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Stability of coherent states

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Abstract. We give an algebraic characterisation of the dynamical systems which preserve the coherence of a generalised coherent state defined for a Lie group. The breaking of coherence is related to singularities appearing in the S matrix. We show a regularisation procedure to eliminate such singularities based on the jet realisation of the diffeomorphism group induced by contact transformations on the state manifold.

1. Introduction

The usual coherent states were introduced as states of a physical system sharply localised in position and momentum around the classical values (Glauber 1963, 1966). They could therefore be thought of as the quantum states more closely related to the classical ones. For these coherent states (customarily referred to as Glauber's coherent states) there is an additional interesting feature connected with their dynamics. Under the action of a harmonic oscillator Hamiltonian they evolve preserving their shape in time, namely remaining coherent (Metha *et al* 1967, Kumar and Metha 1980).

Different methods of generating coherent states have been devised in the literature over recent years, most of them seeking a generalisation capable of extending the above properties to cases of more direct physical significance (Nieto and Simmons 1979a, b, c, Nieto 1980, Gutschick and Nieto 1980, Nieto *et al* 1981).

The most promising among such generalisations is the construction of the coherent states connected to a group (Rasetti 1973, 1975, Perelomov 1972).

The concept of coherent states for an arbitrary Lie group G is based on several ingredients. First the existence of a fixed cyclic vector $|\omega\rangle$ in the Hilbert space $\mathcal{H}$ is required—whose translates under the group action, $T(g)|\omega\rangle$, $g \in \mathrm{G}$ are just the coherent states.

One notices that in the case when G is compact and semi-simple $|\omega\rangle$ certainly exists, and is such that—upon denoting by $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ the complexification of the Lie algebra of G, whose Cartan decomposition writes

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

—it is nothing other than the vector $|\omega\rangle \in \mathcal{H}$ satisfying

$$T(h)|\omega\rangle = e^{\lambda}(h)|\omega\rangle \qquad h \in h$$

§ Also GNSM of the CNR.

$$X|\omega\rangle = 0 \qquad X \in \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$$

where h is the Cartan subgroup of G and for any $X \in \mathfrak{g}$

$$e^\lambda(\exp(X)) = \exp(\lambda(X))$$

$\lambda \in i\mathfrak{g}^*$ denoting the highest weight.

On the other hand the homogeneous space G/h is acted on by G by means of holomorphic transformations, whereby the coherent state representatives can be thought of as holomorphic sections of an homogeneous line bundle. One can check that, if G is simple, G/h is a simply connected Hodge manifold (Hurt 1968) and if, moreover, the maximal root λ is non-degenerate there is a unique fundamental field which can be realised in terms of a G-invariant form on G/h.

Thus the manifold G/h can be viewed in that case as just the (classical) phase space on which G acts through canonical transformations.

All of this suggests that there should be, in general, a dynamical system naturally associated with the manifold of coherent states. In terms of it the evolution of such states should keep the state representative point in the orbit of G. There arises the concept of coherence preserving action.

The aim of the present paper is to investigate the global features of such an action. The language of modern algebraic geometry should be the most appropriate tool for addressing the question in its full formal generality. We prefer however to tackle here the problem in a more physically intuitive—although, of course, rigorous—way, leaving a more formal approach to a later paper.

In § 2 the concept of stability is introduced, using as a reference the coherent states for the Weyl group, which are indeed just the coherent states of Glauber. In § 3 the problem of coherence breaking is taken into consideration and a general technique to avoid the singularities it leads to is described. The latter is based on the jet realisation of the diffeomorphism group induced by contact Bäcklund transformations on the state manifold. Moreover in § 4 a toy model is thoroughly analysed in which the latter technique gives interesting results. Section 5 concludes with a few remarks on the possible role of Morse functions in the framework discussed above.

2. Stability of coherent states

The question of constructing the most general Hamiltonian for which a system—prepared so as to be in a coherent state at the initial time t_0—will remain coherent for all times, has been the object of several discussions (Glauber 1963, 1966, Metha *et al* 1967, Kumar and Metha 1980). Such discussions, however, have always been restricted to the usual Glauber coherent states,

$$|\{z_\lambda\}\rangle = \exp\left(-\frac{1}{2}\sum_{\lambda=1}^{N}|z_\lambda|^2\right)\prod_{\lambda=1}^{N}\exp(z_\lambda a_\lambda^+)|O_N\rangle$$

$$z_\lambda \in \mathbb{C} \qquad \lambda = 1, 2, \ldots, N \tag{2.1}$$

$|O_N\rangle$ denoting the N-particle vacuum state.

In that case the system consists of a set of N quantum harmonic oscillators, described in terms of creation and annihilation operators a_λ^+, a_λ; $[a_\lambda, a_\mu^+] = \delta_{\lambda\mu}$;

$\lambda, \mu = 1, \ldots, N$. The most general coherent-state-preserving Hamiltonian reads:

$$H(t) = \sum_{\lambda,\mu=1}^{N} \omega_{\lambda\mu}(t) a_\lambda^+ a_\mu + \sum_{\lambda=1}^{N} (F_\lambda(t) a_\lambda + \bar{F}_\lambda(t) a_\lambda^+) + \beta(t) \tag{2.2}$$

where $\bar{\omega}_{\lambda\mu} = \omega_{\mu\lambda}$ and $\bar{\beta} = \beta$.

