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Optical Bell measurement by Fock filtering

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Abstract

We describe a nonlinear interferometric setup to perform a complete optical Bell measurement, i.e. to unambiguously discriminate the four polarization-entangled EPR-Bell photon pairs. The scheme is robust against detector inefficiency. © 2000 Published by Elsevier Science B.V.

1. Introduction

Entanglement and entangled states are fundamental concepts of the new field of quantum information [1]. They can be exploited for example in super-dense coding [2,3] or teleportation [4–7] both of which have been demonstrated experimentally. The most important example of entangled states is perhaps given by the four Bell states

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle \pm |10\rangle\right]; \quad |\phi_{\pm}\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle \pm |11\rangle\right] \tag{1}$$

which form a basis of maximally entangled states of two qubits. Many applications of entanglement, including the two mentioned above, rely on the ability to perform a *Bell measurement*, that is, a measurement that distinguishes unambiguously between the four Bell states. For example, in quantum teleportation [4–7], the sending party must realise a Bell measurement on the system formed by the qubit to be teleported together with her half of a Bell state previously shared with the receiving party. In order to properly reconstitute the teleported state, the receiver must know the result of this measurement, which is equally likely to have any of its four outcomes. Any ambiguity in the result leads to the teleportation being imperfect. In this paper we shall be

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concerned with the realization of Bell measurements on a particular implementation of Bell states using two polarization-entangled photons.

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left[|1001\rangle \pm |0110\rangle\right] = \frac{1}{\sqrt{2}}\left[a_\parallel b_\perp \pm a_\perp b_\parallel\right]^\dagger |\mathbf{0}\rangle \tag{2}$$

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\left[|1010\rangle \pm |0101\rangle\right] = \frac{1}{\sqrt{2}}\left[a_\parallel b_\parallel \pm a_\perp b_\perp\right]^\dagger |\mathbf{0}\rangle , \tag{3}$$

where $a_\parallel (b_\parallel)$ denotes horizontally polarized and $a_\perp (b_\perp)$ denotes vertically polarized photons in the two possible directions of propagation denoted by a and b. These states have in fact been produced experimentally via the nonlinear process of spontaneous down-conversion [8]. Our task is to devise an optical setup (i.e., an interferometer) that will, at least in principle, discriminate between these four states. Surprisingly, it has recently been proven that this is impossible to realise if only linear optical elements are used [9]. One method to overcome this conclusion has been suggested in Ref. [10], which, however, requires to embed the state of interest in a larger Hilbert space, and therefore can be applied only in the presence of multiple entanglement (entanglement in more than two degrees of freedom).

2. Bell discrimination

In this paper we consider the nonlinear interferometric setup depicted in Fig. 1. Our starting point is a reliable source of optical EPR-Bell states, i.e., polarization entangled photon pairs. This is usually a birefringent crystal where type-II parametric down-conversion transforms an incoming pump photon into a pair of correlated ordinary and extraordinary photons. We assume that each pulse (the signal) is prepared in one of the four EPR-Bell states, and we want to unambiguously infer from a single measurement which one was actually impinging onto the apparatus. The signal first enters a polarizing beam splitter, which transmits photons with a

