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Quantum walks, deformed relativity and Hopf algebra symmetries

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We show how the Weyl quantum walk derived from principles in D'Ariano & Perinotti (D'Ariano & Perinotti 2014 *Phys. Rev. A* **90**, 062106. (doi:10.1103/PhysRevA.90.062106)), enjoying a nonlinear Lorentz symmetry of dynamics, allows one to introduce Hopf algebras for position and momentum of the emerging particle. We focus on two special models of Hopf algebras–the usual Poincaré and the κ-Poincaré algebras.

1. Introduction

Quantum walks (QWs) [1–5] and more generally quantum cellular automata (QCA) [6–8] have been recently considered not only as a tool for quantum simulation of fields [9–11], but also for the foundation of quantum field theory [12–20]. The QCA framework appears as the natural candidate for the extension of the informational paradigm, which has been crucial in the understanding of foundations of quantum theory [21–27]), to the foundation of quantum field theory.

The free theory has been derived starting from a denumerable set of elementary quantum systems in interaction along with the general assumptions of homogeneity, locality, isotropy and linearity of the interactions [17,18]. The whole framework does not require Lorentz covariance, which results as a subgroup of the dynamical symmetries of the quantum walk/ automaton in the limit of small wavevectors [28,29]. For general wavevectors, the Lorentz transformations are nonlinear, thus realizing a model of doubly special relativity (DSR) [30–32].

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In this paper, we consider the simplest case of the mentioned quantum walk field theory derived from principles, namely the one-particle sector of the free Weyl automaton of reference [17]. We show how the dynamics of this walk enjoys a nonlinear Lorentz symmetry, which allows us to introduce Hopf algebras [33–35] for position and momentum of the quantum walk particle, generalizing the role of the Lie algebra of symmetries. We focus on two special models of Hopf algebras: the usual Poincaré and the κ-Poincaré algebras [36].

After reviewing the derivation of the Weyl quantum walk in §2 along with its symmetries, in §3, we analyse the nonlinear relativity symmetry, within the context of Hopf algebras—the canonical framework in which deformed relativity models are studied [32,37,38]. We expound an analysis, closely related to the one in reference [39], where we study how our non linear deformation of the Lorentz group affects the Hopf algebraic construction of space–time and phase space. We consider the nonlinear deformation in the two alternative scenarios: the usual Poincaré and the κ-Poincaré cases. We will see that the construction of space–time as the dual space to the algebra of translations is left unaffected by any nonlinear deformation that recovers the linear Lorentz transformations at the leading order. Whether we obtain the usual space–time or a non-commutative version is a feature that is independent on the nonlinear transformation that we apply to the momentum operators. This is a slight generalization of the result of reference [39] where only the nonlinear deformations that leave the rotation sector undeformed were considered. On the other hand, we see that the construction of the phase space as the left cross-product algebra between momentum space and space–time, does depend on the nonlinear deformation. We then derive the set of deformed Heisenberg commutation relations emerging in our framework both in the usual Poincaré and in the κ-Poincaré cases. Deformed Heisenberg commutation relations are an ubiquitous feature of quantum gravity models, they were first observed in the context of string theory [40,41], then studied on their own right by many authors [42–46], and recently considered for experimental verification [47].

2. Quantum walk and relativity

A quantum walk describes the discrete time evolution of particle on a discrete set Γ. The Hilbert space of the system is $\mathcal{H} := \ell^2(\Gamma) \otimes \mathbb{C}^s$, where $\ell^2(\Gamma)$ is the Hilbert space of square summable function over Γ and $\mathbb{C}^s$ is the Hilbert space corresponding to the internal degrees of freedom of the evolving particle. We introduce the orthonormal basis $\{|g\rangle\}$ of $\ell^2(\Gamma)$. The physical interpretation is straightforward: the state $|g\rangle \otimes |\psi\rangle$ correspond to a particle which is localized in g with internal state $|\psi\rangle$. The dynamics is described by a unitary operator A ($A^\dagger A = AA^\dagger = I$) on $\mathcal{H}$. As shown in reference [17], the requirements of *homogeneity* and *locality* of the dynamics imply that the set Γ is endowed with a graph structure corresponding to the Cayley graph of a group G[1]. The generators of G are represented by a translation operator T_h acting on $\ell^2(\Gamma)$ as follows: $T_\mathrm{h}|g\rangle = |gh^{-1}\rangle$ (T is the right regular representation of G). Then, the homogeneity and locality assumption imply that the unitary operator corresponding to the quantum walk A can be decomposed as follows:

