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ABSTRACT

A quantum walk describes the discrete unitary evolution of a quantum particle on a discrete graph. Some quantum walks, referred to as the Weyl and Dirac walks, provide a description of the free evolution of relativistic quantum fields in the small wave-vector regime. The clash between the intrinsic discreteness of quantum walks and the continuous symmetries of special relativity is resolved by giving a definition of change of inertial frame in terms of a change of values of the constants of motion, which leaves the walk operator unchanged. Starting from the family of 1 + 1 dimensional Dirac walks with all possible values of the mass parameter, we introduce a unique walk encompassing the latter as an extra degree of freedom, and we derive its group of changes of inertial frames. This symmetry group contains a non-linear realization of $SO^{+}(2,1) \ltimes \mathbb{R}^{3}$; since one of the two space-like dimensions does not correspond to an actual spatial degree of freedom but rather the mass, we interpret it as a 2 + 1 dimensional de Sitter group. This group also contains a non-linear realization of the proper orthochronous Poincaré group $SO^{+}(1,1) \ltimes \mathbb{R}^{2}$ in 1 + 1 dimension, as the ones considered within the framework of doubly special relativity, which recovers the usual relativistic symmetry in the limit of small wave-vectors and masses. Surprisingly, for the Dirac walk with a fixed value of the mass parameter, the group of allowed changes of reference frame does not have a consistent interpretation in the limit of small wave-vectors.

I. INTRODUCTION

Quantum Walks (QWs)[1–5] and Quantum Cellular Automata (QCA)[6–11] describe the discrete lattice of quantum systems whose evolution is given by an update rule that acts in a discrete sequence of steps. The main features of this rule are unitarity, locality (i.e., each system interacts only with a finite number of neighboring ones), and homogeneity (i.e., the evolution it commutes with lattice translations). In the field of quantum computing, quantum walks and quantum cellular auntonata have been studied as universal models for quantum computation,[12–17] and they have been applied to the design of quantum search algorithms.[18–20] Since Feynman's original proposal,[21] these discrete models have also been considered as a framework for quantum simulators[22,23] with a variety of experimental implementations.[24–27] In particular, the simulation of relativistic wave equations with quantum walks has been studied.[28–38]

From a foundational standpoint, quantum walks and quantum cellular automata are the simplest frameworks that allow us to investigate the properties of discrete spacetime, since they are the only unitary evolutions that are compatible with a discrete background and a finite speed of propagation of information. Some recent works[32,33] have shown that symmetry principles such as homogeneity and isotropy single out a class of quantum walks—the Weyl walk and the Dirac walk—which recover the Weyl and Dirac equation in the limit in which the discreteness of the background cannot be probed. Quite surprisingly, this means that a Lorentz invariant dynamics can emerge from a QCA (or QW) discrete framework without assuming the symmetries of Minkowski spacetime. However, recovering a Lorentz invariant dynamics does not imply that one can recover the symmetry transformations of the Lorentz group.

In order to do that, one necessarily has to introduce a notion of change of inertial frame within the QCA model. This was done in Refs. 39–41, where changes of inertial frames have been defined as those changes of representation of the cellular automaton—in terms of the values of its constants of motion—which preserve the update rule. This approach, when applied to the Weyl walk, allows us to recover the Poincaré symmetry, thus giving a proof of principle that a discrete quantum dynamics is consistent with the symmetries of Minkowski spacetime. The key point is that the Poincaré group acts on the space of wave-vectors through a *realization*—in the present case, a group of diffeomorphisms—instead of the usual linear representations of quantum field theory. The usual linear transformations are recovered in the limit of small wave-vectors. Such a non-linear deformation of the Poincaré symmetry is also the distinctive feature of doubly special relativity (DSR) models,[42–44] which consider theories with two observer-independent scales, the speed of light and the Planck energy, which must stay invariant under a change of inertial observer.

It is worth mentioning that experimental tests of violation of Lorentz symmetry have also been proposed.[45–47] In particular, observation of deep space gamma-ray bursts can be sensitive to the vacuum dispersive behavior.[48–51]

A partial classification of the full symmetry group of the Weyl automaton in 3 + 1 dimensions was derived in Ref. 52. In the present paper we provide an extension of the analysis to the case of the Dirac automaton in 1 + 1 dimensions, namely, an automaton where the extra parameter representing mass plays an important dynamical role. If one considers a Dirac automaton with a fixed value of the mass parameter, one finds a symmetry group that is isomorphic to $SO^+(1,1) \rtimes \mathbb{Z}_2$, namely, the Lorentz group in 1 + 1 dimensions. However, the analysis of the action of such a group in terms of its action in the limit of small wave-vectors is inconsistent with the identification of the wave-vector with momentum. Therefore, we introduce a quantum walk in which the mass is an extra degree of freedom on the same footing as the wave-vector.

The symmetry group is proved to be the semidirect product of three groups. The first one is the additive group of smooth functions from the invariant zone to the complex numbers. The second one is a group of diffeomorphisms that act as non-linear dilations of the quantum walk mass shell. The third group is a non-linear realization of $SO^+(2,1)$. We observe that the symmetry group contains a non-linear realization of $ISO^+(1,2) = SO^+(2,1) \ltimes \mathbb{R}^3$, which is interpreted as a variation of the de Sitter group, in the limiting flat case of infinite cosmological constant. The reason for this is that the extra dimension emerging in our case is not a spatial one, but it is associated with the variable mass parameter. This result introduces an inspiring relation between the symmetries of the massive quantum field and those of the emerging spacetime geometry. Note also that the classical-particle interpretation of the conjugated variable of the rest mass is that of proper time.[53]

Within the subgroup $SO^+(1,2) \ltimes \mathbb{R}^3$, we also have a non-linear representation of the Poincaré group $SO^+(1,1) \ltimes \mathbb{R}^2$ in 1 + 1 dimension, which, in contrast to the fixed mass case, consistently recovers the usual (linear) relativistic symmetry in the limit of small wave-vectors. Therefore, the Dirac quantum walk naturally provides a microscopic dynamical model of doubly special relativity.

This paper is organized as follows: Section II begins with a review of basic notions of quantum walks on Cayley graphs and of the one dimensional Dirac quantum walk. Then, in Sec. II A, we introduce the one dimensional Dirac quantum walk with variable mass, whose eigenvalue equation is studied in Sec. II B. In Sec. III, we define a notion of change of inertial frame, which does not rely on a symmetry of a background spacetime. We then characterize the group of changes of inertial frames of the Dirac walk with variable mass, and we show that it consists in a non-linear realization of a semidirect product of the Poincaré group and the group of dilations.

II. THE ONE-DIMENSIONAL DIRAC QUANTUM WALK

A discrete-time quantum walk[54,55] describes the unitary evolution of a particle with s internal degrees of freedom (usually called *coin space*) on a lattice Γ. In the case of interest,[32] the lattice Γ is the *Cayley graph* of a finitely generated group G, i.e., $\Gamma(G, S_+)$ is the edge-colored directed graph having vertex set G and edge set $\{(x, xh), x \in G, h \in S = S_+ \cup S_+^{-1}\}$ (S_+ is a set of generators of G), and a color is assigned to each generator $h \in S_+$. Usually, an edge that corresponds to a generator g such that $h^2 = e$ (e is the identity of G) is represented as undirected. Clearly, each Cayley graph corresponds to a presentation of the group G, where relators are just closed paths over the graph. Within this framework, a discrete-time quantum walk on a Cayley graph $\Gamma(G, S_+)$ with an s-dimensional coin system ($s \geq 1$) is a unitary evolution on the Hilbert space $\ell^2(G) \otimes \mathbb{C}^s$ of the following kind:

$$A := \sum_{h\in S} T_h \otimes A_h, \quad 0 \neq A_h \in M_s(\mathbb{C}),$$
$$T_h|x\rangle := |xh^{-1}\rangle,$$

where, for any $g \in G$, T_g is the right regular representation of G on $\ell^2(G)$ and $\{|x\rangle, x \in G\}$ is an orthonormal basis of $\ell^2(G)$.

