Space-time and special relativity from causal networks

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We show how the Minkowskian space-time emerges from a topologically homogeneous causal network, presenting a simple analytical derivation of the Lorentz transformations, with metric as pure event-counting. The derivation holds generally for $d = 1$ space dimension, however, it can be extended to $d > 1$ for special causal networks.

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Do events happen in space-time or is space-time that is made up of events? This question may be considered a "which came first, the chicken or the egg?" dilemma, but the answer may contain the solution of the main problem of contemporary physics: the reconciliation of quantum theory (QT) with general relativity (GR). Why? Because "events" are central to QT and "space-time" is central to GR. Therefore, the question practically means: which comes first, QT or GR?

In spite of the evidence of the first position—"events happen in space-time"—the second standpoint—"space-time is made up of events"—is more concrete, if we believe à la Copenhagen that whatever is not "measured" is only in our imagination: space-time too must be measured, and measurements are always made-up of events. Thus QT comes first. How? Space-time emerges from the tapestry of events that are connected by quantum interactions, as in a huge quantum computer: this is the Wheeler's *It from bit* [1]. For a theory of quantum gravity a variation of QT may still be needed, such as a "third-quantization" of causal connections, allowing non pre-established causal relations. However, at least for the simplest case of Special Relativity (SR) QT tout court should be sufficient. Ref. [2] showed the mechanism with which space-time emerges endowed with SR from a network of causally connected events, starting only from the topology of the network, and getting the metric from pure event-counting. Ref. [3] later has shown how the Minkowski signature can be derived from the causal poset. Here we will present a simple analytical derivation of the Lorentz transformations from a causal network (CN) in 1 space dimension: generalization to larger dimensions will be discussed at the end of the paper. As we will see, the only thing that is needed in addition to causality is the topological homogeneity of the CN, corresponding to the Galileo relativity principle.

The program of deriving the geometry of space-time from purely causal structure (causal sets) is not new, and was initiated by Sorkin and collaborators more than two decades ago [4]. In this publication and in following ones (see the review [5]) the possibility of recovering the main features of the space-time manifold—topology, differentiable structure and the conformal metric—has been investigated, starting from discrete sets of points endowed with a causal partial ordering. Since from the start, causal sets were an independent research line in quantum gravity, since they naturally possess a space-time discreteness and provide a history-space for a "path integral" formulation [6, 7]. They also fit perfectly the spirit of very recent works on operational probabilistic theories [8, 9].

We now introduce the main notion of causal network (CN) as a partially ordered set of events with the partial order representing the causal relation between two events. As mentioned, the aim is to have the space-time endowed with SR emerging from the network of events, thinking to them not as "happening in space-time", but as making up space-time themselves. Thus the notions of *event* and *causal relation* have to be considered as primitive, similarly to those of "point" and "line" in geometry (for their meaning in an operational framework and in QT, see Refs. [9, 10]). In synthesis, the CN represents the most general structure of "information processing".

A *causal set* is a set N of elements called *events* $a, b, c \ldots, \in \mathsf{N}$ equipped with a partial order relation $\preceq$ which is: (1) *Reflexive:* $\forall a \in \mathsf{N}$ we have $a \preceq a$; (2) *Antisymmetric:* $\forall a, b \in \mathsf{N}$, we have $a \preceq b \preceq a \Rightarrow a = b$; (3) *Transitive:* $\forall a, b, c \in \mathsf{N}$, $a \preceq b \preceq c \Rightarrow a \preceq c$; (4) *Locally finite:* $\forall a, c \in \mathsf{C}$, $|\{b \in \mathsf{N} : a \preceq b \preceq c\}| < \infty$, where $|\mathsf{S}|$ denotes the cardinality of the set S. In the following we will also write $a \prec b$ to state that $a \preceq b$ with $a \neq b$. A causal set is represented by a *graph* with points being the events and the edges drawn between any two points a and b for which $a \preceq b$—i. e. that are causal connected, as in Fig. 1. What we call a *causal network* (CN) is just a causal set that is unbounded in all directions. In order to satisfy transitivity, the CN is a *directed acyclic graph* (DAG), i. e. loops are forbidden (arrows on edges are usually not drawn by orienting the graph e. g. from the bottom to the top).

