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Quantum cellular automaton theory of light

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A B S T R A C T

We present a quantum theory of light based on the recent derivation of Weyl and Dirac quantum fields from general principles ruling the interactions of a countable set of abstract quantum systems, without using space–time and mechanics (D'Ariano and Perinotti, 2014). In a Planckian interpretation of the discreteness, the usual quantum field theory corresponds to the so-called relativistic regime of small wave-vectors. Within the present framework the photon is a composite particle made of an entangled pair of free Weyl Fermions, and the usual Bosonic statistics is recovered in the low photon density limit, whereas the Maxwell equations describe the relativistic regime. We derive the main phenomenological features of the theory in the ultra-relativistic regime, consisting in a dispersive propagation in vacuum, and in the occurrence of a small longitudinal polarization, along with a saturation effect originated by the Fermionic nature of the photon. We then discuss whether all these effects can be experimentally tested, and observe that only the dispersive effects are accessible to the current technology via observations of gamma-ray bursts.

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1. Introduction

A Quantum Cellular Automaton (QCA) describes an evolution step of a discrete set of abstract quantum systems, each one unitarily interacting with a bounded number of neighbors. Since the early work of Feynman [1], which introduced QCAs for describing many body physics and quantum field dynamics, QCAs have become increasingly popular in the theoretical physics community, starting from the

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early works [2–4], followed by mathematical formalizations [5–8], applications to quantum computation [9–12] and quantum field theory QFT [13–16], and experimental implementations [17,18].

In the recent work [19] QCAs have been involved in a formulation of QFT starting from general principles – such as homogeneity, locality, isotropy, and unitarity – ruling the interactions of a countable set of abstract quantum systems. This theory assumes no mechanics, and as such it has no space–time background, and is quantum ab initio, needing no quantization procedure. Remarkably, mechanics and Lorentz covariance emerge from the interactions between the abstract quantum systems. In the mentioned work, along with Refs. [20,21,19], we also assumed linearity of the automaton evolution, which makes the automaton equivalent to a quantum walk, and leading to the free QFT. In these papers the Weyl and Dirac field theories have been derived: the purpose of the present paper is to complete the picture by including the Maxwell field.

In this paper we will see how the electromagnetic field emerges as the relativistic regime of two Weyl QCAs of Ref. [19]. In the ultra-relativistic regime, however, the discreteness of the Planck scale manifests itself in terms of deviations from Maxwell's equations, most notably a wave-vector dependent speed of light. Such a feature has already been considered in some approaches to quantum gravity, and can be in principle experimentally detected in astrophysical observations [22–30]. In the present approach the photon is an entangled pair of non interacting massless Fermions, a scenario resembling the neutrino theory of light of De Broglie [31–36]. The latter theory has been discarded because the composite particle does not obey the exact Bosonic commutation relations [37]. However, as shown in Ref. [36], the non-Bosonic terms introduce negligible contribution at ordinary energy densities, and, as we will see in this paper, in our case the saturation effect originated by the Fermionic nature of the photon is far beyond the current laser technology.

A quantum walk leading to Maxwell's equations was constructed in Ref. [13], by a reverse-engineering technique starting from differential equations. In the present approach we start from two Weyl automata, and find Maxwell's equations as the effective evolution of special entangled pairs, in an appropriate regime. As a consequence, the present model of Maxwell field is not naturally described by a quantum walk, and as such it is very different form the proposal of Ref. [13]. In the present framework, free electrodynamics is recovered without any additional assumption as a special regime of the Weyl QCA, thus emerging from the axioms discussed above. Moreover, this approach allows us to solve the challenging issue of Bosonic statistics without assuming a Bosonic field in the first place, as would be required in the approach of Ref. [13].

In Section 2, after recalling some basic notions about the QCA, we review the Weyl automaton of Ref. [19]. In Section 3 we build a set of Fermionic bilinear operators, which in Section 4 are proved to evolve according to the Maxwell equations. In Section 5 we will show that the polarization operators introduced in Section 4 can be considered as Bosonic operators in a low energy density regime. As a spin-off of this analysis we found a result that completes the proof, given in Ref. [38], that the amount of entanglement quantifies whether pairs of Fermions can be considered as independent Bosons. Section 6 presents the phenomenological consequences of the present QCA theory, the most relevant one being the appearance of a $\mathbf{k}$-dependent speed of light. In the same section we discuss possible experimental tests of such $\mathbf{k}$-dependence in the astrophysical domain, and we compare our result with those from Quantum Gravity literature [22–30]. We conclude with Section 7 where we review the main results and discuss future developments.

2. The Weyl automaton: a review

The basic ingredient of the Maxwell automaton is Weyl's, which has been derived in Ref. [19] from first principles. Here, we will briefly review the construction for completeness.

A QCA represents the evolution of a numerable set G of cells $g \in G$, each one containing an array of Fermionic local modes. The evolution occurs in discrete identical steps, and in each one every cell interacts with the others. The Weyl automaton is derived from the following principles: unitarity, linearity, locality, homogeneity, transitivity, and isotropy. Unitarity means just that each step is a unitary evolution. Linearity means that the unitary evolution is linear in the field. Locality means that at each step every cell interacts with a finite number of others. We call cells interacting in one step *neighbors*. The neighboring notion also naturally defines a graph Γ over the automaton, with g as vertices and the neighboring couples as edges. Homogeneity means both that all steps are the same, all cells are identical systems, and the set of interactions with neighbors is the same for each cell, hence also the number of neighbors, and the dimension of the cell field array, which we will denote by $s > 0$. We will denote by A the matrix representing the linear unitary step. Transitivity means that every two cells are connected by a path of neighbors. Isotropy means that the neighboring relation is symmetric, and there exists a group of automorphisms for the graph for which the automaton itself is covariant. Homogeneity implies that set G has a group structure with the graph Γ being the *Cayley graph* of G. Let S_+ denote the set of generators of G and let S_- be the set of inverse generators. Then, for a given cell g the set of neighboring cells is given by the set $\mathcal{N}_g := \{sg | s \in S := S_+ \cup S_-\}$. Linearity, locality, and homogeneity imply that each step can be described in terms of $s \times s$ transition matrices $A_h \in \mathrm{M}(\mathbb{C}, s)$ $(h \in S)$ as follows:

$$\psi_g(t+1) = \sum_{h \in S} A_h \psi_{hg}(t) \tag{1}$$

where $\psi_g(t)$ is the s-array of field operators at g at step t. Therefore, upon denoting by $T_g, g \in G$ the right-regular unitary representation of G on $\ell^2(G)$, $T_g|f\rangle := |fg^{-1}\rangle$, for $f \in G$, A is a unitary operator on $\ell^2(G) \otimes \mathbb{C}^s$ of the form

$$A := \sum_{h \in S} T_h \otimes A_h. \tag{2}$$

Covariance of the isotropy property means precisely that the group L of automorphisms of the graph is a transitive permutation group of S_+, and there exists a (generally projective) unitary representation $U_l l \in L$ of L such that

$$A = \sum_{h \in S} T_{lh} \otimes U_l A_h U_l^\dagger, \quad \forall l \in L. \tag{3}$$

In Ref. [19] attention was restricted to groups G that are quasi-isometrically embeddable in an Euclidean space. This implies that G is *virtually Abelian* [39], namely it has an Abelian subgroup $G' \subset G$ of finite index, namely with a finite number of cosets. It can be shown the automaton is equivalent to another one with group G' and dimension s' multiple of s. We further assume that the representation of the isotropy group L induced by the embedding is orthogonal, which implies that the graph neighborhood is embedded in a sphere. We call such a property *orthogonal isotropy*.