The one dimensional Dirac quantum walk is a quantum walk on the Cayley graph $\Gamma(\mathbb{Z}, \{0,1\})$ (see Fig. 1) of the group $\mathbb{Z}$ with coin space $\mathbb{C}^2$ (the particle has two internal degrees of freedom). The evolution is the following unitary operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$:

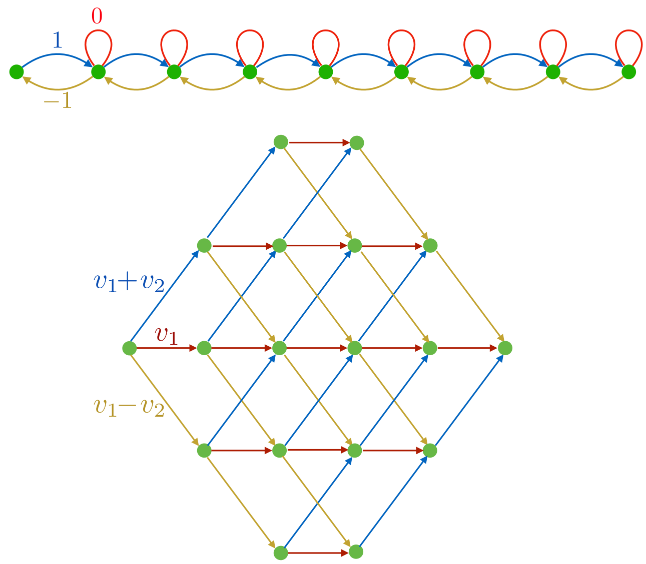

FIG. 1. (Top) Cayley graph of the group $\mathbb{Z}$, where the red loop arrow represents the identity $\{0\}$, and the blue/yellow arrow refer to right/left translation, namely, $\{1, -1\}$. (Bottom) Cayley graph of the group $\mathbb{Z}_2$, where the red arrow is associated with the generator g_1, while the blue and yellow arrows refer to the generators $g_1 + g_2$ and $g_1 - g_2$ respectively.

$$\begin{aligned} A(\mu) &= \begin{pmatrix} T\cos\mu & -iI\sin\mu \\ -iI\sin\mu & T^{\dagger}\cos\mu \end{pmatrix}, \\ |\psi\rangle &= \sum_{s=L,R}\sum_{x\in\mathbb{Z}} \psi(s,x,\mu)|x\rangle|s\rangle, \\ |R\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{1}$$

where $I = T_0$, $T := T_1$, $T_1|x\rangle = |x+1\rangle$, and $T^{\dagger} = T_{-1} = T^{-1}$. Since $A(\mu)$ commutes with the translation operator $T \otimes I_2$, we may represent $A(\mu)$ using the Fourier basis $|\mathrm{k}\rangle = \frac{1}{\sqrt{2\pi}}\sum_{x\in\mathbb{Z}} e^{i\mathrm{k}x}|x\rangle$ and we obtain

$$\begin{aligned} A(\mu) &= \int_{-\pi}^{\pi} d\mathrm{k}\, |\mathrm{k}\rangle\langle\mathrm{k}| \otimes \tilde{A}(\mu,\mathrm{k}), \\ \tilde{A}(\mu,\mathrm{k}) &= \begin{pmatrix} \cos\mu e^{-i\mathrm{k}} & i\sin\mu \\ i\sin\mu & \cos\mu e^{i\mathrm{k}} \end{pmatrix}. \end{aligned} \tag{2}$$

In the limit $\mathrm{k}, \mu \to 0$, the Dirac quantum walk recovers the dynamics of the one dimensional Dirac equation, where k and μ are interpreted as the momentum and mass of the particle, respectively.

A. Variable mass

As we already mentioned in the Introduction, and will be shown in Sec. III, the symmetry group of the Dirac walk cannot recover the relativistic Lorentz symmetry. This obstruction can be overcome by considering the mass no longer as a fixed parameter, but rather as an additional degree of freedom, as follows:

$$\begin{aligned} A &:= \int_{-\pi}^{\pi} d\mu A(\mu) \otimes |\mu\rangle\langle\mu|, \\ |\mu\rangle &:= \frac{1}{\sqrt{2\pi}}\sum_{\tau\in\mathbb{Z}} e^{i\mu\tau}|\tau\rangle, \end{aligned} \tag{3}$$

where $|\tau\rangle$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. The discrete nature of the variable conjugated to the mass agrees with the discreteness of time in the quantum walk, being τ interpreted as the proper time of the classical particle.[53] It is easy to realize that A is a quantum walk on a Cayley graph of $\mathbb{Z}^2$. Indeed, from Eq. (3), we have

$$
\begin{aligned}
&A := \int_B d\mathrm{k}\, d\mu\, \tilde{A}(\mu,\mathrm{k}) \otimes |\mu,\mathrm{k}\rangle\langle\mu,\mathrm{k}|,\\
&\tilde{A}(\mu,\mathrm{k}) = \frac{1}{2}\begin{pmatrix} (e^{i\mu}+e^{-i\mu})e^{-i\mathrm{k}} & (e^{i\mu}-e^{-i\mu}) \\ e^{i\mu}-e^{-i\mu} & (e^{i\mu}+e^{-i\mu})e^{i\mathrm{k}} \end{pmatrix},\\
&|\mu,\mathrm{k}\rangle := |\mu\rangle|\mathrm{k}\rangle,\\
&B := (-\pi,\pi] \times (-\pi,\pi],
\end{aligned} \tag{4}
$$

which in the $|x\rangle|\tau\rangle$ basis reads

$$
\begin{aligned}
&A = \frac{1}{2}\begin{pmatrix} (T^\dagger + T)S & T^\dagger - T \\ T^\dagger - T & (T^\dagger + T)S^\dagger \end{pmatrix},\\
&T|\tau\rangle = |\tau+1\rangle.
\end{aligned} \tag{5}
$$

In the right regular representation of $\mathbb{Z}^2$ with basis $|x\rangle|\tau\rangle$, T and S represent the generators $g_1 := (0,-1)$ and $g_2 := (-1,0)$, respectively. Therefore, A is a quantum walk on the Cayley graph $\Gamma(\mathbb{Z}^2, \{\pm g_1, \pm(g_1+g_2), \pm(g_1-g_2)\})$ (see Fig. 1).

It is worth noting that the previous construction depends on the choice of parameterization for the mass term in Eq. (1) (for example, the change of variables $\mu' = \sin\mu$ would not have led to a quantum walk in the conjugate variables).