Indeed, upon writing the solution of the equation of motion in the form

$$|\psi(t)\rangle = \mathbb{U}(t, t_0)|\psi(t_0)\rangle \qquad \mathbb{U}^+ = \mathbb{U}^{-1} \tag{2.3}$$

we have

$$\mathbb{U}(t, t_0) = \mathbb{P}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} H(t')\,\mathrm{d}t'\right)\right] \tag{2.4}$$

where $\mathbb{P}$ is the time ordering operator. On the other hand, the time evolution operator $\mathbb{U}(t, t_0)$ has a simple representation as a functional integral (path integral over the coherent state manifold $\mathbb{C}^N$):

$$\mathbb{U}(t, t_0) = \int_{\mathbb{C}^2} \frac{\mathrm{d}^2\xi\,\mathrm{d}\zeta}{\pi^2} |\xi\rangle U(\bar{\xi}, \zeta)\langle\zeta| \tag{2.5}$$

where the kernel U is defined by

$$U(\bar{\xi}, \zeta) = \int \mathscr{D}[\bar{z}]\mathscr{D}[z] \exp\left(\frac{\mathrm{i}}{\hbar} S_{\bar{\xi}\zeta}[z, \bar{z}]\right) \qquad z, \bar{z} \in \mathbb{C}^N \tag{2.6}$$

with

$$S_{\bar{\xi}\zeta}[z, \bar{z}] = \int_{t_0}^{t} \mathrm{d}t' \langle z(t')|\mathrm{i}\hbar\partial/\partial t' - H(t')|z(t')\rangle$$

$$= \int_{t_0}^{t} \mathrm{d}t' [\tfrac{1}{2}\mathrm{i}\hbar(\bar{z}\dot{z} - \dot{\bar{z}}z) - \mathbb{H}(z, \bar{z})] \tag{2.7}$$

where $\bar{z}z' \doteq \sum_{\lambda=1}^{N} \bar{z}_\lambda z'_\lambda$; the fixed end-point conditions $z(t_0) = \zeta$, $\bar{z}(t) = \bar{\xi}$ are imposed and $\mathbb{H}(z, \bar{z})$ is the diagonal matrix element of the Hamiltonian in the coherent state basis:

$$\mathbb{H}(z, \bar{z}) = \langle z(t')|H(t')|z(t')\rangle. \tag{2.8}$$

Upon inserting (2.2) into (2.5)–(2.8) one obtains the explicit form of the representation

$$\mathbb{U}(t, t_0) = \exp\left(\mathrm{i}\,\mathrm{Im}\,\bar{z}_\lambda z_0 + \sum_{\lambda=1}^{N} [(z_{\lambda,t} - z_{\lambda,0})a_\lambda^+ - (\bar{z}_{\lambda,t} - \bar{z}_{\lambda,0})a_\lambda]\right) \tag{2.9}$$

given by Kumar and Metha (1980).

Little or no attention at all, on the contrary, has been devoted to the same problem in the case of generalised coherent states for an arbitrary Lie group G (Rasetti 1973, 1975, Perelomov 1972). The latter are defined according to the following scheme. Let $g \in \mathrm{G}$ and denote by $T(g)$ an irreducible unitary representation of g on a suitable Hilbert space $\mathscr{H}$. Let also h be the stability subgroup of some vector $|\omega\rangle \in \mathscr{H}$ and $M = \mathrm{G}/h$ the corresponding factor space. The coherent states $|x\rangle$ for G, labelled by $x \in M$, are constructed as

$$|x\rangle = \exp(-\mathrm{i}\alpha(g))|\phi_g\rangle \tag{2.10}$$

$$|\phi_g\rangle = T(g)|\omega\rangle \tag{2.11}$$

where $\alpha: \mathrm{G} \to \mathbb{R}$.

Notice that the existence of a fixed cyclic vector $|\omega\rangle \in \mathscr{H}$ is guaranteed if G is either a non-compact connected, real semi-simple Lie group with finite centre, or it is solvable. In the latter case h coincides with the maximal compact subgroup K entering the Iwasawa decomposition of G:

$$\mathrm{G} = \mathrm{KAN} \tag{2.12}$$

where A is the Abelian compact subgroup of G and N its maximal nilpotent subgroup.

If G is compact, $|\omega\rangle$ can be chosen to be the highest weight vector of the irreducible representation $T(g)$. Extending $T(g)$ to a holomorphic representation of the complexification $\mathrm{G_c}$ of G in $\mathscr{H}$, the stability subgroup of $|\omega\rangle$ under $T(g)$ becomes a parabolic subgroup $\mathscr{P}$ of $\mathrm{G_c}$ (McDonald 1979). The algebra of $\mathscr{P}$ is $\mathfrak{p} = \mathfrak{b} \oplus \Sigma_{\alpha \in E} \mathfrak{g}_{-\alpha}$ where $\mathfrak{b}$ is the algebra of the Borel subgroup B of $\mathrm{G_c}$ and $E = \{\alpha \in \Delta^+ | (\alpha, \omega) = 0\}$ is the set of positive roots orthogonal to the maximal weight. In this case the stability subgroup h of G is $\mathscr{P} \cap \mathrm{G}$, hence by Bott's theorem (Bott 1953),

$$\mathrm{G}/h \equiv \mathrm{G_c}/\mathscr{P} \tag{2.13}$$

the coherent state representatives can be viewed as holomorphic sections of the homogeneous line bundle $L_\chi(\mathrm{G}/h, \mathbb{C})$ associated with the principal fibre bundle $\mathscr{P} \to \mathrm{G_c} \to \mathrm{G_c}/\mathscr{P}$ by the holomorphic character χ:

$$T(p)|\omega\rangle = \chi(p)|\omega\rangle \qquad p \in \mathscr{P}. \tag{2.14}$$