For $s = 1$ the automaton is trivial, namely $A = I$. For $s = 2$ and for Euclidean space $\mathbb{R}^3$ one has $G = \mathbb{Z}^3$, and the Cayley graphs satisfying orthogonal isotropy are the Bravais lattices. The only lattice that has a nontrivial set of transition matrices giving a unitary automaton is the BCC lattice. We will label the group element as vectors $\mathbf{x} \in \mathbb{Z}^3$, and use the customary additive notation for the group composition, whereas the unitary representation of $\mathbb{Z}^3$ is expressed as follows

$$T_{\mathbf{z}}|\mathbf{x}\rangle = |\mathbf{z} + \mathbf{x}\rangle. \tag{4}$$

Being the group Abelian, we can Fourier transform, and the operator A can be easily block-diagonalized in the $\mathbf{k}$ representation as follows

$$A = \int_B \mathrm{d}^3\mathbf{k}\, |\mathbf{k}\rangle\langle\mathbf{k}| \otimes A_{\mathbf{k}} \tag{5}$$

with $A_{\mathbf{k}} := \sum_{\mathbf{h} \in S} e^{-i\mathbf{k}\cdot\mathbf{h}} A_{\mathbf{h}}$ unitary for every $\mathbf{k} \in B$, and the vectors $|\mathbf{k}\rangle$ given by

$$|\mathbf{k}\rangle := \frac{1}{|B|^{\frac{1}{2}}} \sum_{\mathbf{x} \in G} e^{i\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}\rangle, \tag{6}$$

are a Dirac-notation for the direct integral over $\mathbf{k}$, and the domain B is the first Brillouin zone of the BCC lattice. There are only two QCAs, with unitary matrices

$$A^{\pm}_{\mathbf{k}} := d^{\pm}_{\mathbf{k}} I + \tilde{\mathbf{n}}^{\pm}_{\mathbf{k}} \cdot \boldsymbol{\sigma} = \exp[-i\mathbf{n}^{\pm}_{\mathbf{k}} \cdot \boldsymbol{\sigma}], \tag{7}$$

where

$$
\begin{aligned}
&\tilde{\mathbf{n}}^{\pm}_{\mathbf{k}} := \begin{pmatrix} s_x c_y c_z \mp c_x s_y s_z \\ \mp c_x s_y c_z - s_x c_y s_z \\ c_x c_y s_z \mp s_x s_y c_z \end{pmatrix}, \qquad \mathbf{n}^{\pm}_{\mathbf{k}} := \frac{\lambda^{\pm}_{\mathbf{k}} \tilde{\mathbf{n}}^{\pm}_{\mathbf{k}}}{\sin \lambda^{\pm}_{\mathbf{k}}},\\
&\mathrm{d}^{\pm}_{\mathbf{k}} := (c_x c_y c_z \pm s_x s_y s_z), \qquad \lambda^{\pm}_{\mathbf{k}} := \arccos(d^{\pm}_{\mathbf{k}}),\\
&c_\alpha := \cos(k_\alpha/\sqrt{3}), \qquad s_\alpha := \sin(k_\alpha/\sqrt{3}), \qquad \alpha = x, y, z
\end{aligned}
$$

and $\boldsymbol{\sigma}$ denotes the vector of Pauli matrices. The matrices $A^{\pm}_{\mathbf{k}}$ in Eq. (7) describe the evolution of a two-component Fermionic field,

$$
\psi(\mathbf{k}, t+1) = A^{\pm}_{\mathbf{k}} \psi(\mathbf{k}, t), \qquad \psi(\mathbf{k}, t) := \begin{pmatrix} \psi_R(\mathbf{k}, t) \\ \psi_L(\mathbf{k}, t) \end{pmatrix}. \tag{8}
$$

The adimensional framework of the automaton corresponds to measuring everything in Planck units. In such a case the limit $|\mathbf{k}| \ll 1$ corresponds to the relativistic limit, where one has

$$
\mathbf{n}^{\pm}(\mathbf{k}) \sim \frac{\mathbf{k}}{\sqrt{3}}, \qquad A^{\pm}_{\mathbf{k}} \sim \exp[-i\frac{\mathbf{k}}{\sqrt{3}} \cdot \boldsymbol{\sigma}], \tag{9}
$$

corresponding to the Weyl's evolution, with $\frac{\mathbf{k}}{\sqrt{3}}$ playing the role of momentum.

3. The Maxwell automaton

In order to build the Maxwell dynamics, we need to consider two different Weyl QCAs the first one acting on a Fermionic field $\psi(\mathbf{k})$ by matrix $A_{\mathbf{k}}$ as in Eq. (8), and the second one acting on the field $\varphi(\mathbf{k})$ by the complex conjugate matrix $A^{*}_{\mathbf{k}} = \sigma_y A_{\mathbf{k}} \sigma_y$, i.e.

$$
\varphi(\mathbf{k}, t+1) = A^{*}_{\mathbf{k}} \varphi(\mathbf{k}, t), \qquad \varphi(\mathbf{k}, t) = \begin{pmatrix} \varphi_R(\mathbf{k}, t) \\ \varphi_L(\mathbf{k}, t) \end{pmatrix}. \tag{10}
$$

The matrix $A_{\mathbf{k}}$ can be either one of the Weyl matrices $A^{\pm}_{\mathbf{k}}$, and the whole derivation is independent of the choice.

The Fermionic fields φ and ψ are independent and obey the following anti-commutation relations

$$
\begin{aligned}
[\psi_i(\mathbf{k}), \psi_j(\mathbf{k}')]_+ &= [\varphi_i(\mathbf{k}), \varphi_j(\mathbf{k}')]_+ \\
&= [\varphi_i(\mathbf{k}), \psi_j(\mathbf{k}')]_+ = [\varphi_i(\mathbf{k}), \psi^{\dagger}_j(\mathbf{k}')]_+ = 0 \\
[\psi_i(\mathbf{k}), \psi^{\dagger}_j(\mathbf{k}')]_+ &= [\varphi_i(\mathbf{k}), \varphi^{\dagger}_j(\mathbf{k}')]_+ = \delta_B(\mathbf{k} - \mathbf{k}')\delta_{i,j} \\
i, j = R, L \qquad & \mathbf{k}, \mathbf{k}' \in B,
\end{aligned} \tag{11}
$$

where $\delta_B(\mathbf{k})$ is the 3d Dirac's comb delta-distribution (which repeats periodically with $\mathbb{R}^3$ tessellated into Brillouin zones).

Given now two arbitrary fields $\eta(\mathbf{k})$ and $\theta(\mathbf{k})$ we define the following bilinear function

$$
G^{\mu}_f(\eta, \theta, \mathbf{k}) := \int \frac{\mathrm{d}\,\mathbf{q}}{(2\pi)^3} f_{\mathbf{k}}(\mathbf{q}) \eta\left(\tfrac{\mathbf{k}}{2} - \mathbf{q}\right) \sigma^{\mu} \theta\left(\tfrac{\mathbf{k}}{2} + \mathbf{q}\right) \tag{12}
$$

where $\sigma^0 := I$, $\sigma^1 := \sigma^x$, $\sigma^2 := \sigma^y$, $\sigma^3 := \sigma^z$, $\eta(\mathbf{k}_1)\sigma^{\mu}\theta(\mathbf{k}_2) := \sum_{i,j} \eta_i(\mathbf{k}_1)\sigma^{\mu}_{ij}\theta_j(\mathbf{k}_2)$, and $\int \frac{\mathrm{d}\,\mathbf{q}}{(2\pi)^3} |f_{\mathbf{k}}(\mathbf{q})|^2 = 1$, $\forall \mathbf{k}$. In the following we will also treat the vector part $\boldsymbol{\sigma} := (\sigma^1, \sigma^2, \sigma^3)$ of the four-vector σ^{μ} separately. This allows us to define the following operators

$$
F^{\mu}(\mathbf{k}) := G^{\mu}_f(\varphi, \psi, \mathbf{k}). \tag{13}
$$

In the following sections we study the evolution of the bilinear functions $F^{\mu}(\mathbf{k})$ and their commutation relations and show that, in the relativistic limit and for small particle densities the quantum Maxwell equations are recovered for both choices of $A_{\mathbf{k}} = A^{\pm}_{\mathbf{k}}$.