B. Study of the eigenvalue equation

Let us consider the eigenvalue equation for the Dirac quantum walk with variable mass. From Eq. (4), we have

$$
\begin{aligned}
&A(\mu,\mathrm{k})\psi(\mathrm{k},\mu) = e^{i\omega(\mathrm{k},\mu)}\psi(\mathrm{k},\mu),\\
&\psi(\mathrm{k},\mu) = \begin{pmatrix} \psi(R,\mathrm{k},\mu) \\ \psi(L,\mathrm{k},\mu) \end{pmatrix},
\end{aligned} \tag{6}
$$

which can be rewritten as

$$(\cos\,\mu\,\cos\mathrm{k} - \cos\,\omega)\psi(\mathrm{k},\mu) = 0, \tag{7}$$

$$(\cos\,\mu\,\sin\mathrm{k}\,\sigma_3 - \sin\,\mu\,\sigma_1 + \sin\,\omega\, I)\psi(\mathrm{k},\mu) = 0. \tag{8}$$

From the first equation, we get the expression for the eigenvalue, namely, $\omega = \arccos(\cos\mu\cos\mathrm{k})$, while multiplying the second equation by σ_2, we obtain

$$(\cos\,\mu\,\sin\mathrm{k}\, i\sigma_1 + \sin\,\mu\, i\sigma_3 + \sin\,\omega\,\sigma_2)\psi(\mathrm{k},\mu) = 0. \tag{9}$$

We note that the set $\{\sigma_2, i\sigma_1, i\sigma_3\}$ provides a representation of the generators of the Clifford algebra $C\ell_{1,2}(\mathbb{R})$. Indeed, by renaming the elements of the set as $\{\tau_1,\tau_2,\tau_3\}$, the following relations are satisfied:

$$\{\tau_i,\tau_j\} = 2\eta_{ij},$$

where η_{ij} denotes the Minkowski metric tensor with signature $(+,-,-)$. Hence, we can rewrite Eq. (9) in the relativistic notation

$$
\begin{aligned}
&n_\mu(\mathrm{k},\mu)\tau^\mu\psi(\mathrm{k},\mu) = 0,\\
&n := (\sin\,\omega, \cos\,\mu\,\sin\mathrm{k}, \sin\,\mu),\\
&\tau := (\sigma_2, -i\sigma_1, -i\sigma_3).
\end{aligned} \tag{10}
$$

Furthermore, if Eq. (10) holds, we have

$$n_\nu(\mathrm{k},\mu)n^\nu(\mathrm{k},\mu) = 0, \tag{11}$$

and consequently, Eq. (7) is trivially satisfied, i.e., $\omega(\mathrm{k},\mu) = \arccos(\cos\mu\cos\mathrm{k})$. Now, let us analyze the map

$$\begin{aligned} \overline{n}(\mathrm{k},\mu) &: B \to \mathbb{R}^2 \\ (\mathrm{k},\mu) &\mapsto (\cos\mu\,\sin\mathrm{k}, \sin\mu), \end{aligned} \tag{12}$$

and if we compute the norm of the considered map, we have

$$\|\overline{n}(\mathrm{k},\mu)\|^2 = \sin^2\mathrm{k}\cos^2\mu + \sin^2\mu \le 1, \tag{13}$$

which implies that the Brillouin zone is mapped in the unit disk in $\mathbb{R}^2$. Clearly, $\overline{n}$ is smooth and analytic. The Jacobian of $\overline{n}$ is

$$J_{\overline{n}}(\mathrm{k},\mu) = \det(\partial_i n_j) = \cos^2\mu\,\cos\mathrm{k},$$

and the map results singular for $\mathrm{k} = \pi/2 + m\pi$ and $\mu = \pi/2 + m\pi$, with $m \in \mathbb{N}$. Let us define the following regions $B_i \subset B$:

$$\begin{aligned} B_0 &:= \{(\mathrm{k},\mu)|\mathrm{k} \in (-\frac{\pi}{2}, \frac{\pi}{2}), \mu \in (-\frac{\pi}{2}, \frac{\pi}{2})\}, \\ B_1 &:= B_0 + (\frac{\pi}{2}, 0), \\ B_2 &:= B_0 + (0, \frac{\pi}{2}), \\ B_3 &:= B_0 + (\frac{\pi}{2}, \frac{\pi}{2}), \end{aligned} \tag{14}$$

where $B_0 + (a, b)$ denotes the translation of the set B_0 by the vector (a, b) (see Fig. 2). Denoting by $\overline{n}|_{B_0}$ the map $\overline{n}$ restricted to the region B_0, and referring to Eq. (13), it is easy to note that $\overline{n}|_{B_0}$ is an analytic diffeomorphism between B_0 and the open unit disk in $\mathbb{R}^2$. Then, thanks to the periodicity of the map $\overline{n}$, the property of being an analytic diffeomorphism straightforwardly holds for $\overline{n}|_{B_i}$, $\forall i \in \{0, 1, 2, 3\}$, $\overline{n}|_{B_i}$ denoting the restriction of $\overline{n}$ to the region B_i.

Therefore, for any $i \in \{0, 1, 2, 3\}$ and $(\mathrm{k},\mu) \in B_i$, if $n_\mu(\mathrm{k},\mu)\tau^\mu|\psi(\mathrm{k},\mu)\rangle = 0$, there exists $(\mathrm{k}',\mu') \in B_0$ such that $n_\mu(\mathrm{k},\mu) = n_\mu(\mathrm{k}',\mu')$ and $|\psi(\mathrm{k},\mu)\rangle = |\psi(\mathrm{k}',\mu')\rangle$. We may understand the B_i regions as kinematically equivalent sets, and the quantum walk dynamics is completely specified by the solution of Eq. (10) in any of the regions B_i.

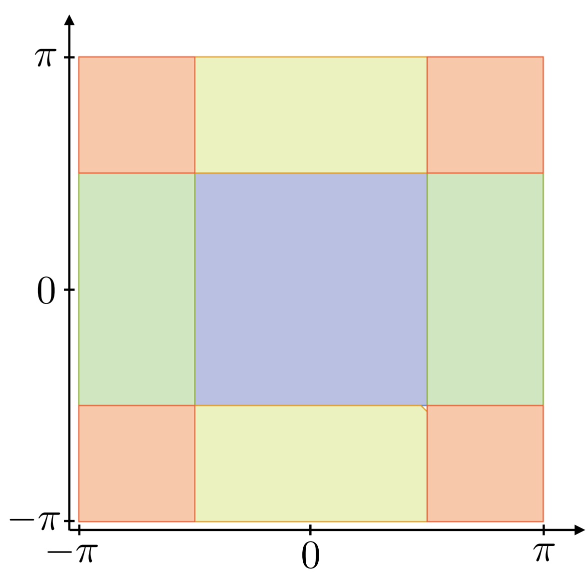

FIG. 2. Brillouin zone: the blue region refers to B_0, while the other colored regions correspond to B_i, obtained by simply translating the vector (a, b) with $a, b \in \{0, \pi/2\}$.

III. CHANGE OF INERTIAL FRAME

As mentioned in Sec. II B, the solution of the eigenvalue Eq. (10) in one of the regions B_i, which were defined in Eq. (14), completely characterizes the quantum walk dynamics. We then require that a change of reference frame leaves invariant the eigenvalue equation (10) restricted to the domain B_0. From now on, unless otherwise specified, we will assume $(\mathrm{k}, \mu) \in B_0$, and consequently, we remove the restriction symbol $\cdot|_{B_0}$ from all the maps. It is also convenient to introduce the notation

$$k := (\omega, \mathrm{k}, \mu).$$

Let us now consider the map

$$n : k \mapsto \begin{pmatrix} \sin\omega \\ \bar{n}(\mathrm{k},\mu) \end{pmatrix} = \begin{pmatrix} \sin\omega \\ \cos\mu \sin\mathrm{k} \\ \sin\mu \end{pmatrix}.$$

The map n defines a diffeomorphism between the quantum walk mass-shell

$$V = \{k \mid \omega = \arccos(\cos\mathrm{k}\cos\mu)\},$$

defined by condition (11), and the truncated cone

$$K := \{(x, y, z) \mid x^2 + y^2 = z, 0 \le z \le 1\},$$

both presented in Fig. 3. In the following, we will also use the non-truncated null mass shell denoted by

$$K_0 := \{(x, y, z) \mid x^2 + y^2 = z\}. \tag{15}$$

We are now ready to give a formal definition of change of reference frame.