We will show that the most general Hamiltonian H preserving the states $|x\rangle \in M$ is an element of the extension of the Lie algebra $\mathfrak{g}$ of G by the algebra $\mathfrak{a} = \mathfrak{s}/\mathfrak{g}$ where $\mathfrak{s}$ is the algebra associated with $\mathscr{G} = \mathrm{Aut}(M)$, the group of automorphisms of M, i.e.

$$H \in \mathfrak{s}. \tag{2.15}$$

If G is semi-simple, $\mathfrak{s}$ coincides—up to identity automorphisms which could be realised only by an extension by a direct sum of ideals—with $\mathfrak{g}$; whereas if G is solvable, $\mathfrak{s}$ is the algebra in whose Levi decomposition the maximal solvable ideal is $\mathfrak{g}$,

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{g} \tag{2.16}$$

where the semi-simple Lie algebra $\mathfrak{a}$ preserves $|\omega\rangle$.

The proof of the above statements is based on the following observations. In order to preserve the coherence of the initial state, due to definitions (2.3), (2.4) the time evolution operators $\mathbb{U}(t, t_0)$ are required to be in one-to-one correspondence with the elements of $\mathrm{Aut}(M)$. We realise $\mathrm{Aut}(M)$ by the adjoint representation $\mathrm{Ad}(\mathscr{G})$. If G is semisimple (Helgason 1968)

$$\mathscr{G} = \mathrm{G} \times \mathrm{D} \tag{2.17}$$

where D denotes a discrete subgroup, and

$$\mathrm{Ad}(\mathscr{G}) \simeq \mathscr{G}. \tag{2.18}$$

It follows that the Hamiltonian H is an element of $\mathfrak{g}$, G is therefore a dynamical group.

If G is solvable,

$$|x_t\rangle \equiv \mathbb{U}(t, t_0)|x_0\rangle = \mathrm{Ad}(s)T(gs)|\omega\rangle \qquad s \in \mathscr{G} \qquad g \in \mathrm{G} \tag{2.19}$$

is a coherent state of G if both G and h are invariant subgroups of $\mathscr{G}$ and the algebra $\mathfrak{s}$ of $\mathscr{G}$ is decomposable according to (2.16).

Indeed in such a case h is certainly an invariant subgroup of $\mathscr{G}$, in that it is contained in the maximal invariant subgroup of $\mathscr{G}$, and G is an ideal of $\mathscr{G}$.

It is instructive to think of the case of Glauber coherent states, synthesised in equations (2.1), (2.3), as a particular case of this general formulation. This is simply done by recognising in $\mathfrak{g}$ the Weyl algebra

$$\{a_\lambda, a_\mu^+, \mathbb{I}; [a_\lambda, a_\mu^+] = \delta_{\lambda\mu}\mathbb{I}; \lambda, \mu = 1, \ldots, N\} \tag{2.20}$$

in $\mathfrak{a}$ the algebra $\mathfrak{gl}(N, \mathbb{C})$ of the extension of the little group (stability subgroup of $|\omega\rangle \equiv |O_N\rangle$) and

$$\begin{aligned}\mathfrak{s} = \{b_{\mu\lambda} = b_{\lambda\mu}^+ \doteq a_\mu^+ a_\lambda\,;\, a_\lambda, a_\mu^+, \mathbb{I}; [b_{\lambda\mu}, b_{\sigma\rho}] = \delta_{\sigma\mu} b_{\lambda\rho} - \delta_{\lambda\rho} b_{\sigma\mu}\,;\\ [b_{\lambda\mu}, a_\sigma] = -\delta_{\lambda\sigma} a_\mu\,; [a_\lambda, a_\mu^+] = \delta_{\lambda\mu}\mathbb{I}; \lambda, \mu, \sigma, \rho = 1, \ldots, N\}.\end{aligned} \tag{2.21}$$

3. Coherence breaking

The discussion of § 2 leads in a natural way to enquiring how the picture is modified when H is perturbed by addition of a small term $W(t)$, breaking the coherence. $W(t)$ as a function of t has a compact support. The problem is somewhat like the quantum analogue of studying the stability of KAM tori under non-integrable perturbations (Moser 1973).

It appears from equation (2.15) that the addition to H of any element of

$$\mathfrak{G} = \bigcup_{2 \leq p \leq k} \mathfrak{e}^{(p)}/\mathfrak{s}$$

where k is a finite integer greater than or equal to 2, and $\mathfrak{e}^{(p)}$ denotes the enveloping algebra of order p of $\mathfrak{g}$, shall, in general, violate the property of preserving the coherent states of G.

The new Hamiltonian $H + W$ is now an element of an infinite dimensional Lie algebra $\mathfrak{l}$, the universal enveloping algebra of $\mathfrak{g}$. On the other hand it was shown by D'Ariano and Rasetti (1985a) that the coherent states for the infinite dimensional Lie algebra $\mathfrak{gl}(\infty)$ coincide with the τ functions, namely the set of all (polynomial) solutions to the hierarchies of equations encountered in soliton theory, and the evolution of such coherent states can be thought of as a succession of infinitesimal Bäckland contact transformations.

The latter have been thoroughly investigated, in particular in view of defining global criteria of integrability. The most promising approach, based on Cartan's theory of exterior differential systems is the prolongation method of Wahlquist and Estabrook (1973, 1975).

When the infinite dimensional Lie algebra of Wahlquist and Estabrook is replaced by a finite dimensional one, the prolongation process coincides with the construction of a connection whose curvature vanishes on solutions of the prolonged equation. The natural framework of theory becomes then that of jet bundles.