Causality of the network naturally suggests the notion of *light-cone* J_a of an event $a \in \mathsf{N}$, along with those of *past/future light-cone* $\mathsf{J}^-_a/\mathsf{J}^+_a$, respectively (see Fig. 1)

$$\mathsf{J}^-_a := \{b \in \mathsf{N} : b \preceq a\}, \quad \mathsf{J}^+_a := \{b \in \mathsf{N} : a \preceq b\}, \qquad (1)$$

and $\mathsf{J}_a := \mathsf{J}^-_a \cup \mathsf{J}^+_a$. Accordingly, one has that $a \preceq b$ is equivalent to $a \in \mathsf{J}^-_b$ and to $b \in \mathsf{J}^+_a$. We will call *independent* or *space-like* two events $a, b \in \mathsf{N}$ that are not causally related—namely $a \notin \mathsf{J}_b$ (or $b \notin \mathsf{J}_a$)—and *causally dependent* or *time-like* otherwise, namely when

$a \in \mathsf{J}_b$ (or $b \in \mathsf{J}_a$). We call a CN *connected* if for every $a, b \in \mathsf{N}$ there exists $c \in \mathsf{J}_a \cap \mathsf{J}_b$, corresponding to the intuitive notion of connectedness. Two events that are

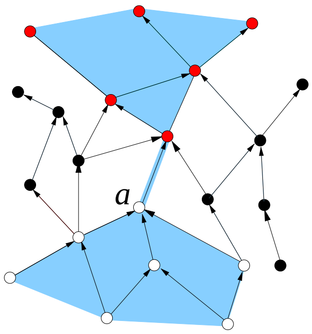

FIG. 1: Causal network: illustration of the set of past/future light-cone of event a.

not space-like are connected by at least a causal chain, e. g. $a \preceq b$ are connected by the causal chain $\mathsf{C}(a,b)$ given by $\mathsf{C}(a,b) := \{c_i\}_{i=1}^N$, with $a \equiv c_1 \prec c_2 \prec \ldots c_N \equiv b$. Being the equivalent of a *world-line*, the causal chain plays also the role of an *observer*. It is convenient to orient the chain, generalizing its definition to include the case $b \preceq a$, writing $\mathsf{C}(a,b)$ for $\mathsf{C}(a,b) := \{c_i\}_{i=1}^N$, with $b \equiv c_1 \prec c_2 \prec \ldots c_N \equiv a$. The verse of the chain is taken into account by a signed cardinality $|\mathsf{C}(a,b)|_\pm := \sigma|\mathsf{C}(a,b)|$ with $\sigma = +$ for $a \prec b$, and $\sigma = -$ for $b \prec a$.

In order to derive SR from the CN, we need the equivalent of the Galileo principle [11], namely the invariance of the physical law with the reference system. Within a single frame the Galileo principle is just uniformity of space and time. In the present purely topological context, this translates to the topological homogeneity of the CN, the physical law being the causal connection-rule of the network, i. e. the tile of the causal pattern. At this point, we need to make more specific the notion of CN, introducing different types of links, e. g. in Fig. 2 we have two generally different kinds of input links—the left and the right ones—for each node. It is now convenient to label links with letters. We then consider the input and the output sets $\mathsf{I}_{in}(a) = \{i_1(a), i_2(a), \ldots i_K(a)\}$ and $\mathsf{I}_{out}(a) = \{o_1(a), o_2(a), \ldots o_H(a)\}$ of links of an event. We now say that a CN is *topologically homogeneous* if for each couple of events $a, b \in \mathsf{N}$ one has the isomorphism $i_j(a) = i_j(b)$ and $o_j(a) = o_j(b)$ for $j = 1, \ldots H = K$. An example of homogeneous CN is given in Fig. 2. There is no loss of generality in considering only homogeneous CN with $H = K$ and with all events isomorphic: in fact, one can always reach this situation, by grouping connected events into single ones, i. e. by *event coarse-graining*.