4. The Maxwell dynamics

In the following we will use the short notations

$$[Z\eta](\mathbf{k}) := Z_{\mathbf{k}}\eta(\mathbf{k}), \qquad [ZW]_{\mathbf{k}} := Z_{\mathbf{k}}W_{\mathbf{k}}, \tag{14}$$

for η a field and Z and W matrices. If the fields ψ and φ evolve according to Eqs. (8) and (10), then the evolution of the bilinear functions $F^{\mu}(\mathbf{k})$ introduced in Eq. (13) obeys the following equation

$$F^{\mu}(\mathbf{k},t) = G^{\mu}_{f}([A^{*t}\varphi],[A^{t}\psi],\mathbf{k}), \tag{15}$$

where we used the notation in (14). Now, let us define

$$\tilde{F}^{\mu}(\mathbf{k},t) := G^{\mu}_{f}([U^{\mathbf{k},t*}\varphi],[U^{\mathbf{k},t}\psi],\mathbf{k}), \qquad U^{\mathbf{k},t}_{\mathbf{q}} := A^{-t}_{\frac{\mathbf{k}}{2}}A^{t}_{\mathbf{q}}, \tag{16}$$

where we remind that $[U^{\mathbf{k},t*}\varphi](\mathbf{q}) := U^{\mathbf{k},t*}_{\mathbf{q}}\varphi(\mathbf{q})$. Clearly, one has $[A^{t}\eta] = [A^{t}_{\frac{\mathbf{k}}{2}}U^{\mathbf{k},t}\eta]$. We now need the identity

$$\begin{aligned}
&\exp(-\tfrac{i}{2}\mathbf{v}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\exp(\tfrac{i}{2}\mathbf{v}\cdot\boldsymbol{\sigma}) = \mathrm{Exp}(-i\mathbf{v}\cdot\mathbf{J})\boldsymbol{\sigma},\\
&\exp(-\tfrac{i}{2}\mathbf{v}\cdot\boldsymbol{\sigma})\sigma^{0}\exp(\tfrac{i}{2}\mathbf{v}\cdot\boldsymbol{\sigma}) = \sigma^{0},
\end{aligned} \tag{17}$$

where the matrix $\mathrm{Exp}(-i\mathbf{v}\cdot\mathbf{J})$ acts on $\boldsymbol{\sigma}$ regarded as a vector, and $\mathbf{J} = (J_x, J_y, J_z)$ is the three dimensional representation of the generators of the group $\mathbb{SO}(3)$. We can then recast Eq. (15) in terms of the following functions

$$\mathbf{F}(\mathbf{k},t) := (F^{1}(\mathbf{k},t), F^{2}(\mathbf{k},t), F^{3}(\mathbf{k},t))^{T}, \tag{18}$$

and $\tilde{\mathbf{F}}(\mathbf{k},t)$ similarly defined, obtaining

$$\begin{aligned}
&F^{0}(\mathbf{k},t) = \tilde{F}^{0}(\mathbf{k},t),\\
&\mathbf{F}(\mathbf{k},t) = \mathrm{Exp}\left(-2i\mathbf{n}_{\frac{\mathbf{k}}{2}}\cdot\mathbf{J}t\right)\tilde{\mathbf{F}}(\mathbf{k},t).
\end{aligned} \tag{19}$$

We now assume that

$$\int_{|\mathbf{q}|\geq\bar{q}(\mathbf{k})} \frac{d\mathbf{q}}{(2\pi)^{3}}|f_{\mathbf{k}}(\mathbf{q})|^{2} \ll 1 \quad \text{for } \bar{q}(\mathbf{k}) \ll |\mathbf{k}|. \tag{20}$$

Taking the Taylor expansion of $\mathbf{n}_{\frac{\mathbf{k}}{2}+\mathbf{q}}$ with respect to $\mathbf{q}$ we can thus make the approximation

$$\begin{aligned}
U^{\mathbf{k},t}_{\frac{\mathbf{k}}{2}\pm\mathbf{q}} &\simeq \exp\left(i\mathbf{n}_{\frac{\mathbf{k}}{2}}\cdot\boldsymbol{\sigma}t\right)\exp\left[-i\left(\mathbf{n}_{\frac{\mathbf{k}}{2}}\pm\mathbf{l}_{\mathbf{k},\mathbf{q}}\right)\cdot\boldsymbol{\sigma}t\right]\\
&\simeq \exp\left(\pm ic_{\mathbf{k},\mathbf{q}}\frac{\mathbf{n}_{\frac{\mathbf{k}}{2}}}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\cdot\boldsymbol{\sigma}t\right) + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right),
\end{aligned} \tag{21}$$

where $\mathbf{l}_{\mathbf{k},\mathbf{q}} := J_{\mathbf{n}}\left(\frac{\mathbf{k}}{2}\right)\mathbf{q}$ and $J_{\mathbf{n}}\left(\frac{\mathbf{k}}{2}\right)$ denotes the Jacobian matrix of the function $\mathbf{n}_{\mathbf{k}}$ evaluated at $\frac{\mathbf{k}}{2}$ and $c_{\mathbf{k},\mathbf{q}} := \frac{\mathbf{n}_{\frac{\mathbf{k}}{2}}}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\cdot\mathbf{l}_{\mathbf{k},\mathbf{q}}$ (the proof of Eq. (21) is given in Appendix A). By introducing the transverse field operators

$$\begin{aligned}
\tilde{\mathbf{F}}_{T}(\mathbf{k},t) &:= \tilde{\mathbf{F}}(\mathbf{k},t) - \left(\frac{\mathbf{n}_{\frac{\mathbf{k}}{2}}}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\cdot\tilde{\mathbf{F}}(\mathbf{k},t)\right)\frac{\mathbf{n}_{\frac{\mathbf{k}}{2}}}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\\
\mathbf{F}_{T}(\mathbf{k},t) &:= \mathbf{F}(\mathbf{k},t) - \left(\frac{\mathbf{n}_{\frac{\mathbf{k}}{2}}}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\cdot\mathbf{F}(\mathbf{k},t)\right)\frac{\mathbf{n}_{\frac{\mathbf{k}}{2}}}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}
\end{aligned} \tag{22}$$

and using Eq. (21) into Eq. (18) we get (see Appendix B)

$$\tilde{\mathbf{F}}_T(\mathbf{k},t) = \mathbf{F}_T(\mathbf{k}) + O\Big(\frac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\Big). \tag{23}$$

Finally, combining Eq. (23) with Eq. (19) we obtain a closed expression for the time evolution of the operator $\mathbf{F}_T(\mathbf{k})$,

$$\mathbf{F}_T(\mathbf{k},t) = \exp\left[\left(2\mathbf{n}_{\frac{\mathbf{k}}{2}}\cdot\mathbf{J}\right)t\right]\mathbf{F}_T(\mathbf{k}) + \Lambda(\mathbf{k},t), \tag{24}$$

where $\|\Lambda(\mathbf{k},t)\| = O\big(\frac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\big)$. Taking the time derivative in Eq. (24) and reminding the definition (22) we obtain

$$\begin{aligned} &\partial_t\mathbf{F}_T(\mathbf{k},t) = 2\mathbf{n}_{\frac{\mathbf{k}}{2}}\times\mathbf{F}_T(\mathbf{k},t) + \partial_t\Lambda(\mathbf{k},t)\\ &2\mathbf{n}_{\frac{\mathbf{k}}{2}}\cdot\mathbf{F}_T(\mathbf{k},t) = 0, \end{aligned} \tag{25}$$

where $\|\partial_t\Lambda(\mathbf{k},t)\| = O\big(\frac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\big)$ (see Appendix B).