$$A = \sum_{h \in \mathrm{S}} T_\mathrm{h} \otimes A_h, \tag{2.1}$$

where S is the set of generators and A_h are operators on $\mathbb{C}^s$.

Given a Cayley graph Γ and a fixed dimension s for the Hilbert space of the internal degrees of freedom, the existence (or not) of a quantum walk on it is a highly non-trivial problem. In reference [17], some authors of the present manuscript addressed the case in which Γ is the Cayley graph of the Abelian group $\mathbb{Z}^3$ and the dimension of the internal degree of freedom is $s = 2$. Moreover, they assumed the quantum walk to be *isotropic*, a condition that translates the idea that all the directions on the lattice are equivalent. In mathematical terms, there must exist a unitary representation U over $\mathbb{C}^2$ of a group L of graph automorphisms, transitive over a set of direct

[1]For the reader's convenience, we remind the definition of Cayley graph. Let G be a group and S be a generating set of G. The Cayley graph $\Gamma = \Gamma(G, S)$ is a coloured directed graph such that (i) each element of G corresponds to a vertex, (ii) each generator S is assigned a colour c_s and (iii) for any $g \in G$, $s \in S$, g and gs are joined by a directed edge of colour c_s.

generators,[2] such that one has $\sum_{h\in S} T_h \otimes A_h = \sum_{l(h)\in S} T_{l(h)} \otimes U_l A_h U_l^\dagger$ for all $l \in L$. Under these assumptions, there is only one admissible Cayley graph of $\mathbb{Z}^3$, which is the one corresponding to the body-centred cubic lattice, and there are only two admissible quantum walks over it (up to a local change of basis). The analytical expression of these quantum walks is easily given in the Fourier transform basis $|\boldsymbol{k}\rangle = (2\pi)^{-3/2} \sum_{\boldsymbol{x}\in\mathbb{Z}^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} |\boldsymbol{x}\rangle$ (where $\boldsymbol{x}$ clearly denotes an element in $\mathbb{Z}^3$)

$$\left.\begin{aligned}
A^{\pm} &:= \int_{\mathsf{B}} d\boldsymbol{k} |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes A^{\pm}_{\boldsymbol{k}} \\
A^{\pm}_{\boldsymbol{k}} &:= (2\pi)^{-3/2} \sum_{\boldsymbol{y}\in S} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} A^{\pm}_{\boldsymbol{y}} \\
A^{\pm}_{\boldsymbol{k}} &:= \lambda^{\pm}(\boldsymbol{k}) I - i\boldsymbol{n}^{\pm}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}^{\pm} \\
\boldsymbol{n}^{\pm}(\boldsymbol{k}) &:= \begin{pmatrix} s_x c_y c_z \pm c_x s_y s_z \\ c_x s_y c_z \mp s_x c_y s_z \\ c_x c_y s_z \pm s_x s_y c_z \end{pmatrix}, \\
\lambda^{\pm}(\boldsymbol{k}) &:= (c_x c_y c_z \mp s_x s_y s_z) \\
\text{and} \qquad c_\alpha &:= \cos\left(\frac{k_\alpha}{\sqrt{3}}\right), \quad s_\alpha := \sin\left(\frac{k_\alpha}{\sqrt{3}}\right), \quad \alpha = x, y, z.
\end{aligned}\right\} \tag{2.2}$$