Definition 1 **(Change of inertial reference frame)**. *A change of inertial reference frame for the Dirac walk is a triple (k', a, M), where*

- $k' : V \to V, k \mapsto k'(k)$, $k' \in \mathrm{Diff}(V)$ *(the diffeomorphism group of the mass shell V),*
- $a \in C^\infty(V, \mathbb{C})$, $k \mapsto a(k)$ *is a smooth complex function,*
- $M \in SL(2, \mathbb{C})$

such that

$$\begin{aligned} &n_\mu(k)\tau^\mu\psi(k) = 0 \Leftrightarrow n_\mu(k')\tau^\mu\psi'(k') = 0, \\ &\psi'(k') = e^{ia(k')}M\psi(k) \end{aligned} \tag{16}$$

for any $k \in V$. We denote by $\mathcal{S}_D$ the group of changes of inertial reference frame (symmetry group for short) for the Dirac quantum walk with variable mass.

According to Definition 1, a change of inertial frame is a relabeling $k'(k)$ of the constants of motion of the quantum walk such that the eigenvalue equation is preserved in the region B_0. The same definition straightforwardly generalizes to the other regions B_i. The crucial assumption in Definition 1 is that the linear transformation M, which acts on the internal degrees of freedom, does not depend on the value

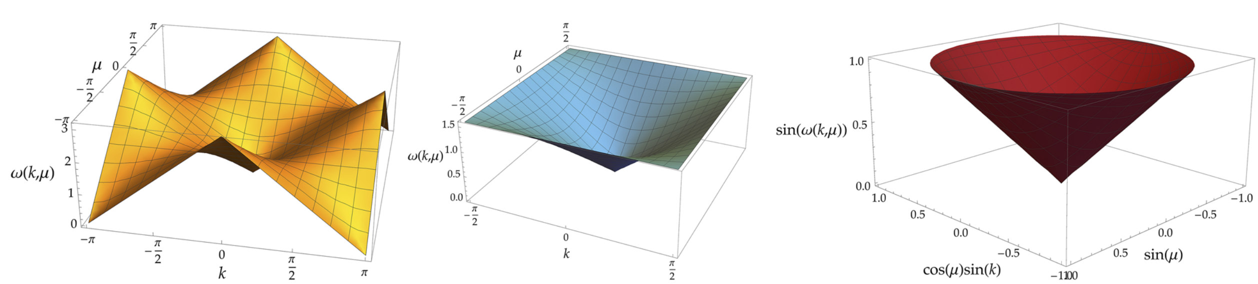

FIG. 3. (Left) Dispersion relation of the Dirac quantum walk with variable mass. (Middle) The quantum walk mass shell V. The surface V is the graph of the dispersion relation restricted to the domain B_0. (Right) The image $K := n(V)$. The truncation is due to the condition $\bar{n}_\nu\bar{n}^\nu < 1$.

of k.[56] Without this limitation, the notion of symmetry group would become trivial, allowing any bijection between the set of solutions. On the other hand, we presently lack a more physically grounded motivation for this assumption.

From Definition 1, the following rules easily follow:

$$\begin{aligned}&[(k',a,M)\psi](k)=e^{ia(k)}M\psi(k'^{-1}(k)),\\&(k'',b,N)\circ(k',a,M)=(k''\circ k',b+a\circ k''^{-1},NM).\end{aligned}\tag{17}$$

Let us now characterize the symmetry group $\mathcal{S}_D$. Clearly, the simplest example of change of inertial frame is the one given by the trivial relabeling $k'=k$, the identity matrix $M=I$, and arbitrary $a(k)$. Moreover, it is easy to realize that when $a(k)$ is a real linear function, i.e., $a(k)=a_\mu k^\mu$ with $a_\mu\in\mathbb{R}^3$, we recover the group of translations in three dimensions (translations in the direction corresponding to the variable τ conjugated to μ are also admissible).

We proceed with a complete characterization of the full symmetry group. The basic result is the following lemma:

Lemma 1. Let (k',a,M) be a change of inertial frame for the Dirac walk. Then, we have

$$f(k')n_\mu(k')=L_\mu^\nu n_\nu(k),\ \ \forall k\in V,\tag{18}$$

where $L\in SO^+(1,2)$ and $f(k')$ is a suitable non-null real function. Moreover, $M\in SL(2,\mathbb{R})$ such that $M^{-1}w_\mu\tau^\mu M=L_\mu^\nu w_\nu\tau^\mu$ for any $w\in\mathbb{R}^3$.

Proof. Clearly, we have that $n_\mu(k)\tau^\mu\psi(k)=0\Leftrightarrow e^{ia(k')}n_\mu(k')\tau^\mu M\psi(k)=0$ for any $k\in V$ if and only if $\sigma_2 n_\mu(k)\tau^\mu\psi(k)=0$ $\Leftrightarrow M^\dagger\sigma_2 n_\mu(k')\tau^\mu M\psi(k)=0$ because $M\in SL(2,\mathbb{C})$. From Eq. (8), we have that $\sigma_2 n_\mu(k)\tau^\mu$ is proportional to a rank one projector, and therefore, we must have

$$g(k')n_\mu(k)\tau^\mu=\sigma_2 M^\dagger\sigma_2 n_\mu(k')\tau^\mu M.\tag{19}$$

Now, the right-hand side is a linear combination of $I,\sigma_x,\sigma_y,\sigma_z$. By the above identity, however, we conclude that the right-hand side must also be a linear combination of τ^μ only. Thus,

$$\sigma_2 M^\dagger\sigma_2 n_\mu(k')\tau^\mu M=:L_\mu^\nu n_\nu(k')\tau^\mu\tag{20}$$

$$\Rightarrow\ g(k')n_\mu(k)=L_\mu^\nu n_\nu(k')\tag{21}$$

for some non-null scalar function $g(k')$ and some linear map L. Then, since $M\in SL(2,\mathbb{C})$, we have that $L\in SO^+(1,2)$ and that $g(k')$ must be a real function. Then, M is a two dimensional representation of $SO^+(1,2)$, which implies $M\in SL(2,\mathbb{R})$. ∎

Corollary 1. Let (k',a,M) be a change of inertial frame for the Dirac walk. Then, we have

$$L_\mu^\nu n_\nu(k)=\mathcal{D}_f n_\mu(k'),\tag{22}$$

$$\mathcal{D}_f:\mathbb{R}^3\to\mathbb{R}^3,\ \ n\mapsto f(n)n,\tag{23}$$

where $L\in SO^+(1,2)$ and $f:\mathbb{R}^3\to\mathbb{R}^3$ is a smooth function such that $\mathcal{D}_f$ is injective.