Let now $\mathbf{E}$ and $\mathbf{B}$ be two Hermitian operators defined by the relation

$$\begin{aligned} &\mathbf{E} := |\mathbf{n}_{\frac{\mathbf{k}}{2}}|(\mathbf{F}_T + \mathbf{F}_T^\dagger), \qquad \mathbf{B} := i|\mathbf{n}_{\frac{\mathbf{k}}{2}}|(\mathbf{F}_T^\dagger - \mathbf{F}_T),\\ &2|\mathbf{n}_{\frac{\mathbf{k}}{2}}|\mathbf{F}_T = \mathbf{E} + i\mathbf{B}. \end{aligned} \tag{26}$$

We now show that in the limit of small wavevectors $\mathbf{k}$ and by interpreting $\mathbf{E}$ and $\mathbf{B}$ as the electric and magnetic field the usual vacuum Maxwell's equations can be recovered. For $|\mathbf{k}| \ll 1$ one has $2\mathbf{n}_{\frac{\mathbf{k}}{2}} \simeq \mathbf{k}/\sqrt{3}$, and Eq. (25) becomes

$$\begin{aligned} &\partial_t\mathbf{F}_T(\mathbf{k},t) = \frac{\mathbf{k}}{\sqrt{3}}\times\mathbf{F}_T(\mathbf{k},t)\\ &\mathbf{k}\cdot\mathbf{F}_T(\mathbf{k},t) = 0. \end{aligned} \tag{27}$$

As in Ref. [19], we recover physical dimensions from the previous adimensional equations using Planck units, taking $c := l_P/t_P$, time measured in Planck times $t \to t * t_P$, and lengths measured in Planck lengths as $x \to x * \sqrt{3}l_P$, the $\sqrt{3}l_P$ corresponding to the distance between neighboring cells. Then Eq. (27) becomes

$$\begin{aligned} &\partial_t\mathbf{F}_T(\mathbf{x},t) = -ic\nabla\times\mathbf{F}_T(\mathbf{x},t)\\ &\nabla\cdot\mathbf{F}_T(\mathbf{x},t) = 0 \end{aligned} \tag{28}$$

which in terms of $\mathbf{E}$ and $\mathbf{B}$ become the vacuum Maxwell's equations

$$\begin{aligned} &\nabla\cdot\mathbf{E} = 0 &&\nabla\cdot\mathbf{B} = 0\\ &\partial_t\mathbf{E} = c\nabla\times\mathbf{B} &&\partial_t\mathbf{B} = -c\nabla\times\mathbf{E}. \end{aligned} \tag{29}$$

Introducing the polarization vectors $\mathbf{u}^1_{\mathbf{k}}$ and $\mathbf{u}^2_{\mathbf{k}}$ satisfying

$$\mathbf{u}^i_{\mathbf{k}}\cdot\mathbf{n}_{\mathbf{k}} = \mathbf{u}^1_{\mathbf{k}}\cdot\mathbf{u}^2_{\mathbf{k}} = 0, \qquad |\mathbf{u}^i_{\mathbf{k}}| = 1, \qquad (\mathbf{u}^1_{\mathbf{k}}\times\mathbf{u}^2_{\mathbf{k}})\cdot\mathbf{n}_{\mathbf{k}} > 0, \tag{30}$$

we can now interpret the following operators

$$\gamma^i(\mathbf{k}) := \mathbf{u}^i_{\mathbf{k}}\cdot\mathbf{F}(\mathbf{k},0), \quad i = 1,2, \tag{31}$$

as the two polarization operators of the field. In the light of this analysis, one can conclude that the automaton discrete evolution leads to modified Maxwell's equations in the form of Eqs. (25), with the electromagnetic field rotating around $\mathbf{n}_{\frac{\mathbf{k}}{2}}$ instead of $\mathbf{k}$. Reminding Eq. (20), we can observe that a photon with a well-defined wave-vector $\mathbf{k}$ corresponds to a state of the two constituent Fermions

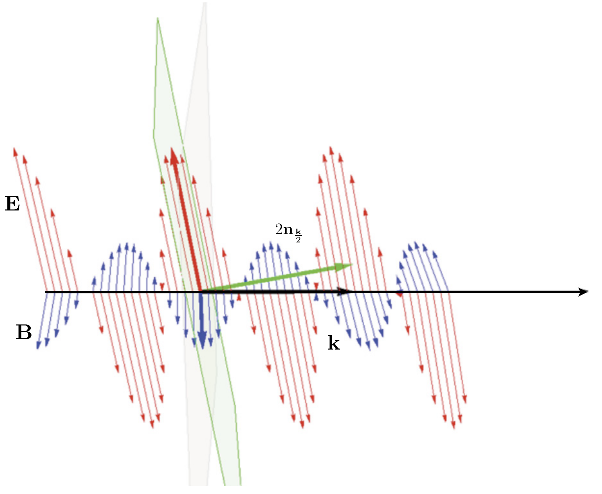

Fig. 1. A rectilinear polarized electromagnetic wave. We notice that the polarization plane (in green) is slightly tilted with respect to the plane orthogonal to $\mathbf{k}$ (in gray). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with a high wave-vector correlation, with very close wave-vectors $\frac{\mathbf{k}}{2} \pm \mathbf{q}$. Notice that correlation in our model of the photon does not require any binding interaction as in Ref. [34]. This correlation forces the wave-vector distribution of the Fermions to be very narrow, with a consequently spread position distribution. In the following we will further discuss the properties of the function $f_{\mathbf{k}}(\mathbf{q})$ requiring specific features that will allow for the correct Bosonic statistics. In particular, it is clear that also for localized photon states, we will need spread Fermionic states in order to avoid the effects of the Pauli exclusion principle on the statistics of the emergent photon. This is not in contradiction with expected particle behavior of the photon, as long as the dynamical and statistical equations coincide with the standard ones in the appropriate small wave-vector regime.

Moreover, since in this framework the photon is a composite particle, the internal dynamics of the constituent Fermions is responsible for an additional term $O\left(\frac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right)$. As a consequence of this distortion, one can immediately see that the electric and magnetic fields are no longer exactly transverse to the wave vector but we have the appearance of a longitudinal component of the polarization (see Fig. 1). In Section 6 we discuss the new phenomenology that emerges from Eqs. (25).

5. Photons as composite Bosons

In the previous section we proved that the operators defined in Eq. (26) dynamically evolve according to the free Maxwell's equation. However, in order to interpret $\mathbf{E}(\mathbf{k})$ and $\mathbf{B}(\mathbf{k})$ as the electric and magnetic fields we need to show that they obey the correct commutation relation. The aim of this paragraph is to show that, in a regime of low energy density, the polarization operators defined in Eq. (31) actually behave as independent Bosonic modes.