where B denotes the Brillouin zone of the body-centred cubic lattice and $\boldsymbol{\sigma}^{+} = \boldsymbol{\sigma}$ denote a vector of the usual Pauli matrices, whereas $\boldsymbol{\sigma}^{-} = \boldsymbol{\sigma}^{T}$ denotes the transposed ones. The unitary constraint implies that $A^{\pm}_{\boldsymbol{k}}$ is unitary for every $\boldsymbol{k} \in \mathsf{B}$. Note that owing to the discreteness of the lattice the quantum walk is band-limited in $\boldsymbol{k}$. The quantum walk dynamics is determined by the solutions of the eigenvalue equation $(A^{\pm} - e^{i\omega})|\psi\rangle = 0$ that is equivalent to

$$(\sin\omega I - \boldsymbol{n}^{\pm}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}^{\pm})\psi(\boldsymbol{k}, \omega) = 0, \tag{2.3}$$

which also implies the identity

$$\sin^2\omega - |\boldsymbol{n}^{\pm}(\boldsymbol{k})|^2 = 0, \tag{2.4}$$

which defines the dispersion relation of the automaton. It is easy to check that, by taking in the limit $\boldsymbol{k} \to \boldsymbol{k}_0 = (0, 0, 0)$ in equation (2.3), the quantum walk A^{+} (resp A^{-}) recovers the dynamics of the right-handed (resp left-handed) Weyl equation. Clearly, taking the same limit in equation (2.4) gives the usual relativistic dispersion relation $\omega^2 - |\boldsymbol{k}|^2 = 0$. We note that the same behaviour occurs in the limit $\boldsymbol{k} \to \boldsymbol{k}_2 = \left(\sqrt{3}\pi/2\right)(-1, -1, -1)$ and in the limits $\boldsymbol{k} \to \boldsymbol{k}_1 = \left(\sqrt{3}\pi/2\right)(1, 1, 1)$, $\boldsymbol{k} \to \boldsymbol{k}_3 = \sqrt{3}\pi(1, 0, 0)$ with the chirality exchanged. Because of this reason, we refer to the quantum walks in equation (2.2) as *Weyl walks*. It is a remarkable result that a Lorentz invariant dynamics is recovered from a dynamical model which follows from the only assumptions of homogeneity, locality and isotropy, without the relativity principle.

In the following, we will consider only the A^{+} Weyl walk and we will drop the $\pm$ apex in order to simplify the notation. The entire analysis can be straightforwardly applied to the A^{-} case.

In the quantum walk, framework space and time are not on an equal footing: space is given by the lattice structure, whereas time comes from the discrete steps of the evolution. It is then far from obvious whether and how it is possible to recover changes of space–time coordinates that mix space and time, like boosts in special relativity. This question was recently addressed and answered in reference [29] where the notion of change of observer for quantum walks was defined as an invertible map $\mathcal{L}_\beta$ over $[-\pi, \pi] \times \mathsf{B}$, as follows:

$$(\omega, \boldsymbol{k}) \to (\omega', \boldsymbol{k}') = \mathcal{L}_\beta(\omega, \boldsymbol{k}), \tag{2.5}$$

where the parameter β labels different changes of reference-frame. The idea is not to focus on the discrete lattice coordinates and the discrete time step, but rather to consider $(\omega, \boldsymbol{k})$–which

[2] The homogeneity assumption guarantees that the set S of generators can be split into disjoint subsets $S_+ \cup e \cup S_-$, where S_- is the set of inverses of S_+ and e is the identity element.

are constants of motion of the quantum walk—as the fundamental variables. In this setting, a symmetry of the dynamics is defined as follows.