A system of nonlinear partial differential equations of order s is defined to be the submanifold $\mathscr{Z}$ of a s-jet bundle $J^{(s)}$, equal to the zero set of a finitely generated ideal of functions on $J^{(s)}$ itself. The pull-back map from $J^{(s)}$ to $\mathbb{R}$ defines the contact module $\Omega^{(s)}$ (Pirani *et al* 1977). If $t > s$ then $\Omega^{(s)}$ is a submodule of $\Omega^{(t)}$.

If the integrability conditions of a map

$$\mathscr{B}: J^{(s)} \times \mathbb{R} \to J^{(1)} \tag{3.1}$$

comprise a system of differential equations on $J^{(s+1)} \times \mathbb{R}$, then $\mathscr{B}$ is a Bäckland map. Prolongations $\mathscr{B}^{(t)}$ are maps of higher jet bundles and systems $\mathscr{X}^{(t)}$ induced from $\mathscr{B}$ and $\mathscr{X}$ (which in coordinates amount merely to taking total derivatives). If there is an integer t such that the image of $\mathscr{B}^{(t)}|\mathscr{X}^{(t)}$ is a system $\mathscr{X}'$ of differential equations on $J^{(t+1)}$, then the Bäckland transformation is nothing but the correspondence between $\mathscr{X}$ and $\mathscr{X}'$. Bäckland maps are then the natural generalisation of contact transformations. They may be thought of as (local) diffeomorphisms of $J^{(1)}$ satisfying $\mathscr{B}^*\Omega^{(1)} = \Omega^{(1)}$ ($\mathscr{B}^*$ is the map, induced by $\mathscr{B}$, of forms and functions—whereas $\mathscr{B}$ is a map of manifolds).

The functions on $J^{(1)}$ can be endowed in a natural way with the infinite Lie algebra structure by considering the vector field $\mathscr{V}$ on $J^{(1)}$ defined by

$$\begin{gathered} \mathscr{V} \lrcorner \vartheta = 0 \\ \mathscr{V} \lrcorner \mathrm{d}\vartheta - \omega \in \Omega^{(1)} \end{gathered} \tag{3.2}$$

for a given contact form ϑ and any 1-form ω; where $\lrcorner$ denotes the interior product of a vector field and a form.

If $\omega = \mathrm{d}f$, where f is a function on $J^{(1)}$, then—writing $\mathscr{V}_f$ for the field $\mathscr{V}$—the Lie bracket is

$$f \circ g \doteq [f, g] = \mathscr{V}_f g. \tag{3.3}$$

$\mathscr{B}$ is in this case an automorphism of the Lie algebra structure characterised by (3.3).

The obvious generalisation of such a scheme is to (local) diffeomorphisms of $J^{(s)}$ which preserve $\Omega^{(s)}$. By a suitable choice of the basis one can identify the functions on $J^{(1)}$ with the components of the cross sections of the tangent bundle $T(\mathscr{K})$ to a differentiable manifold $\mathscr{K}$. I which is isomorphic with the gauge current algebra $\mathscr{A}$ generated over $\mathscr{K}$ by the structure (3.3)—it can be thought of as a submodule of differential operators over the algebra $F(\mathscr{K})$ of C^∞ functions on $\mathscr{K}$ with compact support. On the other hand equations (3.2) imply that $\mathscr{V}_f$ is tangent to the fibres of a maximal rank map $\tilde{\pi}: E \to \mathscr{K}$, where E is the manifold over which the 1-form ϑ is defined.

Let $\mathscr{E}$ be the space of all maps $\tilde{\pi}$, and consider the maps $a: \mathscr{K} \to T(\mathscr{K})$. ϑ induces on $\mathscr{E}$ a contact 1-form ϑ_a which can be written in the functional form

$$\vartheta_a = \int_{\mathscr{K}} \vartheta(a(p))\, \mathrm{d}\mu(p)$$

where $\mathrm{d}\mu(p)$, $p \in \mathscr{K}$ is a suitable measure over $\mathscr{K}$. In such scheme, $\mathscr{A}$ has generators

$$\sigma_a = \int_{\mathscr{K}} \gamma(p) a(p)\, \mathrm{d}\mu(p) \tag{3.4}$$

where $\gamma(p)$ is bilinear in the elements of the Weyl chamber of the Abelian component in the Iwasawa decomposition of $\mathscr{A}$, realised as a module of differential operators over $\mathscr{K}$, such that

$$[\sigma_a, \sigma_b] = \alpha_{ab} \sigma_{a \circ b} \qquad \alpha_{ab} \in \mathbb{C}. \tag{3.5}$$

The map induced by $\tilde{\pi}$, $F(\mathscr{H}) \to F(E)$ enables one to identify $F(\mathscr{H})$ with a subring $\mathscr{R}_0$ of $F(E)$. Define $\mathscr{R}_1$ to be the subspace of functions $f \in F(E)$ such that $\mathscr{V}_f = 0$. One has $[\mathscr{R}_0, \mathscr{R}_0] = 0$, $[\mathscr{R}_1, \mathscr{R}_0] \subset \mathscr{R}_0$, $[\mathscr{R}_1, \mathscr{R}_1] \subset \mathscr{R}_1$. The Lie subalgebra $F^{(1)} = \mathscr{R}_0 + \mathscr{R}_1$ of $F(E)$ has then a representation by first-order differential operators on $\mathscr{H}$ (Hermann 1970).