In order to avoid the technicalities of the continuum we now suppose to confine the system in finite volume $\mathcal{V}$. The finiteness of the volume introduces a discretization of the momentum space and the operators $\psi(\mathbf{k}), \varphi(\mathbf{k})$, obey Eq. (11) where the periodic Dirac delta is replaced by the Kronecker delta. All the integrals over the Brillouin zone are then replaced by sums, and the polarization operators of Eq. (31) become

$$\gamma^i(\mathbf{k}) := \sum_{\mathbf{q}} f_{\mathbf{k}}(\mathbf{q}) \varphi\left(\tfrac{\mathbf{k}}{2} - \mathbf{q}\right) (\mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \sigma) \psi\left(\tfrac{\mathbf{k}}{2} + \mathbf{q}\right). \tag{32}$$

These operators can be simply expressed in terms of the functions $\gamma_{\alpha,\beta}(\mathbf{k})$ defined as follows

$$\gamma_{\alpha,\beta}(\mathbf{k}) := \sum_{\mathbf{q}} f_{\mathbf{k}}(\mathbf{q})\varphi_\alpha\left(\tfrac{\mathbf{k}}{2}-\mathbf{q}\right)\psi_\beta\left(\tfrac{\mathbf{k}}{2}+\mathbf{q}\right),$$
$$\alpha,\beta = R, L. \tag{33}$$

Since the polarization operators $\gamma^i(\mathbf{k})$ are linear combinations of $\gamma_{\alpha,\beta}(\mathbf{k})$, it is useful to compute the commutation relations of the latter. We have

$$[\gamma_{\alpha,\beta}(\mathbf{k}), \gamma_{\alpha',\beta'}(\mathbf{k}')]_- = 0,$$
$$[\gamma_{\alpha,\beta}(\mathbf{k}), \gamma^\dagger_{\alpha',\beta'}(\mathbf{k}')]_- = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\mathbf{k},\mathbf{k}'} - \Delta_{\alpha,\alpha',\beta,\beta',\mathbf{k},\mathbf{k}'},$$
$$\Delta_{\alpha,\alpha',\beta,\beta',\mathbf{k},\mathbf{k}'} := \left(\delta_{\alpha,\alpha'}H^+_{\psi,\beta',\beta,\mathbf{k}',\mathbf{k}} + \delta_{\beta,\beta'}H^-_{\varphi,\alpha',\alpha,\mathbf{k}',\mathbf{k}}\right),$$
$$H^\pm_{\eta,\alpha',\alpha,\mathbf{k}',\mathbf{k}} := \sum_{\mathbf{q}} f_{\mathbf{k}}(\mathbf{q})f^*_{\mathbf{k}'}(\tfrac{\mathbf{k}'-\mathbf{k}}{2}+\mathbf{q})\eta^\dagger_{\alpha'}\left(\tfrac{2\mathbf{k}'-\mathbf{k}}{2}\pm\mathbf{q}\right)\eta_\alpha\left(\tfrac{\mathbf{k}}{2}\pm\mathbf{q}\right). \tag{34}$$

Then the operators $\gamma_{\alpha,\beta}$ fail to be Bosonic annihilation operators because of the appearance of the operator $\Delta_{\alpha,\alpha',\beta,\beta',\mathbf{k},\mathbf{k}'}$ in the commutation relation (34). However, if we restrict attention to states ρ such that $\mathrm{Tr}[\rho H^-_{\varphi,\beta',\beta,\mathbf{k}',\mathbf{k}}] \simeq 0$ and $\mathrm{Tr}[\rho H^+_{\psi,\alpha',\alpha,\mathbf{k}',\mathbf{k}}] \simeq 0$, we could make the approximation $[\gamma_{\alpha,\beta}(\mathbf{k}), \gamma^\dagger_{\alpha',\beta'}(\mathbf{k}')]_- \simeq \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\mathbf{k},\mathbf{k}'}$. Let us then consider the modulus of the expectation value of the operators $H^\pm_{\eta,\beta',\beta,\mathbf{k}',\mathbf{k}}$

$$|\langle H^\pm_{\eta,\beta',\beta,\mathbf{k}',\mathbf{k}}\rangle| \le \sum_{\mathbf{q}} |f_{\mathbf{k}}(\mathbf{q})|\left|f^*_{\mathbf{k}'}(\tfrac{\mathbf{k}'-\mathbf{k}}{2}+\mathbf{q})\right|\left|\left\langle\eta^\dagger_{\beta'}\left(\tfrac{2\mathbf{k}'-\mathbf{k}}{2}\pm\mathbf{q}\right)\eta_\beta\left(\tfrac{\mathbf{k}}{2}\pm\mathbf{q}\right)\right\rangle\right|$$
$$\le \sqrt{\langle\Gamma^\pm_{\eta,\beta,\mathbf{k}}\rangle\langle\Gamma^\pm_{\eta,\beta',\mathbf{k}'}\rangle}, \tag{35}$$

$$\Gamma^\pm_{\eta,\beta,\mathbf{k}} = \sum_{\mathbf{q}} |f_{\mathbf{k}}(\mathbf{q})|^2\,\eta^\dagger_\beta\left(\tfrac{\mathbf{k}}{2}\pm\mathbf{q}\right)\eta_\beta\left(\tfrac{\mathbf{k}}{2}\pm\mathbf{q}\right), \tag{36}$$

where we repeatedly applied the Schwartz inequality. The operators $\Gamma^-_{\varphi,\beta,\mathbf{k}}$ and $\Gamma^+_{\psi,\alpha,\mathbf{k}}$ can be interpreted as number operators "shaped" by the probability distribution $|f_{\mathbf{k}}(\mathbf{q})|^2$.

We then define the set of states

$$\mathcal{S}_\varepsilon := \{\rho \mid \mathrm{Tr}[\rho\Gamma^\pm_{\eta,\beta,\mathbf{k}}] \le \varepsilon\}. \tag{37}$$

It is easy to check that for states in $\mathcal{S}_\varepsilon$ one has $[\gamma_{\alpha,\beta}(\mathbf{k}), \gamma^\dagger_{\alpha',\beta'}(\mathbf{k}')]_- = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\mathbf{k},\mathbf{k}'} + O(\varepsilon)$, and for $\varepsilon \ll 1$ the commutators are well approximated by the Bosonic ones.

If we suppose $|f_{\mathbf{k}}(\mathbf{q})|^2$ to be a constant function over a region $\Omega_{\mathbf{k}}$ which contains $N_{\mathbf{k}}$ modes, i.e. $|f_{\mathbf{k}}(\mathbf{q})|^2 = \frac{1}{N_{\mathbf{k}}}$ if $\mathbf{q} \in \Omega_{\mathbf{k}}$ and $|f_{\mathbf{k}}(\mathbf{q})|^2 = 0$ if $\mathbf{q} \notin \Omega_{\mathbf{k}}$, we have

$$\left\langle\Gamma^+_{\psi,\alpha,\mathbf{k}}\right\rangle = \frac{1}{N_{\mathbf{k}}}\sum_{\mathbf{q}\in\Omega_{\mathbf{k}}}\left\langle\psi^\dagger_\alpha\left(\tfrac{\mathbf{k}}{2}+\mathbf{q}\right)\psi_\alpha\left(\tfrac{\mathbf{k}}{2}+\mathbf{q}\right)\right\rangle = \frac{M_{\psi,\alpha,\mathbf{k}}}{N_{\mathbf{k}}}$$

where we denoted with $M_{\psi,\alpha,\mathbf{k}}$ the number of ψ_α Fermions in the region Ω_k (clearly the same result applies to $\Gamma^-_{\varphi,\beta,\mathbf{k}}$). In this case the subset $\mathcal{S}_\varepsilon$ contains states ρ such that $M_{\xi,\chi,\mathbf{k}}/N_{\mathbf{k}} \le \varepsilon$ for all ξ_χ and $\mathbf{k}$. Then, for states in $\mathcal{S}_\varepsilon$ with $\varepsilon \ll 1$ we can safely assume $[\gamma_{\alpha,\beta}(\mathbf{k}), \gamma^\dagger_{\alpha',\beta'}(\mathbf{k}')]_- = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\mathbf{k},\mathbf{k}'}$ in Eq. (34) which after an easy calculation gives

$$[\gamma^i(\mathbf{k}), \gamma^{j\dagger}(\mathbf{k}')]_- = \delta_{i,j}\delta_{\mathbf{k},\mathbf{k}'} \quad i = 0, 1, 2, 3. \tag{38}$$

In Eq. (38), besides the previously defined transverse polarizations $\gamma^1(\mathbf{k})$ and $\gamma^2(\mathbf{k})$, we considered also the "longitudinal" polarization operator $\gamma^3(\mathbf{k}) := \sum_{\mathbf{q}} f_{\mathbf{k}}(\mathbf{q})\varphi\left(\tfrac{\mathbf{k}}{2}-\mathbf{q}\right)(\mathbf{e}_{\frac{\mathbf{k}}{2}}\cdot\boldsymbol{\sigma})\psi\left(\tfrac{\mathbf{k}}{2}+\mathbf{q}\right)$, where $\mathbf{e}_{\mathbf{k}} := \mathbf{n}_{\mathbf{k}}/|\mathbf{n}_{\mathbf{k}}|$, and the "timelike" polarization operator $\gamma^0(\mathbf{k}) := \sum_{\mathbf{q}} f_{\mathbf{k}}(\mathbf{q})\varphi\left(\tfrac{\mathbf{k}}{2}-\mathbf{q}\right)I\psi\left(\tfrac{\mathbf{k}}{2}+\mathbf{q}\right)$.