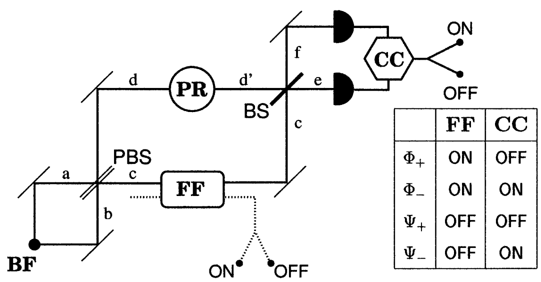

Fig. 1. Schematic diagram of the nonlinear interferometric setup for the discrimination of the four Bell states. One of the four Bell states is produced at the ‘Bell Factory’ (BF) and then impinges onto a polarizing beam splitter, whose action is to transmit photons with a fixed polarization (say vertical) and to reflect photons with the orthogonal one (say horizontal). Inside the interferometer the polarization of the photons in one arm is rotated (PR) by a half wave plate, whereas in the other arm a non-demolition measurement of the photon number is performed by means of a ‘Fock Filter’ (FF) based on Kerr nonlinear interaction. Finally, the photons are recombined by an usual, not polarizing, balanced beam splitter (BS) and then revealed by a couple of avalanche single-photon photo-detectors. A coincidence circuit (CC) tells us whether the photons arrived one for each path or both packed in the same one. The field modes c, d, e and f are the Heisenberg evolute of the input field modes a and b. Their explicit expressions are given in the text. The inset table describes the reaction of the two measurement stages (the Fock Filter FF and the coincidence circuit CC) to the presence of the four Bell states respectively.

given polarization (say vertical) and reflects photons with the orthogonal one (say horizontal). The mode transformations of this element is given by (the notation for the field modes refers to Fig. 1 hereafter)

$$(c_{\parallel},c_{\perp},d_{\parallel},d_{\perp}) = \hat{U}_{\rm PBS}(a_{\parallel},a_{\perp},b_{\parallel},b_{\perp})\hat{U}^{\dagger}_{\rm PBS} = (b_{\parallel},a_{\perp},a_{\parallel},b_{\perp})\,, \tag{4}$$

and the corresponding Schrödinger evolution of the Bell states is

$$\hat{U}_{\rm PBS}(\Phi_+,\Phi_-,\Psi_+,\Psi_-) = (\Phi_+,\Phi_-,\chi_+,\chi_-)\,. \tag{5}$$

In Eq. (5) $\chi_{\pm}$ are superpositions of states with both photons in the same path (arm)

$$|\chi_{\pm}\rangle = \frac{1}{\sqrt{2}}\left[|1100\rangle \pm |0011\rangle\right] = \frac{1}{\sqrt{2}}\left[b_{\parallel}b_{\perp} \pm a_{\perp}a_{\parallel}\right]^{\dagger}|\mathbf{0}\rangle\,. \tag{6}$$

The two sets of states, $\chi_{\pm}$ and $\Phi_{\pm}$, can now be discriminated by the number of photons traveling in one arm of the interferometer, which is either zero or two for $\chi_{\pm}$ and (certainly only) one for $\Phi_{\pm}$. Such a discrimination can be performed by means of a Fock Filter (FF), which is a novel kind of all-optical nonlinear switch [11]. Let us postpone the detailed description of the FF. For the moment we assume that it switches on when a single photon (of any polarization) is present, and does not switch for zero or more than one photons. As we will see, the FF performs a kind of non-demolition measurement [12,13] of the photon number, such that coherence is preserved and the state after the measurement is still available for further manipulations. Indeed, the remaining part of the device should be able to distinguish phases, namely to discriminate between χ_+ and χ_-, or between Φ_+ and Φ_-. For this purpose, first the polarization of photons in the second arm is rotated by $\pi/2$ using a half-wave plate, thus turning $\Phi_{\pm}$ into $\Psi_{\pm}$ respectively, while leaving $\chi_{\pm}$ untouched. In fact, the transformation induced by the polarization rotator reads $\hat{U}_{\rm PR} = \hat{I}\otimes\hat{V}_{\rm PR}$, where $\hat{V}_{\rm PR}$ acts only on two modes

$$(d'_{\parallel},d'_{\perp}) = \hat{V}_{\rm PR}(d_{\parallel},d_{\perp})\hat{V}^{\dagger}_{\rm PR} = (d_{\perp},-d_{\parallel})\,, \tag{7}$$

and thus

$$\hat{U}_{\rm PR}(\chi_+,\chi_-,\Phi_+,\Phi_-) = (\chi_+,\chi_-,\Psi_+,-\Psi_-)\,. \tag{8}$$

The two paths are then recombined into a balanced (not polarizing) beam splitter, whose action on generic field modes x and y is described by

$$\hat{U}_{\rm BS}(x_{\parallel},x_{\perp},y_{\parallel},y_{\perp})\hat{U}^{\dagger}_{\rm BS} = \frac{1}{\sqrt{2}}(x_{\parallel}+y_{\parallel},x_{\perp}+y_{\perp},x_{\parallel}-y_{\parallel},x_{\perp}-y_{\perp})\,. \tag{9}$$