One can then further define a filtration $\{\mathfrak{l}^{(n)}\}$ of $\mathfrak{l}$ by the grading on $F(E)$ given recursively by $\mathscr{R}_n = \mathscr{R}_1 \mathscr{R}_{n-1}$, $n \geq 1$. The filtered algebra $F^{(n)}$ associated to the latter is $F^{(n)} = \mathscr{R}_0 + \cdots + \mathscr{R}_m$, and one has $[F^{(n)}, F^{(m)}] \subset F^{(n+m-1)}$, whence

$$[\mathfrak{l}^{(n)}, \mathfrak{l}^{(m)}] \subset \mathfrak{l}^{(n+m-1)}. \tag{3.6}$$

Notice that the set of elements $\bigcup_{0 \leq n \leq q} \mathfrak{l}^{(n)}$ for a fixed q, in general does not form a subalgebra of $\mathfrak{l}$. It does only if $q = 1$. On the other hand, however,

$$\mathfrak{L}_q = \bigcup_{n=q}^{\infty} \mathfrak{l}^{(n)} \tag{3.7}$$

is a subalgebra of $\mathfrak{l}$. In particular, for $q \geq 1$, $\mathfrak{L}_q$ is an invariant subalgebra of $\mathfrak{L}_1$. One can therefore form the factor

$$\mathfrak{T}_q = \mathfrak{L}_1 / \mathfrak{L}_{q+1} \qquad \forall q \geq 1 \tag{3.8}$$

$\mathfrak{T}_q$ is an algebra. We shall denote by $\{\mathscr{T}_l^{\Lambda_s}\}_{s=1}^{q}$ the infinitesimal generators of the stabiliser subgroup $\mathscr{S}$ of $\mathscr{L}$ ($\mathscr{L}$ denoting the group obtained by exponentation of $\mathfrak{l}$), leaving the point $p \in \mathscr{H}$ fixed. Upon denoting by Λ_s, $\bar{\Lambda}_{t-s}$ the poly-indices $\Lambda_s = \{l_1, \ldots, l_s\}$, $\bar{\Lambda}_{t-s} = \{\bar{l}_{s+1}, \ldots, \bar{l}_t\}$, $t \geq s+1$;

$$[\mathscr{T}_l^{\Lambda_n}, \mathscr{T}_m^{\bar{\Lambda}_{t-n}}] = \delta_{l, \bar{l}_s} \mathscr{T}_{m}^{\bar{\Lambda}_t \setminus \bar{l}_s} - \delta_{m, l_r} \mathscr{T}_l^{\bar{\Lambda}_t \setminus l_r} \tag{3.9}$$

where $\bar{\Lambda}_t = \Lambda_n \cup \bar{\Lambda}_{t-n}$, with $n \geq 1$, $t \geq n+1$, $n+1 \leq s \leq t$, $1 \leq r \leq n$. The $\{\mathscr{T}_l^{\Lambda_s}\}_{s=1}^{q}$ form a representation of $\mathfrak{T}_q$. Let $\mathfrak{T}^{(n)}$ be the subset of all $\mathscr{T}_l^{\Lambda_n}$ for given $n > 0$. Due to the commutation relations (3.9) we have a grading relations similar to (3.6)

$$[\mathfrak{T}^{(n)}, \mathfrak{T}^{(m)}] \subset \mathfrak{T}^{(n+m-1)} \qquad n > 0 \tag{3.10}$$

Now, however, $\mathfrak{T}^{(0)}$ is empty, and it is possible to set, consistently, $\mathfrak{T}^{(n)} = 0$ for all n larger than some fixed integer z.

The representation

$$\mathscr{D}^{(z)} : \mathfrak{l} \to \bigcup_{n=1}^{z} \mathfrak{T}^{(n)}$$

$$\mathscr{D}^{(z)}(\sigma_a) = a \cdot \partial_p + \sum_{n=1}^{z} \sum_{l=1}^{\dim \mathscr{H}} \sum_{\Lambda_n} \frac{1}{n!} c_a^{\Lambda_n} \mathscr{T}_l^{\Lambda_n} \tag{3.11}$$

$$c_a^{\Lambda_n} = \frac{\partial^{L_n} a}{\partial p^{l_1} \ldots \partial p^{l_n}} \in \mathbb{C} \qquad L_n = \sum_{s=1}^{n} l_s$$

is the so-called jet representation of order z of $\mathfrak{l}$. In this representation $H + W$ still preserves a set of coherent states. The latter, however, are not the coherent states of G, but of the group $\tilde{G}_q$ obtained by exponentiation of $\mathfrak{T}_q$. In general only a set of measure zero of the orbit of $\tilde{G}_q$ belongs to the orbit of G.

4. Application to a simple solvable model

We now show the structure outlined in the previous section for a simple solvable example, once more related to the case when G is the Weyl group. Consider the choice

$$W(t)=\sum_{n=1}^{\nu}\delta(t-\tau_n)W_n$$
$$=\varepsilon\sum_{n=1}^{\nu}\hbar_n\delta(t-\tau_n)\sum_{\lambda,\mu=1}^{N}(a_\lambda+a_\lambda^+)^2(a_\mu+a_\mu^+)^2 \tag{4.1}$$

where $0\leq \hbar_n\leq 1$ are anisotropy factors, ε the characteristic coupling energy, $\{\tau_n\}$ an arbitrary ordered sequence of times in the interval t_0-t_1 and H is given by (2.2).