This result tells us that, as far as we restrict ourselves to states in $\mathcal{S}_\varepsilon$ we are allowed to interpret the operators $\gamma^i(\mathbf{k})$ as 4 independent Bosonic field modes and then to interpret $\mathbf{E}$ and $\mathbf{B}$ defined in Eq. (26) as the electric and the magnetic field operators. This fact together with the evolution given by Eq. (27) proves that we realized a consistent model of free quantum electrodynamics in which the photons are composite particles made by correlated Fermions whose evolution is described by a cellular automaton.

5.1. Composite Bosons and entanglement

The results that we had in this section are in agreement with the recent works [40–43] which studied the conditions under which a pair of Fermionic fields can be considered as a Boson. In Refs. [44,38] it was shown that a sufficient condition is that the two Fermionic fields ψ, ϕ are sufficiently entangled. More precisely, for a composite Boson $c := \sum_i f(i)\psi_i\phi_i$, $\sum_i |f(i)|^2 = 1$ one has

$$[c, c^\dagger] = 1 - (\Gamma_\psi + \Gamma_\phi), \tag{39}$$

where

$$\Gamma_\psi = \sum_i |f(i)|^2 \psi_i^\dagger \psi_i, \qquad \Gamma_\phi = \sum_i |f(i)|^2 \phi_i^\dagger \phi_i, \tag{40}$$

and in Ref. [38] it was shown that the following bound holds

$$\forall N \geq 1, \quad NP \geq \langle N|\Gamma_\psi|N\rangle \geq P, \tag{41}$$

and the same holds for Γ_ϕ, where $P = \sum_{i=1}^N |f(i)|^4$ is the purity of the reduced state of a single particle and $|N\rangle = \frac{1}{\sqrt{N!}}\chi_N(c^\dagger)^N|0\rangle$ (χ_N is a normalization constant). From this result, the authors of Ref. [38] concluded that, as far as $P, NP \approx 0$, c and $c^\dagger$ can be safely considered as a Bosonic annihilation/creation pair. Our criterion of Eq. (37), applied to states $\rho = |N\rangle\langle N|$ reduces to the criterion in Refs. [44,38]. Moreover it is interesting to show that the technique applied in the derivation of Eq. (35) can be used to answer an open question raised in Ref. [38]. There the authors conjecture that, given two different composite Bosons $c_1 = \sum_i f_1(i)\psi_i\phi_i$ and $c_2 = \sum_i f_2(i)\psi_i\phi_i$ such that $\sum_i f_1(i)f_2(i)^* = 0$, the commutation relation $[c_1, c_2^\dagger]$ should vanish as the two purities P_1 and P_2 ($P_a = \sum_{i=1}^N |f_a(i)|^4$) decrease. Since $[c_1, c_2^\dagger] = -\sum_i f_1(i)f_2(i)^*(\psi_i^\dagger\psi_i + \phi_i^\dagger\phi_i)$ we have

$$|\langle[c_1, c_2^\dagger]\rangle| \leq \sum_x \sqrt{\langle\Gamma_x^{(1)}\rangle\langle\Gamma_x^{(2)}\rangle}, \tag{42}$$

by the same reasoning that we followed in the derivation of Eq. (35). Combining this last inequality with the condition $\langle N|\Gamma_x^{(i)}|N\rangle \leq NP$ we have $|\langle N|[c_1, c_2^\dagger]|N\rangle| \leq 2NP$ which proves the conjecture.

6. Phenomenological analysis

We now investigate the new phenomenology predicted from the modified Maxwell equations (25) and the modified commutation relations (34), with a particular focus on practically testable effects. A natural assumption in the following discussion is to set the discrete scales of the QCA to the corresponding Planck units, e.g. taking the lattice step of the order of the Planck length.

Let us first have a closer look at the dynamics described by Eq. (24). If $\mathbf{u}_+$ and $\mathbf{u}_-$ are the two eigenvectors of the matrix $\mathrm{Exp}[(2\mathbf{n}_{\frac{\mathbf{k}}{2}} \cdot \mathbf{J})t]$, corresponding to eigenvalues $e^{\mp i2|\mathbf{n}_{\frac{\mathbf{k}}{2}}|t}$, Eq. (24) can be written as

$$\mathbf{F}_T(\mathbf{k}, t) = e^{-i2|\mathbf{n}_{\frac{\mathbf{k}}{2}}|t}\gamma_+(\mathbf{k})\mathbf{u}_+ + e^{i2|\mathbf{n}_{\frac{\mathbf{k}}{2}}|t}\gamma_-(\mathbf{k})\mathbf{u}_- \tag{43}$$

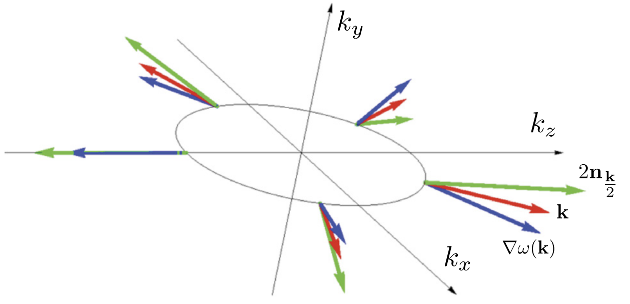

Fig. 2. The graphics shows the vector $2\mathbf{n}_{\frac{\mathbf{k}}{2}}$ (in green), which is orthogonal to the polarization plane, the wavevector $\mathbf{k}$ (in red) and the group velocity $\nabla\omega(\mathbf{k})$ (in blue) as function of $\mathbf{k}$ for the value $|\mathbf{k}| = 0.8$ and different directions. Notice that the three vectors are not parallel and the angles between them depend on $\mathbf{k}$. Such anisotropic behavior can be traced back to the anisotropy of the dispersion relation of the Weyl automaton. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where the corresponding polarization operators $\gamma_\pm(\mathbf{k})$ are defined according to Eq. (31). According to Eq. (43) the angular frequency of the electromagnetic waves is given by the modified dispersion relation

$$\omega(\mathbf{k}) = 2|\mathbf{n}_{\frac{\mathbf{k}}{2}}|. \tag{44}$$

The usual relation $\omega(\mathbf{k}) = |\mathbf{k}|$ is recovered in the $|\mathbf{k}| \ll 1$ regime. The speed of light is the group velocity of the electromagnetic waves, i.e. the gradient of the dispersion relation. The major consequence of Eq. (44) is that the speed of light depends on the value of $\mathbf{k}$, as for Maxwell's equations in a dispersive medium.