Definition 2.1. Let A be a quantum walk on $\mathbb{Z}^3$. A *symmetry of the dynamics* for A is a triple $(\mathcal{L}_\beta, \Gamma_\beta, \tilde{\Gamma}_\beta)$, with $\mathcal{L}_\beta$ defined in equation (2.5) and Γ_β, $\tilde{\Gamma}_\beta$ invertible matrix functions of $(\omega, \boldsymbol{k})$, such that

$$(\sin\omega I - \boldsymbol{n}(\boldsymbol{k})\cdot\boldsymbol{\sigma}) = \tilde{\Gamma}_\beta^{-1}(\sin\omega' I - \boldsymbol{n}(\boldsymbol{k}')\cdot\boldsymbol{\sigma})\Gamma_\beta. \tag{2.6}$$

The set of symmetries $\mathbf{S}^A$ is a group which we refer to as the *symmetry group of the quantum walk A*.

The next step is then to explore whether the symmetry group of the Weyl walk A contains a representation of the Lorentz group which recovers the usual one in the regime in which the walk approaches the Weyl equation (i.e. near $\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2$, and $\boldsymbol{k}_3$). In other words, we are asking whether there exists a *deformed relativity model* which preserves the dynamics of the Weyl walk A.

Deformed (or doubly) special relativity is a theoretical proposal in which one modifies the linear Lorentz transformations in order to have an invariant energy scale in addition to the speed of light. Such a theory has been proposed by Amelino-Camelia [30] and developed by other authors [31] as a kinematic structure which may underlie quantum theory of gravity. Indeed, if the Planck length were a threshold beyond which quantum gravity effects would become relevant, this length should be the same for all the observers, a statement which clearly disagrees with special relativity. A deformed relativity model consist of replacing the usual (linear) Lorentz transformation L_β in momentum space as follows:

$$\left.\begin{aligned} &L_\beta \to \mathcal{L}_\beta,\\ &\mathcal{L}_\beta = \mathcal{D}^{-1}\circ L_\beta \circ \mathcal{D}\\ \text{and}\quad &(\omega,\boldsymbol{k}) \to \mathcal{L}_\beta(\omega,\boldsymbol{k}),\end{aligned}\right\} \tag{2.7}$$

where the map $\mathcal{D}$ is a singular invertible map such that its Jacobian $J_\mathcal{D}$ equals the identity in $(\omega,\boldsymbol{k}) = 0$. These conditions are needed in order to have an invariant energy, while recovering the usual phenomenology at energy scales much smaller than the Planck scale.

For a complete derivation where we refer to reference [29]. Apart from a null measure set, we split the Brillouin zone B into four parts B_i, $i = 0,\ldots 3$. Each vector $\boldsymbol{k}_i$ belongs to the corresponding region B_i. The regions B_i are chosen such that the compositions $\mathcal{L}_\beta^{(i)} = \mathcal{D}^{(i)-1}\circ L_\beta \circ \mathcal{D}^{(i)}$ are well defined, with $\mathcal{D}^{(i)}$ given by

$$\left.\begin{aligned} &\mathcal{D}^{(i)}: \Sigma_i \to \Gamma_0,\quad \mathcal{D}^{(i)}: \begin{pmatrix}\omega\\ \boldsymbol{k}\end{pmatrix} \mapsto g(\omega,\boldsymbol{k})\begin{pmatrix}\sin\omega\\ \boldsymbol{n}^{(i)}(\boldsymbol{k})\end{pmatrix},\\ &\Sigma_i := \{(\omega,\boldsymbol{k}) \text{ s.t. } \boldsymbol{k}\in\mathsf{B}_i, \sin^2\omega - |\boldsymbol{k}|^2 = 0\}\\ \text{and}\quad &\Gamma_0 := \{p\in\mathbb{R}^4 \text{ s.t. } p_\mu p^\mu = 0\},\end{aligned}\right\} \tag{2.8}$$

for a suitably defined function $g(\omega,\boldsymbol{k})$.[3] The maps $\mathcal{L}_\beta^{(i)}$ provide a well-defined nonlinear representation of the Lorentz group on each set Σ_i.