We intend to compute

$$\mathbb{U}(\bar{\xi},\zeta)=\left\langle\xi\left|\mathbb{P}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1}(H(t)+W(t))\,\mathrm{d}t\right)\right]\right|\zeta\right\rangle. \tag{4.2}$$

As we discussed before, the unperturbed Hamiltonian H preserves coherence and one expects to have to deal essentially with the effect of W. This is more effectively done in terms of the intermediate Dirac representation instead of the Schrödinger representation implicitly adopted so far. Indeed the transformation from the latter representation to the former is an element of G, and all the relevant information is therefore contained in

$$S(\bar{\eta},\vartheta)=\left\langle\eta\left|\mathbb{P}\left[\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1}\tilde{W}(t)\,\mathrm{d}t\right)\right]\right|\vartheta\right\rangle \tag{4.3}$$

where

$$\tilde{W}(t)=\exp\left(\frac{\mathrm{i}t}{\hbar}\operatorname{Ad}H(t)\right)W(t). \tag{4.4}$$

Now, due to the form of (4.1),

$$\mathbb{P}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1}\tilde{W}(t)\,\mathrm{d}t\right)\right]=\overset{\frown}{\prod}_n\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\tilde{W}_n\right)\right] \tag{4.5}$$

where $\overset{\frown}{\prod}_n$ denotes the ordered product over the discrete sequence $\{\tau_n\}$ and

$$\tilde{W}_n=\exp\left(\frac{\mathrm{i}\tau_n}{\hbar}\operatorname{Ad}H(\tau_n)\right)W(\tau_n). \tag{4.6}$$

One has therefore

$$S(\bar{\eta},\vartheta)=\int_{\mathbb{C}^{(\nu-1)N}}\left(\prod_{j=1}^{\nu-1}\frac{\mathrm{d}^2z_j}{\pi}\prod_{k=0}^{\nu-1}\left\langle z_k\left|\exp\left(\frac{\mathrm{i}\tau_{k+1}}{\hbar}\operatorname{Ad}H(\tau_{k+1})\right)\right.\right.\right.$$
$$\left.\left.\left.\times\exp\left(-\frac{\mathrm{i}}{\hbar}W_{k+1}\right)\right|z_{k+1}\right\rangle \tag{4.7}$$

where $|z_0\rangle\equiv|\eta\rangle$; $|z_\nu\rangle\equiv|\vartheta\rangle$. Since $H(t)$ preserves the coherence,

$$\left\langle z_k\left|\exp\left(\frac{\mathrm{i}\tau_{k+1}}{\hbar}\operatorname{Ad}H(\tau_{k+1})\right)\exp\left(-\frac{\mathrm{i}}{\hbar}W_{k+1}\right)\right|z_{k+1}\right\rangle$$
$$=\left\langle\xi_k\left|\exp\left(-\frac{\mathrm{i}}{\hbar}W_{k+1}\right)\right|\zeta_k\right\rangle \tag{4.8}$$

for some ξ_k; ζ_k; $k=0,\ldots,\nu-1$. The integrand in (4.7) is thus a product of ν factors of the form (we define $g_k \equiv \varepsilon \hbar_{k+1}/\hbar$, and hence drop the index k for the sake of simplicity of notation):

$$\langle\xi|\mathbb{S}|\zeta\rangle = \left\langle \xi \left| \exp\left[-\mathrm{i}g\left(\sum_{\lambda=1}^{N} (a_\lambda + a_\lambda^+)^2 \right)^2 \right] \right| \zeta \right\rangle. \tag{4.9}$$

The typical factor (4.9) can be computed explicitly by first expanding

$$\exp\left[-\mathrm{i}g\left(\sum_{\lambda=1}^{N} (a_\lambda + a_\lambda^+)^2 \right)^2 \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}g)^n}{n!} \sum_{\{\nu_\lambda\}} \frac{(2n)!}{\prod_{\lambda=1}^{N} \nu_\lambda!} \delta\left(\sum_{\lambda=1}^{N} \nu_\lambda - 2n \right) \prod_{\lambda=1}^{N} (a_\lambda + a_\lambda^+)^{2\nu_\lambda} \tag{4.10}$$

where $\{\nu_\lambda\}$ are all possible collections of N non-negative integers. By the Wilcox formula (Wilcox 1967), whereby

$$(a_\lambda + a_\lambda^+)^{2\nu_\lambda} = \sum_{l=0}^{2\nu_\lambda} \sum_{s=0}^{l} \begin{Bmatrix} 2\nu_\lambda, l \\ s \end{Bmatrix} (a_\lambda^+)^s a_\lambda^{l-s} \tag{4.11}$$

where the generalised binomial coefficient $\left\{ {a, b \atop c} \right\}$ reads

$$\begin{Bmatrix} a, b \\ c \end{Bmatrix} = \begin{cases} \dfrac{a!}{(2k)!!} \dbinom{b}{c} & \text{if } a-b=2k \\ 0 & \text{if } a-b \text{ is odd} \end{cases} \tag{4.12}$$

and upon setting $\omega_\lambda = (\bar{\xi}_\lambda + \zeta_\lambda)^2$, one gets in a straightforward manner

$$\begin{aligned} \langle\xi|\mathbb{S}|\zeta\rangle &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}g)^n}{n!} \sum_{\{\nu_\lambda\}} \frac{(2n)!}{\prod_{\lambda=1}^{N} \nu_\lambda!} \delta\left(\sum_{\lambda=1}^{N} \nu_\lambda - 2n \right) \prod_{\lambda=1}^{N} \sum_{q=0}^{\nu_\lambda} \frac{(2\nu_\lambda)!}{[2(\nu_\lambda - q)]!!} \omega_\lambda^q \langle\xi|\zeta\rangle \\ &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}g)^n}{n!} (2n)! \sum_{\{\nu_\lambda\}} \delta\left(\sum_{\lambda=1}^{N} \nu_\lambda - 2n \right) \prod_{\lambda=1}^{N} \frac{(2\nu_\lambda)!}{\nu_\lambda!} \omega_\lambda^{\nu_\lambda} E^{(\nu_\lambda)}(\tfrac{1}{2}\omega_\lambda^{-1}) \langle\xi|\zeta\rangle \end{aligned} \tag{4.13}$$