The phenomenon of a $\mathbf{k}$-dependent speed of light was already analyzed in the context of quantum gravity where many authors considered the hypothesis that the existence of an invariant length (the Planck scale) could manifest itself in terms of modified dispersion relations [22–24,26,28]. In these models the $\mathbf{k}$-dependent speed of light $c(\mathbf{k})$, at the leading order in $k := |\mathbf{k}|$, is expanded as $c(\mathbf{k}) \approx 1 \pm \xi k^\alpha$, where ξ is a numerical factor of order 1, while α is an integer. This is exactly what happens in our framework, where the intrinsic discreteness of the quantum cellular automata $A^\pm$ leads to the dispersion relation of Eq. (44) from which the following $\mathbf{k}$-dependent speed of light

$$c^\mp(\mathbf{k}) \approx 1 \pm 3\frac{k_x k_y k_z}{|\mathbf{k}|^2} \approx 1 \pm \tfrac{1}{\sqrt{3}}k, \tag{45}$$

can be obtained by computing the modulus of the group velocity and power expanding in $\mathbf{k}$ with the assumption $k_x = k_y = k_z = \frac{1}{\sqrt{3}}k$, ($k = |\mathbf{k}|$). It is interesting to observe that depending on the automaton $A^+(\mathbf{k})$ of $A^-(\mathbf{k})$ in Eq. (7) we obtain corrections to the speed of light with opposite sign. Moreover the correction is not isotropic and can be superluminal, though uniformly bounded for all $\mathbf{k}$ as shown for the Weyl automaton in Ref. [19].

Models leading to modified dispersion relations recently received attention because they allow one to derive falsifiable predictions of the Planck scale hypothesis. These can be experimentally tested in the astrophysical domain, where the tiny corrections to the usual relativistic dynamics can be magnified by the huge time of flight. For example, observations of the arrival times of pulses originated at cosmological distances, like in some γ-ray bursts [25,45–47], are now approaching a sufficient sensitivity to detect corrections to the relativistic dispersion relation of the same order as in Eq. (45).

A second distinguishing feature of Eq. (25) is that the polarization plane is neither orthogonal to the wavevector, nor to the group velocity, which means that the electromagnetic waves are no longer exactly transverse (see Figs. 1 and 2). However the angle θ between the polarization plane and the plane orthogonal to $\mathbf{k}$ or $\nabla\omega(\mathbf{k})$ is of the order $\theta \approx 2k$, which gives 10^{-15}rad for a γ-ray wavelength, a precision which is not reachable by the present technology. Since for a fixed $\mathbf{k}$ the polarization plane is constant, exploiting greater distances and longer times does not help in magnifying this deviation from standard electrodynamics.

Finally, the third phenomenological consequence of our modeling is that, since the photon is described as a composite Boson, deviations from the usual Bosonic statistics are in order. As we proved in Section 5, the choice of the function $f_{\mathbf{k}}(\mathbf{q})$ determines the regime where the composite photon can be approximately treated as a Boson. However, independent of the details of function $f_{\mathbf{k}}(\mathbf{q})$ one can easily see that a Fermionic saturation of the Boson is not visible, e.g. for the most powerful laser [48] one has approximately an Avogadro number of photons in 10^{-15} cm^3, whereas in the same volume on has around 10^{90} Fermionic modes.

Another test for the composite nature of photons is provided by the prediction of deviations from the Planck's distribution in Blackbody radiation experiments. A similar analysis was carried out in Ref. [36], where the author showed that the predicted deviation from Planck's law is less than one part over 10^{-8}, well beyond the sensitivity of present day experiments.

7. Conclusions

In this paper we complete the derivation from principles of the free quantum field theory initiated in [19] for the Weyl and Dirac fields, deriving the quantum automaton theory of the Maxwell field. Within the present framework the electromagnetic field emerges from two entangled free massless Fermionic fields whose evolution is given by the Weyl automaton. Then the electric and magnetic fields are described in terms of bilinear operators of the two constituent Fermionic fields. The automaton evolution leads to a set of modified Maxwell's equations whose dynamics differs from the usual one for ultra-high wavevectors. This model predicts a longitudinal component of the polarization and a $\mathbf{k}$-dependent speed of light. This last effect could be observed by measuring the arrival times of light originated at cosmological distances, like in some γ-ray bursts, exploiting the huge distance scale to magnify the tiny corrective terms to the relativistic kinematics. This prediction agrees with the one presented in Ref. [25] where γ-ray bursts were for the first time considered as tests for physical models with non-Lorentzian dispersion relations. Within this perspective, our quantum cellular automaton singles out a specific modified dispersion relation as emergent from a Planck-scale microscopic dynamics.

Another major feature of the proposed model, is the composite nature of the photon which leads to a modification of the Bosonic commutation relations. Because of the Fermionic structure of the photon we expect that the Pauli exclusion principle could cause saturation effects when a critical energy density is achieved. However, an order of magnitude estimation shows that the effect is very far from being detectable with the current laser technology.

As a spin-off of the analysis of the composite nature of the photons, we proved a result that strengthens the thesis that the amount of entanglement quantifies whether a pair of Fermions can be treated as a Boson [44,38]. Indeed we showed that, even in the case of several composite Bosons, the amount of entanglement for each pair is a good measure of how much the different pair of Fermions can be treated as independent Bosons. This question was proposed as an open problem in Ref. [38].

The results of this work leave room for future investigation. The major question is the study of how symmetry transformations can be represented in the model. The scenario we considered is restricted to a fixed reference frame and in order to properly recover the standard theory we should discuss how the Poincarè group acts on our physical model. This analysis could be done following the lines of Ref. [49] where it is shown how a QCA dynamical model is compatible with a deformed relativity model [50,28] which exhibits a non-linear action of the Poincarè group.

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Appendix A. Proof of Eq. (21)

Given two vectors $\mathbf{a}, \mathbf{a}' \in \mathbb{R}^3$, we define

$$\begin{aligned} &U = R_{-\boldsymbol{a}} R_{\mathbf{a}+\mathbf{a}'} \\ &R_{-\boldsymbol{a}} = \exp(i\mathbf{a}\cdot\boldsymbol{\sigma}\, t) \\ &R_{\mathbf{a}+\mathbf{a}'} \exp(-i\left(\mathbf{a}+\mathbf{a}'\right)\cdot\boldsymbol{\sigma}\, t). \end{aligned} \tag{A.1}$$

By explicit computation $R_{\mathbf{a}+\mathbf{a}'}$ can be written as

$$\begin{aligned} R_{\mathbf{a}+\mathbf{a}'} &= \exp(-i(\mathbf{a}+\mathbf{a}')\cdot\boldsymbol{\sigma}\, t) \\ &= \exp(-it|\mathbf{a}+\mathbf{a}'|\mathbf{e}_{\mathbf{a}+\mathbf{a}'}\cdot\boldsymbol{\sigma}) \\ &= I\cos(|\mathbf{a}+\mathbf{a}'|t) - i\sin\left(|\left(\mathbf{a}+\mathbf{a}'\right)|t\right)\mathbf{e}_{\mathbf{a}+\mathbf{a}'}\cdot\boldsymbol{\sigma} \end{aligned} \tag{A.2}$$

where we introduced $\mathbf{e}_{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$ and $\mathbf{e}_{\mathbf{a}+\mathbf{a}'} = \frac{\mathbf{a}+\mathbf{a}'}{|\mathbf{a}+\mathbf{a}'|}$. For $|\mathbf{a}'| \ll |\mathbf{a}|$ we have

$$\begin{aligned} |\mathbf{a}+\mathbf{a}'| &= \sqrt{|\mathbf{a}|^2+|\mathbf{a}'|^2+2\mathbf{a}\cdot\mathbf{a}'} \\ &= |\mathbf{a}| + \frac{\mathbf{a}\cdot\mathbf{a}'}{|\mathbf{a}|^2} + O\left(\frac{|\mathbf{a}'|^2}{|\mathbf{a}|^2}\right) \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} |\mathbf{e}_{\mathbf{a}+\mathbf{a}'} - \mathbf{e}_{\mathbf{a}}| &= \left|\frac{\mathbf{a}+\mathbf{a}'}{|\mathbf{a}+\mathbf{a}'|} - \frac{\mathbf{a}}{|\mathbf{a}|}\right| = \left|\frac{\mathbf{a}(-|\mathbf{a}+\mathbf{a}'|+|\mathbf{a}|)+\mathbf{a}'|\mathbf{a}|}{|\mathbf{a}+\mathbf{a}'||\mathbf{a}|}\right| \\ &\leq \frac{|\mathbf{a}'|}{|\mathbf{a}+\mathbf{a}'|} + 1 - \frac{|\mathbf{a}|}{|\mathbf{a}+\mathbf{a}'|} = O\left(\frac{|\mathbf{a}'|}{|\mathbf{a}|}\right). \end{aligned} \tag{A.4}$$