If the transformation (9) is applied to the field modes c and d' we have, using Eqs. (7) and (4),

$$\begin{aligned}(e_{\parallel},e_{\perp},f_{\parallel},f_{\perp}) &= \hat{U}_{\rm BS}(c_{\parallel},c_{\perp},d'_{\parallel},d'_{\perp})\hat{U}^{\dagger}_{\rm BS} = \frac{1}{\sqrt{2}}(c_{\parallel}+d'_{\parallel},c_{\perp}+d'_{\perp},c_{\parallel}-d'_{\parallel},c_{\perp}-d'_{\perp}) \\ &= \frac{1}{\sqrt{2}}(b_{\parallel}+b_{\perp},a_{\perp}-a_{\parallel},b_{\parallel}-b_{\perp},a_{\perp}+a_{\parallel})\,. \end{aligned} \tag{10}$$

In terms of the state just before the BS this corresponds to the following Schrödinger evolution

$$\hat{U}_{\rm BS}(\chi_+,\chi_-,\Psi_+,\Psi_-) = (\chi_+,\Psi_+,\chi_-,-\Psi_-)\,. \tag{11}$$

Finally, the photons are measured by single-photons avalanche photodetectors, where the last stage of the setup is a coincidence circuit (CC). In fact, $\Psi_{\pm}$ correspond to the presence of one photon in each channel, whereas $\chi_{\pm}$ are superpositions with both photons in the same path. In terms of the states *before* the BS, this means that superpositions with the minus sign (χ_- and Ψ_-) lead to coincident clicks at photodetectors (CC ON), whereas superpositions with the plus sign (χ_+ and Ψ_+) do not switch on the coincidence circuit. The whole chain of transformations of our setup can be summarized by the following diagram:

$$\begin{array}{llllllll}
\Psi_+ & \xrightarrow{PBS} & \chi_+ & \text{[FF OFF]} & \xrightarrow{PR} & \chi_+ & \xrightarrow{BS} & \chi_+ \quad \text{[CC OFF]} \\
\Psi_- & \xrightarrow{PBS} & \chi_- & \text{[FF OFF]} & \xrightarrow{PR} & \chi_- & \xrightarrow{BS} & \Psi_+ \quad \text{[CC ON]} \\
\Phi_+ & \xrightarrow{PBS} & \Phi_+ & \text{[FF ON]} & \xrightarrow{PR} & \Psi_+ & \xrightarrow{BS} & \chi_- \quad \text{[CC OFF]} \\
\Phi_- & \xrightarrow{PBS} & \Phi_- & \text{[FF ON]} & \xrightarrow{PR} & -\Psi_- & \xrightarrow{BS} & \Psi_- \quad \text{[CC ON]}
\end{array}$$

which illustrates the unambiguous discrimination of the four Bell states.

Our scheme is minimal, as it involves only two measurements, and thus only four possible outcomes, which equals the number of states to be discriminated.

3. Fock filtering

The Fock Filter is schematically depicted in Fig. 2. The signal under examination is coupled to a high-Q ring cavity by a nonlinear crystal with relevant third-order susceptibility $\chi \equiv \chi^{(3)}$, which imposes cross-Kerr phase modulation. We assume that the coupling is independent of polarization, such that the evolution operator of the Kerr medium is given by

$$U_K = \exp\left\{ -ig\left(a_\parallel^\dagger a_\parallel + a_\perp^\dagger a_\perp \right) c^\dagger c \right\}, \tag{12}$$

where $g = \chi t$ is the coupling constant, the a's are the two polarization modes of the signal, and c describes the cavity mode. The required nonlinearity is relatively large and can be obtained by light pulses propagating in a coherently prepared atomic gases [14]. Eq. (12) states that, as a result of the Kerr interaction, the cavity mode is subjected to a phase-shift proportional to the number of photons passing through the arm of the interferometer. The FF is complemented by a further, tunable, phase-shift ψ, and operates with the ring cavity fed by a relatively strong coherent state $|z\rangle$, i.e. a laser beam provided by a stable source (the second port of the cavity is