For $i = 0, 2$, one can easily check that the conditions of definition 2.1 are met if we set $\Gamma_k = \Lambda_\beta$ and $\tilde{\Gamma}_k = \tilde{\Lambda}_\beta$, provided that Λ_β is the right-handed spinor representation of the Lorentz group, and $\tilde{\Lambda}_\beta$ is the left-handed representation. For $i = 1, 3$, the same holds provided we exchange the two representations. The four vector $(\omega,\boldsymbol{k})\in\Sigma_i$ transforms under the nonlinear representation $\mathcal{L}_\beta^{(i)}$. Because $\cup_{i=0}^{3}\mathsf{B}_i = \mathsf{B}$ (apart from a zero-measure set), we have that the maps $\mathcal{L}_\beta^{(i)}$ provide a notion of Lorentz transformation for any solution of the Weyl QCA dynamics.

[3] An admissible expression of the function $g(\omega,\boldsymbol{m})$ is explicitly given in reference [29]. For the following consideration, it suffices to know that $g(\boldsymbol{k}_i) = 1$ and $\nabla g(\boldsymbol{k}_i) = \boldsymbol{0}$ for all $i = 0,\ldots,3$.

We note that the choice of the map (2.8) is not unique, because there are many admissible choices for the function $g(\omega, \boldsymbol{k})$. The symmetry group $\mathbf{S}^A$ of the Weyl walk A contains then many different instances of deformed relativity. However, all of them will recover the usual Lorentz transformations near the points $\boldsymbol{k}_i$. The four invariant regions are interpreted as four different particles (this is the phenomenon of Fermion doubling).

Finally, it is worth stressing the reversed perspective of this approach with respect to the usual one in relativistic quantum mechanics. The Weyl walk dynamics has been singled out without requiring Lorentz invariance, whereas the Lorentz invariance is recovered as a symmetry of the dynamics.

3. Hopf Algebra, κ-Poincare and non-commutative space–time

Here, we explore how the deformation of the Lorentz group given by the nonlinear deformation (2.8) manifests itself at the level of the Poincaré algebra. We will restrict to the $\mathcal{D}^{(0)}$ case and then drop the $^{(0)}$ apex in order to simplify the notation, the generalization for $i = 1, 2, 3$ is trivial. In order to perform this analysis, we will need to consider the framework of Hopf algebras (for a comprehensive introduction to the subject, we suggest reference [34]). The notion of Hopf algebra generalizes that of Lie algebra to a less 'rigid' object, which is can accommodate a nonlinear version of the Lorentz group, which is incompatible with a Lie algebra structure. Unfortunately, any specific nonlinear deformation of the Lorentz group, of the kind in equation (2.7), is not sufficient to select a unique Hopf algebra, because there are many compatible coproduct structures. Nevertheless, it is interesting to study the role that our deformed Lorentz transformation plays within the context of Hopf algebras, because this is the canonical context in the specialized literature on deformed relativity [32,37,38].

(a) Classical Poincaré and κ-Poincaré–Hopf algebras

The Lie algebra of the Poincare group is given by the relations

$$\left.\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k \quad [M_i, p_j] = i\epsilon_{ijk}p_k \\ [M_i, N_j] &= i\epsilon_{ijk}N_k \quad [M_i, p_0] = 0 \\ [N_i, N_j] &= -i\epsilon_{ijk}M_k \quad [N_i, p_j] = i\delta_{ij}p_0 \\ \text{and}\qquad [N_i, p_0] &= -ip_0 \quad [p_\mu, p_\nu] = 0, \end{aligned}\right\} \tag{3.1}$$

where we denoted with M_i the generators of spatial rotations, with N_i the generators of boosts and with p_μ the generators of translations—p_0 denoting the generator of time translation. Clearly, if we apply a nonlinear map to the generators p_μ, then the set of commutation relations (3.1) is spoiled, and generally does not define a Lie algebra anymore. However, it is possible to treat such deformations on formal grounds, within the more general setting of Hopf algebras. The universal enveloping algebra of the Lie algebra (3.1) can be endowed with a Hopf algebra structure by defining the primitive coproduct Δ, antipode S and co-unit ϵ as

$$\left.\begin{aligned} \Delta(O) &= 1 \otimes O + O \otimes 1, \\ S(O) &= -O, \quad S(1) = 1 \\ \text{and}\qquad \epsilon(O) &= 0, \quad \epsilon(1) = 1. \end{aligned}\right\} \tag{3.2}$$