Then, for $|\mathbf{a}'| \ll |\mathbf{a}|$ we obtain

$$\begin{aligned} R_{\mathbf{a}+\mathbf{a}'} &= I\cos\left(\left(|\mathbf{a}| + \frac{\mathbf{a}\cdot\mathbf{a}'}{|\mathbf{a}|}\right)t\right) + -i\sin\left(\left(|\mathbf{a}| + \frac{\mathbf{a}\cdot\mathbf{a}'}{|\mathbf{a}|}\right)t\right)\mathbf{e}_{\mathbf{a}}\cdot\boldsymbol{\sigma} + \Lambda'(\mathbf{a},\mathbf{a}') + \Theta'(\mathbf{a},\mathbf{a}') \\ &= \exp\left(-it\left(|\mathbf{a}| + \frac{\mathbf{a}\cdot\mathbf{a}'}{|\mathbf{a}|}\right)\mathbf{e}_{\mathbf{a}}\cdot\boldsymbol{\sigma}\right) + \Lambda'(\mathbf{a},\mathbf{a}') + \Theta'(\mathbf{a},\mathbf{a}',t) \end{aligned}$$

where $\Lambda'(\mathbf{a},\mathbf{a}')$ + $\Theta'(\mathbf{a},\mathbf{a}',t)$ are a couple of operators such that

$$|\Lambda'(\mathbf{a},\mathbf{a}')| = O\left(\frac{|\mathbf{a}'|}{|\mathbf{a}|}\right), \qquad |\Theta'(\mathbf{a},\mathbf{a}')| = O\left(\frac{|\mathbf{a}'|^2}{|\mathbf{a}|^2}t\right)$$

from which we finally get

$$\begin{aligned} &U = \exp\left(-it\frac{\mathbf{a}\cdot\mathbf{a}'}{|\mathbf{a}|}\mathbf{e}_{\mathbf{a}}\cdot\boldsymbol{\sigma}\right) + \Lambda(\mathbf{a},\mathbf{a}') + \Theta(\mathbf{a},\mathbf{a}',t) \\ &|\Lambda(\mathbf{a},\mathbf{a}')| = O\left(\frac{|\mathbf{a}'|}{|\mathbf{a}|}\right), \qquad |\Theta(\mathbf{a},\mathbf{a}')| = O\left(\frac{|\mathbf{a}'|^2}{|\mathbf{a}|^2}t\right) \end{aligned} \tag{A.5}$$

which leads to Eq. (21) if we identify $\mathbf{a} = \mathbf{n}_{\mathbf{k}}$, $\mathbf{a}' = \mathbf{l}_{\mathbf{k},\mathbf{q}}$.

Appendix B. Proof of Eq. (22)

Let us introduce the vectors $\mathbf{u}^1_{\mathbf{k}}, \mathbf{u}^2_{\mathbf{k}} \in \mathbb{R}^3$ such that

$$\mathbf{u}^1_{\mathbf{k}}\cdot\mathbf{n}_{\mathbf{k}} = 0 \qquad \mathbf{u}^2_{\mathbf{k}} := \mathbf{e}_{\mathbf{k}} \times \mathbf{u}^1_{\mathbf{k}} \qquad \mathbf{e}_{\mathbf{k}} := |\mathbf{n}_{\frac{\mathbf{k}}{2}}|^{-1}\mathbf{n}_{\frac{\mathbf{k}}{2}}. \tag{B.1}$$

The transverse field $\tilde{\mathbf{F}}_T(\mathbf{k},t)$ defined in Eq. (22) can then be written in the basis $\{\mathbf{u}^i_{\mathbf{k}}\}$ as

$$\tilde{\mathbf{F}}_T(\mathbf{k},t) = \begin{pmatrix} \mathbf{u}^1_{\frac{\mathbf{k}}{2}}\cdot\tilde{\mathbf{F}}^{\mathbf{u}_1}(\mathbf{k},t) \\ \mathbf{u}^2_{\frac{\mathbf{k}}{2}}\cdot\tilde{\mathbf{F}}^{\mathbf{u}_2}(\mathbf{k},t) \end{pmatrix}. \tag{B.2}$$

Reminding the definition (18) we have

$$\mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \tilde{\mathbf{F}}(\mathbf{k}, t) = \int \frac{\mathrm{d}\,\mathbf{q}}{(2\pi)^3} f_{\mathbf{k}}(\mathbf{q}) \varphi\left(\tfrac{\mathbf{k}}{2} - \mathbf{q}\right) Q^i \psi\left(\tfrac{\mathbf{k}}{2} + \mathbf{q}\right)$$
$$Q^i(\mathbf{k}, \mathbf{q}, t) := (U^{\mathbf{k},t}_{\frac{\mathbf{k}}{2}-\mathbf{q}})^\dagger \mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \boldsymbol{\sigma} U^{\mathbf{k},t}_{\frac{\mathbf{k}}{2}+\mathbf{q}}. \tag{B.3}$$

If we insert Eq. (21), which can be written as

$$U^{\mathbf{k},t}_{\frac{\mathbf{k}}{2}\pm\mathbf{q}} = R_{\pm\xi\mathbf{e}} + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right) \tag{B.4}$$

$$R_{\pm\xi\mathbf{e}} := \exp(\pm i\xi\mathbf{e} \cdot \boldsymbol{\sigma}) \quad \xi := c_{\mathbf{k},\mathbf{q}} t, \tag{B.5}$$

inside Eq. (B.3) we have

$$\begin{aligned} Q^i(\mathbf{k}, \mathbf{q}, t) &= R_{-\xi\mathbf{e}} \mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \boldsymbol{\sigma} R_{-\xi\mathbf{e}} + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right) \\ &= \mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \boldsymbol{\sigma} + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right) = Q^i(\mathbf{k}, \mathbf{q}, 0) + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right), \end{aligned} \tag{B.6}$$

where we used the identity

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma})(\mathbf{a} \cdot \boldsymbol{\sigma}) = -\mathbf{b} \cdot \boldsymbol{\sigma} \tag{B.7}$$

holding for $\mathbf{a} \cdot \mathbf{b} = 0$, $|\mathbf{a}| = |\mathbf{b}| = 1$, which implies

$$\exp(i\xi\mathbf{e} \cdot \boldsymbol{\sigma}) \mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \boldsymbol{\sigma} \exp(i\xi\mathbf{e} \cdot \boldsymbol{\sigma}) = \mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \boldsymbol{\sigma} \quad \forall \xi \in \mathbb{R}. \tag{B.8}$$

Inserting Eq. (B.6) in Eq. (B.3) we have

$$\mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \tilde{\mathbf{F}}(\mathbf{k}, t) = \mathbf{u}^i_{\frac{\mathbf{k}}{2}} \cdot \tilde{\mathbf{F}}(\mathbf{k}, 0) + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right) \quad i = 1, 2$$

which then implies

$$\tilde{\mathbf{F}}_T(\mathbf{k}, t) = \tilde{\mathbf{F}}_T(\mathbf{k}, 0) + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right) = \mathbf{F}_T(\mathbf{k}) + O\left(\tfrac{\bar{q}(\mathbf{k})}{|\mathbf{n}_{\frac{\mathbf{k}}{2}}|}\right).$$

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