Proof. Let $f(k')$ be as in Lemma 1. Since $n(k')$ is a diffeomorphism, we may consider f as a function of n, namely, $f(n):=f(k'(n))$. Let us now assume that $\mathcal{D}_f$ is not injective. Then, we would have $\mathcal{D}_f\circ n(k_1')=\mathcal{D}_f\circ n(k_2')$ for some $k_1'\neq k_2'$. From Eq. (18), we then have $L_\mu^\nu n_\nu(k_1)=L_\mu^\nu n_\nu(k_2)$. However, since both maps $k'(k)$ and L are invertible, this would imply $k_1=k_2$. ∎

We can finally prove the characterization of the symmetry group of the Dirac walk.

Proposition 1. The triple (k',a,M) is a change of inertial frame for the Dirac walk if and only if

$$k'(k)=[n^{-1}\circ\mathcal{D}_f{}^{-1}\circ L\circ\mathcal{D}_g\circ n](k),\tag{24}$$

$$M\in SL(2,\mathbb{R}),M^{-1}w_\mu\tau^\mu M=L_\mu^\nu w_\nu\tau^\mu,\forall w\in\mathbb{R}^3,\tag{25}$$

$$a\in C^\infty(V,\mathbb{C}),\tag{26}$$

where $\mathcal{D}_f$ and $\mathcal{D}_g$ are two diffeomorphisms between K and K_0, of the form of Eq. (23), and $L\in SO^+(1,2)$.

Proof. From Corollary 1, we have that

$$k'(k) = [n^{-1} \circ \mathcal{D}_f{}^{-1} \circ L \circ n](k),$$

where $L \in SO^{+}(1,2)$ and $\mathcal{D}_f$ is of the form of Eq. (23). Let now $\mathcal{D}_g$ be any diffeomorphism of the same form between K and K_0. Since K is star shaped such a $\mathcal{D}_g$ exists (see Appendix A for an example). Then,

$$k'(k) = [n^{-1} \circ \mathcal{D}_{\bar{f}}^{-1} \circ L \circ \mathcal{D}_g \circ n](k),$$
$$\mathcal{D}_{\bar{f}}^{-1} := \mathcal{D}_f{}^{-1} \circ L \circ \mathcal{D}_g^{-1} \circ L^{-1},$$

where $\mathcal{D}_{\bar{f}}$ is a diffeomorphism between K and K_0. ■

We provide a pictorial representation of the change of inertial frame in Fig. 4.

From Eq. (24), it follows that the diffeomorphisms $n \circ k' \circ n^{-1}$ form a subgroup G of Diff(K), which is the product of a non-linear realization of $SO^{+}(1,2)$ and a group M_K of non-linear dilations of K.

Lemma 2. Let $G \subseteq \mathrm{Diff}(K)$ such that $\mathcal{G} \in G$ if and only if $\mathcal{G} = n \circ k' \circ n^{-1}$, where k' obeys Eq. (24). Then, we have

$$G = D_K \rtimes SO_f^{+}(1,2), \tag{27}$$

$$D_K := \{\mathcal{M} \in \mathrm{Diff}(K) \mid \mathcal{M}(v) = m(v)v \ \forall v \in K\}, \tag{28}$$

$$SO_f^{+}(1,2) := \{\mathcal{L} \in \mathrm{Diff}(K) \mid \mathcal{L} = \mathcal{D}_f^{-1} \circ L \circ \mathcal{D}_f\}, \tag{29}$$

where $m(v)$ is a real function on K, $L \in SO^{+}(1,2)$, and $\mathcal{D}_f : v \mapsto (1 - (v_x^2 + v_y^2))^{-1} v$.

Proof. Let us fix an arbitrary $\mathcal{G} \in G$. From Eq. (24), we have that $\mathcal{G} = \mathcal{D}_h^{-1} \circ L \circ \mathcal{D}_g$, where $\mathcal{D}_h$ and $\mathcal{D}_g$ are two diffeomorphisms between K and K_0, of the form of Eq. (23), and $L \in SO^{+}(1,2)$. Let us define

$$\mathcal{L} := \mathcal{D}_f^{-1} \circ L \circ \mathcal{D}_f,$$
$$\mathcal{M} := \mathcal{L}^{-1} \circ \mathcal{G} = \mathcal{D}_f^{-1} \circ L^{-1} \circ \mathcal{D}_f \circ \mathcal{D}_h^{-1} \circ L \circ \mathcal{D}_g.$$

One can verify that $\mathcal{L} \in SO_f^{+}(1,2)$ and $\mathcal{M} \in D_K$. Therefore, as $\mathcal{G} = \mathcal{L} \circ \mathcal{M}$, we have $G = SO_f^{+}(1,2)D_K$. Now, let $\mathcal{L} \in SO_f^{+}(1,2)$ and $\mathcal{M} \in D_K$, and $\mathcal{L} = \mathcal{M}$. Then,

$$Lf(v)v = \mathcal{D}_f[m(v)v] = g(v)v, \quad \forall v \in K,$$

for some $g : K \to \mathbb{R}$. This implies that $L = I$, namely, the intersection between $SO_f^{+}(1,2)$ and D_K is only the identity map. Finally, $\mathcal{L} \circ \mathcal{M} \circ \mathcal{L}^{-1} \in D_K$ for any $\mathcal{L} \in SO_f^{+}(1,2)$ and $\mathcal{M} \in D_K$, i.e., D_K is normal in G. ■

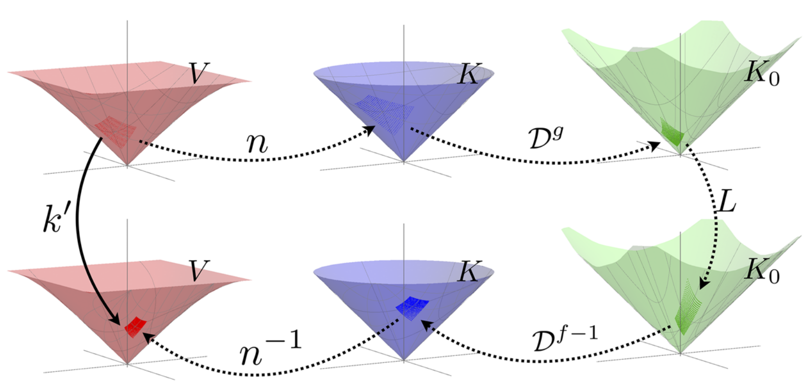

FIG. 4. Pictorial representation of a change of inertial frame for the Dirac walk with variable mass.

We remark that the choice of $\mathcal{D}_f$ in the definition of the subgroup $SO_f^+(1,2)$ is arbitrary. One can choose any other diffeomorphism between K and K_0 that satisfies Eq. (23). The subscript f in $SO_f^+(1,2)$ is just a reminder that this group is a nonlinear realization of $SO^+(1,2)$, i.e., a homomorphism of $SO^+(1,2)$ on Diff(V). Clearly, $SO^+(1,2)$ and $SO_f^+(1,2)$ are isomorphic.

The decomposition of the symmetry group of the Dirac quantum walk with variable mass is now easily provided by the following proposition:

Proposition 2. Let $\mathcal{S}_D$ be the symmetry group of the Dirac quantum walk with variable mass. Then, we have

$$\mathcal{S}_D = C^\infty(V,\mathbb{C}) \rtimes (D_K \rtimes SO^+(1,2)). \tag{30}$$

Proof. The result follows from Lemma 2 and Eq. (17). ∎

Thanks to the decomposition (30), we easily see that the symmetry group $\mathcal{S}_D$ contains $\mathbb{R}^3 \rtimes SO^+(1,2)$ as a subgroup, i.e., the Poincaré group in 2 + 1 dimensions, where the Lorentz transformations are nonlinearly deformed. Since one of the dimensions is not spatial, but associated with the mass parameter, the subgroup $\mathbb{R}^3 \rtimes SO^+(1,2)$ is interpreted as a variation of the de Sitter group, which occurs in 3 + 1 spacetime dimensions.