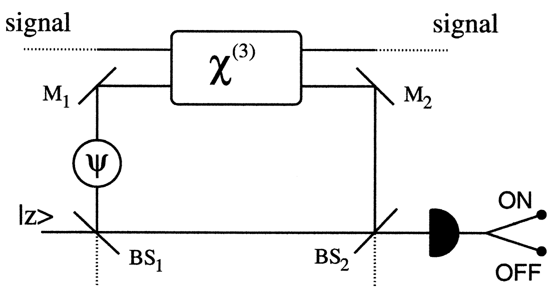

Fig. 2. Schematic diagram of the Fock Filter. The signal modes are coupled to the cavity mode by a nonlinear crystal with relevant third-order susceptibility $\chi^{(3)}$. The resulting cross-phase modulation imposes to the cavity mode a phase-shift proportional to the number of photons of the signal. The ring cavity is built by the mirrors M_1 and M_2, and by the low transmittivity beam splitters BS_1 and BS_2, whereas ψ denotes an externally tunable phase-shift. The cavity is fed by a strong coherent state $|z\rangle$, and its output is monitored by an avalanche photo-detector.

left unexcited). The input state of the whole device can be written as $|\varphi_{\rm in}\rangle = |z\rangle|0\rangle|\nu\rangle$, where $|\nu\rangle = \sum_{n_\perp n_\parallel} \nu_{n_\perp n_\parallel}|n_\perp\rangle|n_\parallel\rangle$ denotes a generic preparation of the signal mode. The output state is given by

$$|\varphi_{\rm out}\rangle = \sum_{n_\perp n_\parallel} \nu_{n_\perp n_\parallel}|\sigma_{n_\perp+n_\parallel} z\rangle|\kappa_{n_\perp+n_\parallel} z\rangle|n_\perp\rangle|n_\parallel\rangle\,, \tag{13}$$

where

$$\sigma_n = \frac{\tau}{1-[1-\tau]e^{i\phi_n}}\,, \quad \kappa_n = \frac{\sqrt{1-\tau}\,(e^{i\phi_n}-1)}{1-[1-\tau]e^{i\phi_n}}\,, \tag{14}$$

are the overall photon-number-dependent transmittivity and reflectivity of the cavity. In Eq. (14) $\phi_n = \psi - gn$, whereas τ denotes the transmittivity of the cavity beam splitters BS_1 and BS_2. For a good cavity (i.e. a cavity with large quality factor), τ should be quite small, usual values achievable in quantum optical labs are about $\tau \simeq 10^{-4}$–10^{-6} (losses, due to absorption processes, are about 10^{-7}). At the cavity output one mode is ignored, whereas the other one is monitored by an avalanche single-photon photo-detector, which checks whether or not photons are present. This kind of ON/OFF measurement is described by a two valued POM

$$\hat{\Pi}_{\sf OFF} = \sum_k (1-\eta)^k |k\rangle\langle k|\,, \quad \hat{\Pi}_{\sf ON} = \hat{1} - \hat{\Pi}_{\sf OFF}\,, \tag{15}$$

where η is the quantum efficiency of the photo-detector and the operator is understood to act on the state of the first port in Eq. (13). If the state traveling through the interferometer is either Φ_+ or Φ_- we have $|\varphi_{\rm in}\rangle = |z\rangle|0\rangle|\Phi_\pm\rangle$ and thus

$$|\varphi_{\rm out}\rangle = \frac{1}{\sqrt{2}}\left[|\sigma_1 z\rangle|\kappa_1 z\rangle|1010\rangle \pm |\sigma_1 z\rangle|\kappa_1 z\rangle|0101\rangle\right] = |\sigma_1 z\rangle|\kappa_1 z\rangle|\Phi_\pm\rangle\,. \tag{16}$$