In a homogeneous causal network we can also easily see how causality is sufficient to guarantee a maximum speed of "information flow". Such speed is just "one-event per step", corresponding to a line at 45^o in Fig. 2 (to connect events along a line making an angle $< 45^o$ with the horizontal, one needs to follow some causal connections in the backward direction from the output to the input).

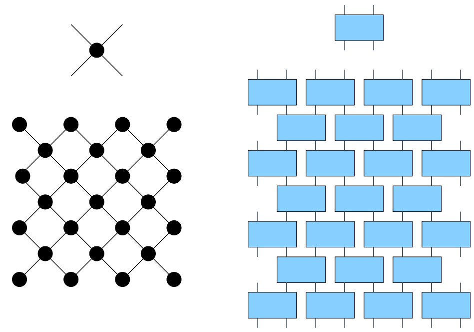

FIG. 2: Right: homogeneous causal network. Left: equivalent representation as a quantum circuit. .

We will now introduce the notion of simultaneity in relation to an observer. The observer is just a causal chain (conveniently taken as unbounded). We label the events of the chain with relative numbers, choosing an event for the zero. Hence, an observer will be denoted as $\mathsf{O}_a = \{o_i\}_{i\in\mathbb{Z}}$, with $o_i \preceq o_{i+1}$ $\forall i \in \mathbb{Z}$, and with $a = o_0$ representing the origin. The index $i \in \mathbb{Z}$ plays the role of the observer's proper time. Thanks to the topological homogeneity, we can translate the observer O_a to any event $a' \in \mathsf{N}$. We will denote by O the equivalence class of all observers translated over all events of the CN. We will also denote by $\mathsf{O}_a(b,c)$ the causal chain $\mathsf{C}(b,c) \subset \mathsf{O}_a$. We now define simultaneity of events a and b—denoted as $a \sim_{\mathsf{O}} b$—as follows

$$a \sim_{\mathsf{O}} b \Leftarrow \inf_{b^* \in \mathsf{J}_b^+} |\mathsf{O}_a(a, b^*)|_\pm = \inf_{a^* \in \mathsf{J}_a^+} |\mathsf{O}_b(b, a^*)|_\pm. \quad (2)$$

Depending on the shape of the observer chain, one may have situations in which there are no synchronous events. However, it is easy to see that for an observer that is topologically homogeneous (i. e. periodic) there always exist infinitely many simultaneous events. Moreover, modulo event coarse-graining, without loss of generality we can restrict only to observers with a zig-zag with a single period, with $\alpha \geq 1$ steps to the right and $\beta \geq 1$ steps to the left (we will call them *simply periodic*). Each ziz-zag is the equivalent of a tic-tac of an Einstein clock made with light bouncing between two mirrors. All events on the same mirror lay on a line, and for such events there always exist (infinitely many) synchronous events.

The given notion of simultaneity allows us to associate each observer with a *foliation* of the CN. For each event $o_i \in \mathsf{O}_a$ there is a *leaf* $\mathsf{L}_i(\mathsf{O}_a)$, which is the set of events simultaneous to o_i with respect to the observer O_a, namely

$$\mathsf{L}_i(\mathsf{O}_a) := \{b \in \mathsf{N} : a \sim_{\mathsf{O}} o_i\}. \quad (3)$$

The collection of all leaves for all the events in O_a is the *foliation* $\mathsf{L}(\mathsf{O}_a)$ of N associated to the observer O_a

$$\mathsf{L}(\mathsf{O}_a) := \{\mathsf{L}_i(\mathsf{O}_a), \forall i \in \mathbb{Z}\}. \quad (4)$$

The foliation has an "origin" a defined by the observer O_a. Homogeneity of foliations follows from that of the observer. Notice that a foliation does not generally contain all the events of the CN (it certainly does for $\alpha = \beta = 1$): this fact is related to the sparsness issue raised in Ref. [12] for Lorentz-transformed regular lattices of points.