Moreover, we remark that the subgroup given by

$$\begin{aligned} k'(k) &:= n^{-1} \circ \mathcal{D}_f^{-1} \circ L \circ \mathcal{D}_f \circ n, \\ L &= \begin{pmatrix} \cosh(\xi) & \sinh(\xi) & 0 \\ \sinh(\xi) & \cosh(\xi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{31}$$

is a non-linear representation of the 1 + 1 dimensional Lorentz group as the ones considered within the context of doubly special relativity.[42–44] If the Jacobian matrix of the non-linear map $\mathcal{D}_f$ is the identity at the origin, it means that the non-linear Lorentz transformations recover the usual linear ones in the limit of small wave-vectors. This is the case for the non-linear map given in Appendix A.

We have therefore characterized the full symmetry group of the Dirac walk with variable mass, and we showed that in the small wave-vector limit, it contains the usual Poincaré symmetry. Now, we can proceed giving an alternative definition of change of inertial frame, starting from Definition 1 with the additional requirement that the mass term is left unchanged.

Definition 2 **(Change of Inertial frame with fixed $\boldsymbol{\mu}$)**. *A change of inertial frame, which leaves unchanged the third component μ, is a triple (k', a, M), where*

$$k' : V \to V,\ k := \begin{pmatrix} \omega \\ \mathrm{k} \\ \mu \end{pmatrix} \mapsto k'(k) := \begin{pmatrix} \omega'(k,\mu) \\ \mathrm{k}'(k,\mu) \\ \mu \end{pmatrix}$$

is a diffeomorphism, $a : V \to \mathbb{C},\ k \mapsto a(k)$ is a smooth map, and $M \in SL(2,\mathbb{C})$ such that

$$\begin{aligned} &n_\mu(k)\tau^\mu\psi(k) = 0 \Leftrightarrow n_\mu(k')\tau^\mu\psi'(k') = 0, \\ &\psi'(k') = e^{ia(k)}M\psi(k) \end{aligned} \tag{32}$$

for any $k \in V$.

The analysis of Appendix B allows one to show that starting from Definition 2, the group of changes of inertial frame with fixed μ is characterized in terms of the group

$$G \cong SO^+(1,1) \ltimes \mathbb{Z}_2 \tag{33}$$

generated by the matrices

$$L = SDS^{-1},\ L_+ = SFS^{-1}, \tag{34}$$

with

$$S = \begin{pmatrix} 1 & 1 & \sin\mu \\ -\cos\mu & \cos\mu & 0 \\ \sin\mu & \sin\mu & 1 \end{pmatrix},$$

and

$$D = \begin{pmatrix} e^{-\beta} & 0 & 0 \\ 0 & e^{\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proposition 3. The triple (k', a, M) is a change of inertial frame for the Dirac walk with fixed μ if and only if

$$k'(k) = [n^{-1} \circ \mathcal{D}_f{}^{-1} \circ L \circ \mathcal{D}_f \circ n](k), \tag{35}$$

where

$$\mathcal{D}_f(w) := \frac{\sin\mu}{T^3_\nu w^\nu} w,$$

where $T = \{T^\mu_\nu\}$, $T \in SO^+(1,1) \ltimes \mathbb{Z}_2$.

As shown in Sec. III, the above group does not provide the expected phenomenology of a Lorenz group of boosts in 1 + 1 dimensions. This result then justifies the analysis of the full symmetry group $SO(1,2)$, starting from a definition of change of inertial frame that involves also μ as a dynamical degree of freedom.

IV. CONCLUSION

In this paper, we address the problem of reconciling special relativity and the discrete dynamics of a quantum walk by defining a change of inertial frame as a coordinate transformation in momentum space that leaves invariant the eigenvalue equation of the evolution operator. This approach, already explored in Refs. 40, 41, and 57, is somewhat opposite to the usual one, where the pre-existing notions of spacetime and of inertial frame of reference constrain the admissible dynamical laws.

We derived the group of changes of inertial reference frame for the Dirac walk in 1 + 1 dimension. If the mass of the walk is fixed, the group of admissible symmetries is inconsistent with the interpretation of the wave-vector as momentum. Therefore, we defined a Dirac walk with variable mass and studied the symmetry group of the latter. As a result, one finds a group of transformations that, along with ω and k, modify also the variable μ, which defines the mass term. Such a group can be considered as the 1 + 1-dimensional counterpart of the de Sitter group, which acts on the walk mass-shell by a realization in terms of a group of diffeomorphisms. Along with the de Sitter group, one is forced to consider a group of non-linear rescaling maps so that the final group is a semidirect product of these two components. This symmetry group implements the doubly special relativity model that recovers the Lorenz group in the limit of small wave-vectors and masses in the limit of small energy and mass.

In this paper, we restricted our analysis to momentum space and we did not approach the sensitive problem of how a deformed Poincaré symmetry would be described in position space.[57] This change of representation is highly non-trivial since a non-linear deformation of the Poincaré heavily affects our notion of spacetime, which could become an observer dependent entity on its own, as suggested in Ref. 58. A standard approach in the deformed relativity literature (see e.g., Refs. 59 and 60) is to describe the symmetry of spacetime as a κ-Poincaré algebra,[61] and Ref. 62 provides a good account of the problematic issues and some tentative solutions of this proposal.

The issue of position space Lorentz covariance was also addressed within the quantum walk literature. In Refs. 63 and 64, the authors studied a model in which each point of the lattice is mapped to a lightlike rectangular spacetime patch and Lorentz covariance is allowed up to a global rescaling.

In the approach of Ref. 65, each point of the lattice is labeled by auxiliary spacetime coordinates that are treated as independent dynamical variables of covariant Lagrangians. The quantum walk equation of motion is then derived from an action principle that involves a preferred reference frame.

Regardless of the problematic issues (the presence of a rescaling in the first approach and a privileged reference frame in the second one), both of these approaches has the advantage that spacetime is an undeformed regular lattice, and this would, in principle, allow us to apply this proposal also in the presence of local interaction, as in the models studied in Refs. 37, 38, 66, and 67.

On the other hand, extending the analysis of the present paper to interacting dynamics is not trivial, since, in general, we cannot completely diagonalize the evolution. A more viable option would be to find a class of interacting terms that are invariant under the symmetry transformation that we characterize in this work. This is an approach that could be carried on in momentum space, disregarding the interpretation on position space in the first instance. However, in any of these approaches that aim at a reconciliation of special relativity and a discrete structure, it is still largely unclear what would be a sensible notion of an observer-independent local interaction.

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APPENDIX A: EXAMPLE OF A RESCALING FUNCTION

We now provide an example of a real function f such that the map $\mathcal{D}_f$ is a diffeomorphism between K and K_0. In order to have $\mathcal{D}_f$ surjective, f must be singular at the superior border of the truncated cone. Hence, we define

$$f : n \to \mathbb{R}, \qquad f(n) := \frac{1}{1 - n_2^2 - n_3^2}. \tag{A1}$$

The latter function is manifestly singular at the border of K and is monotonic vs $\|n\|_E$ ($\|.\|_E$ is the Euclidean norm). With this choice of f, it easy to very that $\mathcal{D}_f$ is a diffeomorphism between K and K_0.

