand, the choice between Quantum Roulette and heterodyne detection is a matter of convenience, depending on the state under examination. For coherent states, the Quantum Roulette becomes convenient with respect to heterodyne detection when the field intensity is lower than threshold value $\langle \hat{n} \rangle = \eta^{-1}$. In general, Quantum Roulette is more convenient for low excited states, and is more robust in presence of low efficient photodetectors.

Acknowledgements

The work of M.G.A.P. is supported by a post-doctoral grant of the University of Pavia.

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