For a given foliation $\mathsf{L}(\mathsf{O}_a)$ we can now define a pair of coordinates $\mathbf{z}(b)$ for any event $b \in \mathsf{L}(\mathsf{O}_a)$ via the map

$$K_{\mathsf{O}_a} : \mathsf{N} \to \mathbb{Z}^2, \quad b \mapsto K_{\mathsf{O}_a}(b) := \mathbf{z}(b) = \begin{bmatrix} z_1(b) \\ z_2(b) \end{bmatrix},$$
$$z_1(b) := \inf_{b^* \in \mathsf{J}_b^+} |\mathsf{O}_a(a, b^*)|_\pm, \quad z_2(b) := \inf_{a^* \in \mathsf{J}_a^+} |\mathsf{O}_b(b, a^*)|_\pm. \tag{5}$$

Thus, to each observer O_a it corresponds a coordinate

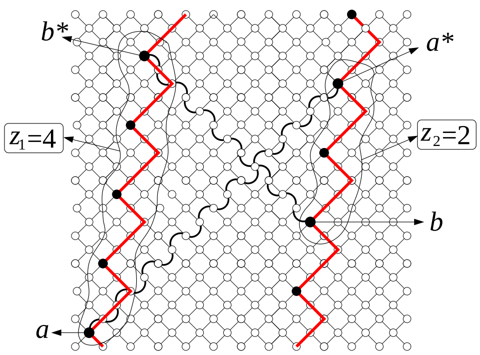

FIG. 3: Illustration of the coordinate map in Eq. (5) (the observer has $\alpha = 3$ and $\beta = 2$).

map, and this is what is commonly called a *reference frame*—shortly *frame*. The coordinates z_1 and z_2 do not have an immediate meaning, but get an simple interpretation thanks to the following Lemma.

Lemma 1 *An event $b \in \mathsf{L}(\mathsf{O}_a)$ belongs to the t-th leaf $\mathsf{L}_t(\mathsf{O}_a)$ for $t = (z_1 - z_2)/2$, and the number of events on such leaf between b and O_a is given by $s = (z_1 + z_2)/2$.*

Proof. There exists $t \in \mathbb{Z}$ such that o_t is simultaneous to b. By definition one has $b \in \mathsf{L}_t(\mathsf{O}_a)$, and

$$\inf_{b^* \in \mathsf{J}_b^+} |\mathsf{O}_{o_t}(o_t, b^*)|_\pm = \inf_{o_t{}^* \in \{j f o_t} |\mathsf{O}_b(b, o_t{}^*)|_\pm. \tag{6}$$

One has

$$z_1(b) = t + \inf_{b^* \in \mathsf{J}_b^+} |\mathsf{O}_{o_t}(o_t, b^*)|_\pm, \tag{7}$$

whereas

$$z_2(b) = \inf_{o_t^* \in \mathsf{J}_{o_t}^+} \inf_{a^* \in \mathsf{J}_a^+} \Big(|\mathsf{O}_b(b, o_t^*)|_\pm + |\mathsf{O}_b(o_t^*, a^*)|_\pm \Big). \tag{8}$$

Topological homogeneity implies that

$$z_2(b) = \inf_{o_t{}^* \in \mathsf{J}_{o_t}^+} |\mathsf{O}_b(b, o_t^*)|_\pm - t. \tag{9}$$

Using the simultaneity condition in Eq. (6) we can combine Eqs. (7) and (9) to get $t = \frac{1}{2}(z_1 - z_2)$. ■

According to the last Lemma the coordinates

$$\begin{bmatrix} t(b) \\ s(b) \end{bmatrix} := 2^{\frac{1}{2}} \mathbf{U}(\pi/4) \begin{bmatrix} z(b) \\ z(b) \end{bmatrix}, \tag{10}$$

where $\mathbf{U}(\theta)$ is the matrix performing a θ-rotation, can be interpreted as the space-time coordinates of the event b in the frame $\mathsf{L}(\mathsf{O}_a)$.