APPENDIX B: SYMMETRY OF THE DIRAC WALK WITH FIXED μ

The analysis of the symmetry transformations of the Dirac walk with fixed μ follows the same steps as in the case of variable mass. The condition that the third component μ of the vector k in the eigenvalue equation of the Dirac walk is fixed implies that, for a fixed value of μ and any $\mathrm{k} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, we need to satisfy the system of equations

$$\begin{cases} n^\sigma(\mathrm{k}', \mu) = \varphi(\mathrm{k}, \mu, L) L^\sigma_\nu n^\nu(\mathrm{k}, \mu), \\ n^3(\mathrm{k}', \mu) = n^3(\mathrm{k}, \mu) = \sin \mu, \end{cases} \tag{B1}$$

where $L \in SO^+(1,2)$ and φ is a non-null function that may generally depend on L. Hence, the transformed n is

$$\frac{1}{\varphi(\mathrm{k}, \mu, L)} \begin{pmatrix} \sin \omega(\mathrm{k}') \\ \cos \mu \sin(\mathrm{k}') \\ \sin \mu \end{pmatrix} = L \begin{pmatrix} \sin \omega(\mathrm{k}) \\ \cos \mu \sin \mathrm{k} \\ \sin \mu \end{pmatrix}. \tag{B2}$$

Considering the equation for the third component, we can easily obtain a form for the dilation function, namely,

$$\frac{1}{\varphi(\mathrm{k}, \mu, L)} = \frac{L^3_\nu n^\nu(\mathrm{k}, \mu)}{\sin \mu}. \tag{B3}$$

Notice that the image of the map n (for fixed value of μ) is the hyperbolic arc given by the intersection of K and the plane of constant μ. The extremal points u, v correspond, respectively, to $\mathrm{k} = \pm\pi/2$ and are given by

$$u = (1, \cos \mu, \sin \mu), \quad v = (1, -\cos \mu, \sin \mu). \tag{B4}$$

Since the transformation L is linear, it maps extremal points to extremal points, and from Eq. (B2), we must have

$$\begin{cases} Lu = \eta u \\ Lv = \xi v \end{cases} \quad \text{or} \quad \begin{cases} Lu = \eta v, \\ Lv = \xi u \end{cases} \tag{B5}$$

with $\eta, \xi \in \mathbb{R}$. We start focusing our attention on the leftmost conditions in (B5).

At this point, we can characterize the subgroup starting from a complete set of eigenstates $\{u, v, w\}$. The vector w is such that

$$\forall a, b \in \mathbb{R},\ w_\nu (au + bv)^\nu = 0,$$

and hence, $w = (\sin \mu, 0, 1)$. Moreover, it is an eigenvector of L, since

$$0 = L^{\sigma}_{\nu} w_{\sigma} L^{\nu}_{\tau} (au + bv)^{\tau} = L^{\sigma}_{\nu} w_{\sigma} (\eta a u + \xi b v)^{\nu},$$

then $L^{\sigma}_{\nu} w^{\nu} = \theta w^{\sigma}$ for some real θ. Considering that $L \in SO^{+}(1,2)$, we have $\det L = 1$, thus the product of the eigenvalues $\eta\theta\xi = 1$. Moreover,

$$L^{\sigma}_{\nu} v_{\sigma} L^{\nu}_{\tau} u^{\tau} = \eta\xi v_{\nu} u^{\nu} \implies \eta\xi = 1,$$

and then, $\theta = 1$. Considering the parameterization of $\eta = e^{\beta}, \xi = e^{-\beta}$, where $\beta \in \mathbb{R}$, we can diagonalize L as

$$D := S^{-1} L S = \begin{pmatrix} e^{-\beta} & 0 & 0 \\ 0 & e^{\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B6}$$

$$S = \begin{pmatrix} 1 & 1 & \sin\mu \\ -\cos\mu & \cos\mu & 0 \\ \sin\mu & \sin\mu & 1 \end{pmatrix}.$$

Let us now consider the alternative transformations, defined by the rightmost condition in (B5). Repeating a similar analysis as before, we recover the two following transformations:

$$N_{\pm} = \begin{pmatrix} 0 & \pm e^{\beta} & 0 \\ \pm e^{-\beta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad O_{\pm} = \begin{pmatrix} 0 & \pm e^{\beta} & 0 \\ \mp e^{-\beta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B7}$$

Computing the square of transformations on the right, we obtain

$$O^2_{\pm} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and then, representing $O^2_{\pm}$ in the canonical basis, we have

$$T^2_{\pm} = S O^2_{\pm} S^{-1},$$

$$T^2_{\pm} = \begin{pmatrix} -\dfrac{3 - \cos(2\mu)}{2\cos^2\mu} & 0 & \dfrac{2\sin\mu}{\cos^2\mu} \\ 0 & -1 & 0 \\ \dfrac{2\sin\mu}{\cos^2\mu} & 0 & \dfrac{3-\cos(2\mu)}{2\cos^2\mu} \end{pmatrix}.$$

We easily note that the following inequality holds:

$$-\frac{3 - \cos(2\mu)}{2\cos^2\mu} < 0, \quad \forall \mu,$$

namely, the orthochronicity condition is not verified, and then, $T^2_{\pm} \notin SO^{+}(1,2)$. Hence, we are left with the transformations $N_{\pm}$ in (B7). Their representation in the canonical basis is $L_{\pm} = S N_{\pm} S^{-1}$. By an explicit calculation, we see that $(L_{\pm})^1_1 = \pm\sec^2\mu \cosh\beta + \tan^2\mu$. We can then exclude the transformation L_{-}, since it is manifestly not orthochronous. Moreover, it is clear that the transformations can be obtained as follows:

$$L_{+} = LSFS^{-1}, \qquad F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for some L. Therefore, the allowed subgroup is $SO^{+}(1,1) \rtimes \mathbb{Z}_2$, where $SO^{+}(1,1)$ is the group of matrices L in Eq. (B6).

Considering the rescaling in Eq. (B3), we obtain the following expression for the changes of inertial frame:

$$k'(k) = (n^{-1} \circ \mathcal{D}_\varphi^{-1} \circ L \circ \mathcal{D}_\varphi \circ n)(k) \tag{B8}$$

with $L \in SO^+(1,1) \rtimes \mathbb{Z}_2$. At this point, we want to study the resulting group in the relativistic regime for small values of the mass parameter μ. Deriving the expressions in Eq. (B8) with respect to β in $\beta = 0$ and expanding the generators to the first order in μ, we obtain the following group generators:

$$\tilde{J} := \varphi(\mathrm{k}, \mu, 0)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\mu \\ 0 & \mu & 0 \end{pmatrix} + \partial_\mu \varphi(\mathrm{k}, \mu, 0) I.$$

It is thus clear that we do not recover the Lorentz group in 1 + 1 dimension.