These relations are just a rephrasing of the usual Poincaré Lie algebra structure (3.1) in the language of Hopf algebras, where the additional co-algebra structure allows one to express the Leibniz rule for the infinitesimal action of the group on products of functions through the coproduct. This rule can be easily accounted for using the tensor product structure and the theory of group representations. On the other hand, within the context of Hopf algebras, any invertible analytical map that transforms momenta as $p'_\nu = f_\nu(p_\mu)$ can be treated as a change of basis in an infinite dimensional algebra. Even if, from a mathematical perspective, this transformation is just a change of basis, it may have significant physical consequences like, e.g. a deformation of the dispersion relation.

Nonlinear modifications of the translation generators are not the only possible deformation of the classical Poincaré symmetry. It is indeed possible to consider scenarios in which the Hopf-algebraic structure itself is different (up to any change of basis) from the classical one given by equations (3.1) and (3.2). Of particularly interest are those deformations of the classical Poincaré–Hopf algebra that reduce to the usual one in a suitable limit of values of the deformation parameters. The classification of all the possible deformation of Poincaré–Hopf algebra is still an open problem.

Up to now, the most studied example is the so-called κ-Poincaré–Hopf algebra [33,36], which in the so-called classical basis [37,48] takes the following form:

$$\left.\begin{aligned}&\text{the same algebraic sector}\\ &\Delta(p_0)=\frac{\kappa}{2}(K\otimes K-K^{-1}\otimes K^{-1})+\frac{1}{2\kappa}(K^{-1}|p|^2\otimes K^{-1})\\ &\qquad+(K^{-1}p_i\otimes p_i+K^{-1}\otimes K^{-1}|p|^2)\\ \text{and}\quad &\Delta(p_i)=p_i\otimes K+1\otimes p_i,\end{aligned}\right\}\tag{3.3}$$

where $K:=(1/\kappa)(p_0+(p_0^2-|p|^2+\kappa^2)^{1/2})$ and κ is a real parameter. One can check that the usual classical Poincaré–Hopf algebra is recovered in the limit $\kappa\to\infty$.

Then, starting from the enveloping algebra of the Poincaré Lie algebra we have two different roads that can be explored: (i) assume the co-algebra structure (3.2) and consider the classical Poincaré–Hopf algebra or (ii) assume equation (3.3) and study the κ-Poincaré–Hopf algebra. On the one hand, our scenario singles out a set of generators k_μ that are defined in terms of the classical one p_μ by the nonlinear deformation $p=\mathcal{D}(k)$. On the other hand, our model does not prefer any of the different algebraic models, and it is interesting to consider the consequences of the nonlinear deformation given by the map $\mathcal{D}$ in both the classical Poincaré and in the κ-Poincaré cases.

(b) From Poincaré–Hopf algebra to space–time

One of the most popular speculations concern the relation between the algebra of position coordinate and the algebra of translation.