where $E^{(\mu)}(x) = \sum_{k=0}^{\mu} x^k/k!$ is the truncated exponential function. Upon representing now the inner sum in (4.13) constrained by the Kronecker delta as

$$\sum_{\{\nu_k\}} \delta\left(\sum_{k=1}^{N} \nu_k - P \right) \prod_{k=1}^{N} c^{(k)}(\nu_k) = \frac{1}{P!} \frac{\mathrm{d}^P}{\mathrm{d}z^P} \prod_{k=1}^{N} \mathscr{F}^{(k)}(z) \bigg|_{z=0} \tag{4.14}$$

with

$$\mathscr{F}^{(k)}(z) = \sum_{\nu=0}^{\infty} c^{(k)}(\nu) z^\nu \tag{4.15}$$

one finally obtains

$$\langle\xi|\mathbb{S}|\zeta\rangle = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}g)^n}{n!} \frac{\mathrm{d}^{2n}}{\mathrm{d}z^{2n}} \prod_{\lambda=1}^{N} \mathscr{F}^{(\lambda)}(z) \bigg|_{z=0} \langle\xi|\zeta\rangle \tag{4.16}$$

where the coefficients $c^{(k)}(\nu)$ are defined by

$$c^{(k)}(\nu) = \frac{(2\nu)!}{\nu!} \omega_k^\nu E^{(\nu)}(\tfrac{1}{2}\omega_k^{-1}). \tag{4.17}$$

The S matrix $\mathbb{S}$, as given by (4.16), is non-analytic at least in the ground state. Indeed for $\xi_\lambda = \zeta_\lambda = 0$, i.e. $\omega_\lambda = 0$; $\lambda = 1, \ldots, N$, (4.13) reads

$$\langle O_N|\mathbb{S}|O_N\rangle = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!}(2n)! \sum_{\{\nu_k\}} \delta\left(\sum_{\lambda=1}^{N} \nu_\lambda - 2n\right) \prod_{\lambda=1}^{N} \frac{(2\nu_\lambda - 1)!!}{\nu_\lambda !} \langle O_N|O_N\rangle. \tag{4.18}$$

Using once more the formula (4.14) which holds now with

$$\mathscr{F}(z) = (1-2z)^{-1/2} \tag{4.19}$$

as well as the normalisation $\langle O_N|O_N\rangle = 1$, equation (4.18) becomes

$$\begin{aligned}\langle O_N|\mathbb{S}|O_N\rangle &= \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!}\frac{(N+4n-2)!!}{(N-2)!!} \\ &= (8ig)^{-N/4} \exp[(32ig)^{-1}] U(\tfrac{1}{2}(N-1), (8ig)^{-1/2})\end{aligned} \tag{4.20}$$

where $U(\alpha, x)$ is the Weber parabolic cylinder function regular at infinity ($g \to 0$). Equation (4.20) shows that the S matrix is not analytic for $g = 0$, i.e. the radius of convergence of the power series in g in (4.20) itself is zero. The above results implies that as one performs the integration in (4.7), at those values of z_k and z_{k+1} such that the ket

$$|\zeta_k\rangle = \exp\left(-\frac{i}{\hbar}\tau_{k+1} H(\tau_{k+1})\right)|z_{k+1}\rangle$$

coincides with

$$|\xi_k\rangle = \exp\left(-\frac{i}{\hbar}\tau_{k+1} H(\tau_{k+1})\right)|z_k\rangle$$

and both are equal to $|O_N\rangle$, there the orbit describing the time evolution of the system, in the presence of a perturbation ε (possibly infinitesimal) does not coincide with the unperturbed orbit even in the limit $\varepsilon \to 0$. This is the breaking of coherence. The point along the trajectory where this happens is a branch point which corresponds to a possible bifurcation of the process. The manifold M in its neighbourhood is to be replaced by the universal covering manifold. The latter has locally the structure of a Riemann surface, and the continuation of the orbit implies a choice about the sheet along which the process takes place. In other words one should equip the propagator with a rule about how to go around the singularity induced by the branching when ε is vanishingly small. This is most naturally done by following the procedure discussed at the beginning of the present section, i.e. expressing the propagator, which is not an element of $\mathfrak{l}$, in the jet representation of some finite order z of $\mathfrak{l}$. One can easily check that for the example discussed in (4.11) to (4.20), such a procedure is simply realised with $z = 1$ and $q = 1$, by the following scheme. We introduce first the generalised k-boson operators (Rasetti 1972, Katriel 1979)

$$B_\lambda^{(k)} = \sum_{j=0}^{\infty} \alpha_j^{(k)} [a_\lambda^+]^j a_\lambda^{j+k} \tag{4.21}$$

where

$$\alpha_j^{(k)} = \sum_{s=0}^{j} \frac{(-1)^{j-s}}{(j-s)!}\left(\frac{1+[\![s/k]\!]}{s!\,(s+k)!}\right)^{1/2} \exp(i\vartheta_s) \tag{4.22}$$

as well as their conjugate $B_\lambda^{(k)+}$, $\lambda = 1, \ldots, N$, ($[\![x]\!]$ denotes integer part of x; ϑ_s are arbitrary phases). We restrict here our attention to the two-boson operators $B_\lambda := B_\lambda^{(2)}$, B_λ^+. Such operators annihilate and create respectively two bosons of type a_λ, in the sense that