The probability of having a click is given by

$$P({\sf ON}|\Phi_\pm) = {\rm Tr}\left\{|\varphi_{\rm out}\rangle\langle\varphi_{\rm out}|\,\hat{\Pi}_{\sf ON}\right\} = 1 - \exp\left(-\eta|\sigma_1|^2|z|^2\right), \tag{17}$$

whereas the conditional output signal state, after a click has been actually registered, turns out to be

$$\hat{\nu}_{\rm out}({\sf ON}|\Phi_\pm) = \frac{1}{P({\sf ON}|\Phi_\pm)}\,{\rm Tr}_{\rm cavity}\left\{|\varphi_{\rm out}\rangle\langle\varphi_{\rm out}|\,\hat{\Pi}_{\sf ON}\right\} = |\Phi_\pm\rangle\langle\Phi_\pm|. \tag{18}$$

By setting $\psi = g$ and for $\eta|z|^2 \gg 1$ we have $P({\sf ON}|\Phi_\pm) \simeq 1$. On the other hand, if the signal is either χ_+ or χ_- we have $|\varphi_{\rm in}\rangle = |z\rangle|0\rangle|\chi_\pm\rangle$ and

$$|\varphi_{\rm out}\rangle = \frac{1}{\sqrt{2}}\left[|\sigma_2 z\rangle|\kappa_2 z\rangle|1100\rangle \pm |\sigma_0 z\rangle|\kappa_0 z\rangle|0011\rangle\right] = \frac{1}{\sqrt{2}}\left[|\sigma z\rangle|\kappa z\rangle|1100\rangle \pm |\sigma^* z\rangle|\kappa^* z\rangle|0011\rangle\right], \tag{19}$$

where the second equality comes from the fact that by setting $\psi = g$ we have $\sigma_0^* = \sigma_2 \equiv \sigma$. Finally, from Eqs. (15) and (19) we have

$$P({\sf OFF}|\chi_\pm) = \exp\left(-\eta|\sigma|^2|z|^2\right) \tag{20}$$

$$\hat{\nu}_{\rm out}({\sf OFF}|\chi_\pm) = \tfrac{1}{2}(|1100\rangle\langle 1100| + |0011\rangle\langle 0011|) \pm \tfrac{1}{2}(\alpha|1100\rangle\langle 0011| + {\rm h.c.}), \tag{21}$$

where

$$\alpha = \exp\left(\eta|\sigma|^2|z|^2\right)\langle\sigma^* z|\hat{\Pi}_{\sf OFF}|\sigma z\rangle\langle\kappa^* z|\kappa z\rangle\,. \tag{22}$$

For small g and τ we have $\sigma \simeq i\tau/g$. Therefore, when $\tau \ll g$ we have $P(\mathsf{OFF}|\chi_{\pm}) \simeq 1$. As an explicit example, if $\eta|z|^2 = 4.6$, so that $P(\mathsf{ON}|\Phi_{\pm}) = 0.99$ in Eq. (17), then $\tau/g \simeq 0.047$ is sufficient to have also $P(\mathsf{OFF}|\chi_{\pm}) = 0.99$. This means that a click at the Fock Filter unambiguously implies that either Φ_+ or Φ_- was traveling through the interferometer, where having no click indicates the state was either χ_+ or χ_-. Furthermore, within this same limit and choosing the coherent amplitude z to be real, we also have

$$\alpha \simeq \exp\left(-2z^2\left((1-\eta)|\sigma|^2 - 2\sigma\right) \simeq \exp\left(-2iz^2\tau/g\right) \simeq 1. \tag{23}$$

Remarkably, the measurement does not destroy the coherence of the input state, merely adding an easily corrected phase in the conditional output state $\hat{\nu}_{out}(\mathsf{OFF}|\chi_{\pm})$. This occurs because the cavity states correlated with each signal state in Eq. (19) have almost unit overlap. Note also that this result is quite robust against detector inefficiency (as only the product $\eta|z|^2$ is relevant). We can conclude that for both the possible outcomes (either ON or OFF) the state after the measurement remains unaffected, and is still available for further manipulations.

4. Conclusions

In conclusion, we have described an interferometric setup to perform a complete optical Bell measurement. It consists of a Mach–Zehnder interferometer with the first beam splitter a polarizing one, and the second a normal one, and where inside the interferometer a non-demolition photon number measurement is performed by the Fock filtering technique. resulting scheme is robust against detectors inefficiency, and provides a reliable method to unambiguously discriminate among the four polarization-entangled EPR-Bell photon pairs.

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