If we denote by T the Hopf algebra generated by the translation generators p_μ, then one can define the position algebra as the dual Hopf algebra T^* on which T acts covariantly [36]. T^* is determined by introducing the generators x_μ and the pairing

$$\langle f(p_\mu),x_\nu\rangle=f\left(\frac{\partial}{\partial x_\mu}\right)[x_\nu](0).\tag{3.4}$$

This way of introducing the pairing follows the classical pairing between the enveloping algebra of $\mathbb{R}^4$ with the algebra of functions on $\mathbb{R}^4$, i.e. the translation generators act as derivatives evaluated at the origin. The structure of T^* is then determined by the axioms of Hopf algebra duality

$$\left.\begin{aligned}&\langle p,xy\rangle=\langle\Delta(p),x\otimes y\rangle\\ \text{and}\quad&\langle pq,x\rangle=\langle p\otimes q,\Delta(x)\rangle.\end{aligned}\right\}\tag{3.5}$$

Because the momenta commute, we have that positions co-commute with co-commutators

$$\Delta x_\mu=1\otimes x_\mu+x_\mu\otimes 1.\tag{3.6}$$

The commutation relations $[x_\mu,x_\nu]$ are different from 0 only if the coproducts for the p_μ are not co-commutative. Then, if we are dealing with the usual Poincaré algebra, then we will always have a commutative space–time, independently of the nonlinear mapping we are using to define the generators, as their coproduct will still be co-commutative.

The scenario is different in the κ-Poincaré case where it has been proved that the Hopf algebra defined by equations (3.3) leads to the following commutation relations for positions

$$[x_i, x_j] = 0 \quad [x_0, x_i] = -\frac{i}{\kappa} x_i. \tag{3.7}$$

In this case, it could happen that a different choice of the generators p_μ could lead to different commutation relations. In the literature [39], it is proved that the commutation relations (3.7) do not depend on the choice of basis as long as it is rotationally invariant and such that the usual generators are recovered in the limit $\kappa \to \infty$. It is possible to slightly generalize this result by dropping the assumption of rotational invariance.

Lemma 3.1. *Let* $\mathcal{M} : p \mapsto p' = \mathcal{M}(p)$ *be a transformation of the translation generators such that* $J_{\mathcal{M}}(0) = I$. *Then, the commutation relations* (3.7) *remain unchanged.*

Proof. First, we observe that, from the pairing (3.4), we have that the only terms in the co-commutators (3.3) that are relevant for computing the commutators $[x_\mu, x_\nu]$ are the ones that are at most bilinear, i.e. $\Delta(p_0) = 1 \otimes p_0 + p_0 \otimes 1 + (1/\kappa) \sum_i p_i \otimes p_i$ and $\Delta(p_i) = p_i \otimes 1 + (1/\kappa) p_i \otimes p_0 + 1 \otimes p_i$. By power expanding $\mathcal{M}$, we have $p'_\mu = p_\mu + (1/\kappa) m_{\alpha\beta} p_\alpha p_\beta$ and by power expanding the inverse function $\mathcal{M}^{-1}$, we have $p_\mu = p'_\mu + (1/\kappa) n_{\alpha\beta} p'_\alpha p'_\beta$. It is then easy to verify that, up to the bilinear terms, the coproduct $\Delta(p'_0)$ is co-commutative, whereas the coproducts $\Delta(p'_i)$ are the sum of a co-commutative term and $(1/\kappa) p'_i \otimes p'_0$. Because the non-co-commutative term $(1/\kappa) p'_i \otimes p'_0$ has the same expression independently of the nonlinear mapping $\mathcal{M}$, the commutation relation for the space–time variables remains the same. ∎

This result tells us that our nonlinear mapping, which satisfies the hypotheses of lemma 3.1, does not change the commutation relations for the space–time variables.

(c) From Poincaré–Hopf algebra to phase space

We have seen in the preceding section that a notion of space–time can be introduced as the dual T^* to the Hopf algebra of translations T. The additional notion of *left coregular action*

$$p \triangleright x := \langle p, x_{(2)} \rangle x_{(1)} \tag{3.8}$$

allows to introduce a notion of phase space [37,49] as the *left cross-product* algebra $T^* \rtimes T$ where the multiplication is defined as

$$(x \otimes p)(x' \otimes p') = x(p_{(1)} \triangleright x') \otimes p_{(2)} p'. \tag{3.9}$$

If we define the isomorphisms

$$x \sim x \otimes 1 \quad p \sim 1 \otimes p, \tag{3.10}$$

it makes sense to consider the commutation relation

$$[p_\mu, x_\nu] = x_\nu \otimes p_\mu - \langle p_{\mu(1)}, x_\nu \rangle 1 \otimes p_{\mu(2)} - \langle p_{\mu(1)}, 1 \rangle x_\nu \otimes p_{\mu(2)}. \tag{3.11}$$

We will see that the commutation relations (3.11) will depend on the choice of the generators, i.e. they depend on the nonlinear deformation.