REFERENCES

[1] G. Grossing and A. Zeilinger, Complex Syst. **2**(2), 197 (1988).
[2] D. A. Meyer, J. Stat. Phys. **85**(5), 551 (1996).
[3] A. Nayak and A. Vishwanath, arXiv:quant-ph/0010117 (2000).
[4] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, in *Proceedings of 33rd annual ACM symposium on Theory of computing* (ACM, 2001), pp. 37–49.
[5] D. Aharonov, A. Ambainis, J. Kempe, and U. Vazirani, in *Proceedings of 33rd annual ACM symposium on Theory of computing* (ACM, 2001), pp. 50–59.
[6] B. Schumacher and R. Werner, arXiv:quant-ph/0405174 (2004).
[7] D. Gross, V. Nesme, H. Vogts, and R. Werner, Commun. Math. Phys. **310**, 419–454 (2012).
[8] P. Arrighi, V. Nesme, and R. Werner, J. Comput. Syst. Sci. **77**(2), 372 (2011).
[9] M. Freedman, M. B. Hastings, Commun. Math. Phys. **376**, 1171–1222 (2020).
[10] T. Farrelly, arXiv:1904.13318 (2019).
[11] P. Arrighi, Nat. Comput. **18**, 885–899 (2019).
[12] A. M. Childs, Phys. Rev. Lett. **102**, 180501 (2009).
[13] A. M. Childs, D. Gosset, and Z. Webb, Science **339**(6121), 791 (2013).
[14] N. B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon, Phys. Rev. A **81**, 042330 (2010).
[15] R. Raussendorf, Phys. Rev. A **72**, 022301 (2005).
[16] D. Shepherd, T. Franz, and R. Werner, Phys. Rev. Lett. **97**(2), 020502 (2006).
[17] P. Arrighi and J. Grattage, J. Comput. Syst. Sci. **78**(6), 1883 (2012).
[18] A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, and D. A. Spielman, in *Proceedings of 35th Annual ACM Symposium on Theory of Computing, STOC '03* (ACM, New York, NY, USA, 2003), pp. 59–68.
[19] A. Ambainis, SIAM J. Comput. **37**(1), 210 (2007).
[20] F. Magniez, M. Santha, and M. Szegedy, SIAM J. Comput. **37**(2), 413 (2007).
[21] R. P. Feynman, Int. J. Theor. Phys. **21**(6), 467 (1982).
[22] C. M. Dawson, J. Eisert, and T. J. Osborne, Phys. Rev. Lett. **100**, 130501 (2008).
[23] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, J. Stat. Mech.: Theory Exp. **2017**(8), 083105.
[24] M. Genske, W. Alt, A. Steffen, A. H. Werner, R. F. Werner, D. Meschede, and A. Alberti, Phys. Rev. Lett. **110**, 190601 (2013).
[25] A. Alberti, W. Alt, R. Werner, and D. Meschede, New J. Phys. **16**(12), 123052 (2014).
[26] L. Sansoni, F. Sciarrino, G. Vallone, P. Mataloni, A. Crespi, R. Ramponi, and R. Osellame, Phys. Rev. Lett. **108**(1), 010502 (2012).
[27] A. Crespi, R. Osellame, R. Ramponi, V. Giovannetti, R. Fazio, L. Sansoni, F. De Nicola, F. Sciarrino, and P. Mataloni, Nat. Photonics **7**, 322 (2013).
[28] S. Succi and R. Benzi, Physica D **69**(3), 327 (1993).
[29] I. Bialynicki-Birula, Phys. Rev. D **49**, 6920 (1994).
[30] D. A. Meyer, J. Stat. Phys. **85**(5), 551 (1996).
[31] J. Yepez, Int. J. Mod. Phys. C **12**(09), 1285 (2001).
[32] G. M. D'Ariano and P. Perinotti, Phys. Rev. A **90**, 062106 (2014).
[33] A. Bisio, G. M. D'Ariano, and A. Tosini, Ann. Phys. **354**, 244 (2015).
[34] A. Bisio, G. M. D'Ariano, and P. Perinotti, Ann. Phys. **368**, 177 (2016).
[35] P. Arrighi and S. Facchini, EPL (Europhys. Lett.) **104**(6), 60004 (2013).
[36] G. Di Molfetta, M. Brachet, and F. Debbasch, Phys. Rev. A **88**, 042301 (2013).
[37] P. Arnault and F. Debbasch, Phys. Rev. A **93**, 052301 (2016).
[38] A. Bisio, G. M. D'Ariano, P. Perinotti, and A. Tosini, Phys. Rev. A **97**, 032132 (2018).
[39] A. Bibeau-Delisle, A. Bisio, G. M. D'Ariano, P. Perinotti, and A. Tosini, EPL (Europhys. Lett.) **109**(5), 50003 (2015).
[40] A. Bisio, G. M. D'Ariano, and P. Perinotti, Philos. Trans. R. Soc., A **374**(2068), 20150232 (2016).
[41] A. Bisio, G. M. D'Ariano, and P. Perinotti, Phys. Rev. A **94**, 042120 (2016).
[42] G. Amelino-Camelia, Phys. Lett. B **510**(1), 255 (2001).
[43] J. Magueijo and L. Smolin, Phys. Rev. D **67**(4), 044017 (2003).
[44] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, Phys. Rev. D **84**, 084010 (2011).
[45] M. Moyer, Sci. Am. **306**, 30–37 (2012).
[46] C. Hogan, Phys. Rev. D **85**, 064007 (2012).

[47] I. Pikovski, M. Vanner, M. Aspelmeyer, M. Kim, and C. Brukner, Nat. Phys. **331**(1), 393 (2012).
[48] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature **393**, 763 (1998).
[49] A. A. Abdo *et al.*, Nature **462**, 331 (2009).
[50] V. Vasileiou, J. Granot, T. Piran, and G. Amelino-Camelia, Nat. Phys. **11**, 344 (2015).
[51] G. Amelino-Camelia, G. D'Amico, G. Rosati, and N. Loret, Nat. Astron. **1**, 0139 (2017).
[52] A. Bisio, G. M. D'Ariano, and P. Perinotti, Found. Phys. **47**(8), 1065 (2017).
[53] D. M. Greenberger, J. Math. Phys. **11**(8), 2329 (1970).
[54] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, in *Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, STOC '01* (ACM, New York, NY, USA, 2001), pp. 37–49.
[55] R. Portugal, *Quantum Walks and Search Algorithms* (Springer Science & Business Media, 2013).
[56] We could also have considered the case $M \in GL(2,\mathbb{C})$. However, this choice would only have introduced a k-independent multiplicative constant. The symmetry group would have been $\mathcal{S}'_D = \mathcal{S} \times \mathbb{C}$, i.e., the symmetry group of the $M \in SL(2,\mathbb{C})$ case trivially extended by the direct product with the multiplicative action of $\mathbb{C}$.
[57] A preliminary analysis was done in Ref. 39.
[58] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, Int. J. Mod. Phys. D **20**(14), 2867 (2011).
[59] G. Amelino-Camelia, Symmetry **2**(1), 230 (2010).
[60] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. D **12**(02), 299 (2003).
[61] S. Majid and H. Ruegg, Phys. Lett. B **334**(3), 348 (1994).
[62] E. R. Livine and D. Oriti, J. High Energy Phys. **2004**(06), 050.
[63] G. Mauro D'Ariano and A. Tosini, Stud. Hist. Philos. Sci., Part B **44**(3), 294 (2013).
[64] P. Arrighi, S. Facchini, and M. Forets, New J. Phys. **16**(9), 093007 (2014).
[65] F. Debbasch, Ann. Phys. **405**, 340 (2019).
[66] P. Arnault, G. Di Molfetta, M. Brachet, and F. Debbasch, Phys. Rev. A **94**, 012335 (2016).
[67] P. Arrighi, C. Bény, and T. Farrelly, Quantum Inf. Process. **19**(3), 88 (2020).