Frames in standard configuration (boosted). Consider now two observers $\mathsf{O}_a^1 = \{o_i^1\}$ and $\mathsf{O}_a^2 = \{o_j^2\}$ sharing the same origin (homogeneity guarantees the existence of observers sharing the origin). We will shortly denote the two frames as $\mathfrak{R}^1$ and $\mathfrak{R}^2$, and the corresponding coordinate maps as K^1 and K^2. We will say that the two frames $\mathfrak{R}^1$ and $\mathfrak{R}^2$ are in *standard configuration* if there exist positive α^{12}, β^{12}, such that $\forall i \in \mathbb{Z}$

$$K^1(o_i^2) = \mathbf{D}^{12} K^2(o_i^2), \ \mathbf{D}^{12} := \mathrm{diag}(\alpha^{12}, \beta^{12}). \tag{11}$$

It turns out that having chosen only simply periodic observers, one has $\alpha^{ij} = \alpha^j/\alpha^i \in \mathbb{Z}^+$, $\beta^{ij} = \beta^j/\beta^i \in \mathbb{Z}^+$.

Examples of observers corresponding to frames in standard configuration are shown in Fig. 4. Clearly different frames correspond to generally different sets of events, and what follows applies to the events in their intersection: thus, again, the transformation includes an implicit event coarse-graining. We now see how it is possible to

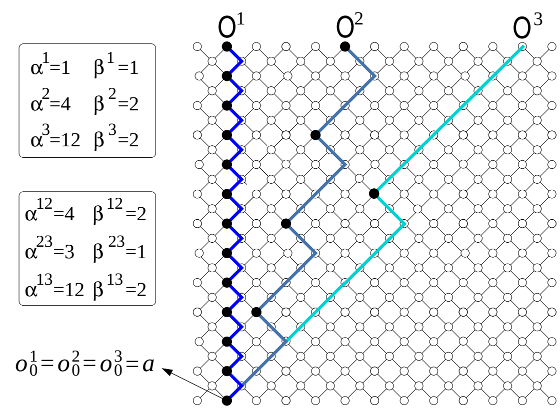

FIG. 4: Example of three observers related as in Eq. (11) and then generating reference frames in standard configuration.

define a relative velocity between two frames in standard configuration. It is readily seen that $K^2(o_n^2) = (n, -n)$, whence $K^1(o_l^2) = (l\alpha^{12}, -l\beta^{12})$. We can now define the relative velocity between $\mathfrak{R}^1$ and $\mathfrak{R}^2$ as the quotient between the space and time coordinates of the observer O^2 with respect to observer O^1, namely, by Lemma 1

$$v^{12} = \frac{n\alpha^{12} - n\beta^{12}}{n\alpha^{12} + n\beta^{12}} = \frac{\alpha^{12} - \beta^{12}}{\alpha^{12} + \beta^{12}}. \tag{12}$$

Of course one has $K^2(o_i^2) = \mathbf{D}^{21} K^1(o_i^2) \ \forall i \in \mathbb{Z}$, with $\mathbf{D}^{21} = \mathbf{D^{12}}^{-1} = \mathrm{diag}(1/\alpha^{12}, 1/\beta^{12})$, whence upon rewriting Eq. (12) for v^{21} one obtains $v^{21} = -v^{12}$.

Velocity-composition rule. Consider three frames $\mathfrak{R}^1$, $\mathfrak{R}^2$, $\mathfrak{R}^3$ in pairwise standard relation, associated to observers O^1, O^2, O^3 sharing the origin a, corresponding to the coordinate maps K^1, K^2, K^3 (see for example the situation illustrated in Fig. 4). Let $\mathbf{D}^{12} = \mathrm{diag}(\alpha^{12}, \beta^{12})$ and $\mathbf{D}^{23} = \mathrm{diag}(\alpha^{23}, \beta^{23})$ be the matrixes relating respectively the coordinates of the second observer with respect to the first one and the coordinates of the third observer with respect to the second one, according to

$$K^1(o_i^1) = \mathbf{D}^{12} K^2(o_i^2), \quad K^2(o_j^3) = \mathbf{D}^{23} K^3(o_j^3). \tag{13}$$