We will now compute the commutation relation (3.11) for the choice of generators given by the map $\mathcal{D}$. Because we cannot derive an analytic expression for the inverse map $\mathcal{D}^{-1}$, we will consider just the terms up to the first order in $1/\kappa$. We have then

$$\left.\begin{aligned} E &= \omega & \omega &= E \\ p_x &= k_x + \frac{1}{\kappa}k_y k_z & k_x &= p_x - \frac{1}{\kappa}p_y p_z \\ p_y &= k_y - \frac{1}{\kappa}k_x k_z & k_y &= p_y + \frac{1}{\kappa}p_x p_z \\ \text{and}\quad p_z &= k_z + \frac{1}{\kappa}k_x k_y & k_z &= p_z - \frac{1}{\kappa}p_x p_y. \end{aligned}\right\} \tag{3.12}$$

This result holds the same for any choice of $g(\omega, \mathbf{k})$ such that $\nabla g(\mathbf{0}) = \mathbf{0}$.

After some cumbersome but straightforward calculation, we have, in the classical Poincaré–Hopf algebra case

$$\left.\begin{aligned} &[k_i, x_j] = -i\delta_{ij} - i\frac{(-1)^{\delta_{i,2}}}{\kappa}(\delta_{i+1,j}k_{i+2} + \delta_{i+2,j}k_{i+1}) \\ \text{and}\quad &[\omega, x_j] = [k_i, t] = 0 \quad [\omega, t] = i, \end{aligned}\right\} \tag{3.13}$$

where we used the notation $x = 1, y = 2, z = 3$ and the sums are meant to be modulo 3. Similarly, in the κ-Poincaré–Hopf algebra case, we get

$$\left.\begin{aligned} &[k_i, x_j] = -i\delta_{ij}\left(1 - \frac{\omega}{\kappa}\right) - i\frac{(-1)^{\delta_{i,2}}}{\kappa}(\delta_{i+1,j}k_{i+2} + \delta_{i+2,j}k_{i+1}) \\ &[\omega, x_j] = \frac{i}{\kappa}k_j - \frac{1}{2\kappa}x_j|k|^2 \quad [k_i, t] = 0 \\ \text{and}\quad &[\omega, t] = i - \frac{1}{2\kappa}x_j|k|^2. \end{aligned}\right\} \tag{3.14}$$

Different from the space–time commutation relations, the commutation relation between position and momentum are affected by the choice of the basis. As one could expect, in both cases, we recover the usual commutation relations between position and momentum as the deformation parameter κ goes to infinity.

4. Conclusion

In this paper, we have studied the dynamical symmetries of the Weyl quantum walk. As explained in the paper such walk is particularly interesting, because it was derived from general principles without assuming Lorentz covariance, but nevertheless it recovers a Lorentz-invariant dynamics in the limit of small wavevectors. For large wavevectors the Lorentz group becomes nonlinear, and we have a model of doubly special relativity. We introduced the Hopf algebras for position and momentum of the quantum walk particle, and evaluated the structure constants of the algebras for the usual Poincaré and the κ-Poincaré cases. Generalizing a result of reference [39], we have shown that the space–time commutators are left unaffected by any nonlinear deformation that recovers the linear Lorentz transformations at the leading order. Finally, we derived the analytical expression up to the first order in the inverse Planck-energy κ^{-1} of the deformed Heisenberg commutation relations.

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