$$\begin{aligned} &[B_\lambda, B_\mu^+] = \delta_{\lambda\mu} \\ &[B_\lambda, N_\mu] = 2B_\lambda\delta_{\lambda\mu} \qquad [B_\lambda^+, N_\mu] = -2B_\mu^+\delta_{\lambda\mu} \end{aligned} \tag{4.23}$$

where $N_\mu = a_\mu^+ a_\mu$. The B_λ and their conjugates, as given by (4.21), are manifestly elements of $\mathfrak{L}_2$ (see (3.7)). Thus, due to definition (3.8), they form the algebra $\mathfrak{T}_1$. The latter is isomorphic with the original Weyl algebra, as (4.23) show.

A jet representation of order 1 of $\mathfrak{l}$ matches the operators of $\mathfrak{g}$, thought of as differential operators, up to the second order; and one can regularise $H + W$ by simply replacing $(a_\lambda + a_\lambda^+)^2$ in (4.9) with $(B_\lambda + B_\lambda^+)$, $\lambda = 1, \ldots, N$. The new $\mathbb{S}$ operator thus obtained reads:

$$\tilde{\mathbb{S}} = \exp\left[-\mathrm{i}g\left(\sum_{\lambda=1}^{N} (B_\lambda + B_\lambda^+)\right)^2\right]. \tag{4.24}$$

In general the coherent states connected with the algebra (4.23) are obviously in one-to-one correspondence with those given in (2.1) and can be constructed out of the same fixed vector $|O_N\rangle$ in the Hilbert space. Therefore all we need to check is that $\langle O_N|\tilde{\mathbb{S}}|O_N\rangle$ as a function of g has a finite radius of convergence.

Repeating the procedure utilised in (4.10)-(4.13), upon replacing $(a_\lambda + a_\lambda^+)^2$ by $(B_\lambda + B_\lambda^+)$ and noticing that the Wilcox formula can once more be used due to (4.23), we get instead of (4.20)

$$\begin{aligned} \langle O_N|\tilde{\mathbb{S}}|O_N\rangle &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}g)^n}{n!} (2n)! \sum_{\{\nu_k\}} \delta\left(\sum_{k=1}^{N} \nu_k - n\right) \prod_{k=1}^{N} \frac{1}{(2\nu_k)!!} \\ &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}g)^n}{(n!)^2} (2n)! \frac{\mathrm{d}^n}{\mathrm{d}z^n} \exp(\tfrac{1}{2}Nz)\bigg|_{z=0} \\ &= (1 + \mathrm{i}gN)^{-1/2} \end{aligned} \tag{4.25}$$

which is clearly analytic in g for $g \to 0$.

In conclusion we notice that the procedure of regularisation of the S matrix discussed before suggests the possibility of generalising the squeeze operators whereby the squeezed coherent states are generated. It was shown by Fisher *et al* (1984) that the customary squeeze operators

$$u_k(z) := \exp[z_k(a^+)^k - \bar{z}_k a^k] \tag{4.26}$$

are unbounded in the Hilbert space of the simple harmonic oscillator for $k > 2$. On the contrary, operators of the form

$$\tilde{u}_k(z) := \exp(z_k B^{(k)+} - \bar{z}_k B^{(k)}) \tag{4.27}$$

clearly do not suffer similar pathologies and their spectrum is both measurable and bounded (D'Ariano *et al* 1985b).

5. Conclusions

In § 3 of this paper we characterised dynamical systems in algebraic terms, identifying which algebraic structure the corresponding Hamiltonians should have in order that a system—prepared so as to be in a state described by a generalised coherent state for a given group G at time zero—during its time evolution under such a Hamiltonian, would pass through a continuous sequence of states, each described by a generalised coherent state.

In § 4, moreover, we showed how, when a coherence breaking term is added to the previous Hamiltonian, singularities appear in the propagator. A regularisation procedure could, however, be devised, whereby such singularities are removed. Essentially this amounts to matching locally the group of diffeomorphisms of the jet bundle describing the local evolution of the system over the coherent state manifold (by Bäckland contact transformations), with the jet realisation of a group included in $\mathscr{L}$ (the exponentiation of the enveloping algebra of the Lie algebra of G).

In such a case coherence is expected to be preserved for states corresponding to a different group G′ than the original one (even though we were able to exhibit, in an example thoroughly discussed in § 4 itself, a case in which the new group is indeed isomorphic with G). In general, the conservation of coherence in this latter case holds only for a finite neighbourhood of each point of the new coherent state manifold M', and the latter turns out to be split into the union of a set of non-overlapping stability domains (whose boundaries are the coherence breaking submanifolds).

If a symplectic Kähler structure can be constructed over M' (which should be always the case, if M' is to be interpreted as a classical phase space), globally extendable over the whole manifold, then the latter is a homogeneous Kähler variety (and the corresponding stability subgroup h' is the centraliser of the maximal toral subgroup of G′). The resulting coherent state manifold has then a homology without torsion, with vanishing odd Betti numbers.

On the other hand Betti numbers can be found in a straightforward manner from the diagram of G′ utilising Morse theory (Bott 1958).

We conjecture that the resulting Borel–Morse cells (Borel 1954) analytic subvarieties whereby the space M' is decomposed in such a procedure, indeed coincide with just the structural stability basins for coherent states (labelled in M') defined before; and that the corresponding Morse functions themselves can be written in terms of generalised coherent state representatives. Work is in progress along these lines.

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