We are interested in the relation between the coordinates of frame $\mathfrak{R}^3$ with respect to frame $\mathfrak{R}^1$. This is given by

$$K^1(o_j^3) = \mathbf{D}^{13} K^3(o_j^3), \tag{14}$$

with matrix $\mathbf{D}^{13} = \mathbf{D}^{12}\mathbf{D}^{23} = \mathrm{diag}(\alpha^{12}\alpha^{23}, \beta^{12}\beta^{23})$. From Eq. (12) it immediately follows that

$$v^{13} = \frac{\alpha^{12}\alpha^{23} - \beta^{12}\beta^{23}}{\alpha^{12}\alpha^{23} + \beta^{12}\beta^{23}}, \tag{15}$$

which by simple algebraic manipulations gives

$$v^{13} = \frac{\left(\frac{\alpha^{12}-\beta^{12}}{\alpha^{12}+\beta^{12}}\right) + \left(\frac{\alpha^{23}-\beta^{23}}{\alpha^{23}+\beta^{23}}\right)}{1 + \left(\frac{\alpha^{12}-\beta^{12}}{\alpha^{12}+\beta^{12}}\right)\left(\frac{\alpha^{23}-\beta^{23}}{\alpha^{23}+\beta^{23}}\right)} = \frac{v_{12} + v_{23}}{1 + v_{12}v_{23}}, \tag{16}$$

namely the velocity composition rule of special relativity.

Lorentz transformations. Again using Lemma 1 we can derive the space-time coordinate transformations between the two frames $\mathfrak{R}^1$ and $\mathfrak{R}^2$ in standard relation. Using the topological homogeneity of N it follows that Eq. (11) holds for any event $b \in \mathfrak{R}^1 \cap \mathfrak{R}^2$. One has $z_1^1 = \alpha^{12} z_1^2$ and $z_2^1 = \beta^{12} z_2^2$, and after easy manipulations we get

$$\frac{z_1^1 \pm z_2^1}{2} = \frac{\alpha^{12} + \beta^{12}}{2}\left[\frac{z_1^2 \pm z_2^2}{2} + \left(\frac{\alpha^{12} - \beta^{12}}{\alpha^{12} + \beta^{12}}\right)\frac{z_1^2 \mp z_2^2}{2}\right], \tag{17}$$

where we can easily identify the space-time coordinates of the event in the two frames and their relative velocity, in terms of which Eqs. (17) become

$$\begin{aligned} t^1 &= \tfrac{1}{2}(\alpha^{12} + \beta^{12})\left(t^2 + v^{12}s^2\right), \\ s^1 &= \tfrac{1}{2}(\alpha^{12} + \beta^{12})\left(s^2 + v^{12}t^2\right). \end{aligned} \tag{18}$$

Using the simple relation

$$\tfrac{1}{2}(\alpha^{12} + \beta^{12}) = \frac{\chi_{12}}{\sqrt{1 - (v^{12})^2}}, \quad \chi_{12} := \sqrt{\alpha^{12}\beta^{12}}, \tag{19}$$

we obtain the identities

$$t^1 = \chi_{12}\frac{t^2 + v^{12}s^2}{\sqrt{1 - (v^{12})^2}}, \qquad s^1 = \chi_{12}\frac{s^2 + v^{12}t^2}{\sqrt{1 - (v^{12})^2}}, \tag{20}$$

which differ from the Lorentz transformations only by the multiplicative factor χ_{12}. The factor χ_{12} can be removed by rescaling the coordinate map in Eq. (10) using the factor $(2\alpha\beta)^{\frac{1}{2}}$ in place of $2^{\frac{1}{2}}$, with the constants α and β of the observer. The relative velocity between two frames $\mathfrak{R}^1$ and $\mathfrak{R}^2$ does not change in this representation because the common factor simplifies in Eq. (12). Consequently also the velocity-composition rule is left unchanged. A multiplicative factor $\sqrt{\frac{\alpha^1\beta^1}{\alpha^2\beta^2}} = \chi_{12}^{-1}$ now shows up after the factor 1/2 in both transformations (18), and, using relation (19) we get the usual Lorentz transformations.

We emphasize that the whole procedure for defining the space-time coordinates is made only with event-counting on the CN. For each transformation a corresponding coarse-graining (of the starting or the ending foliation) seems essential (corresponding to the usual rescaling in the Minkowski space, due to reciprocity between the observers). Finally, it is clear that our derivation could be extended to $d > 1$ space dimensions, for CN that are embeddable in $d+1$ dimensions, with leaves that can be embedded in d dimensions, e. g. for a $d+1$-dimensional diamond lattice.

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