

Symmetries of the Dirac quantum walk and emergence of the de Sitter group

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A quantum walk describes the discrete unitary evolution of a quantum particle on a graph. Some quantum walks, referred to as the Weyl and Dirac quantum walks, provide a description the free evolution of relativistic quantum fields in a regime where the wave-vectors involved in the particle state are small. The clash between the intrinsic discreteness of quantum walks and the symmetries of the dynamic equations they give rise to, can be resolved by rethinking the notion of a change of inertial reference frame. We give here a definition of the latter that avoids a pre-defined space-time geometry, and apply it to the case of the Dirac walk in 1+1 dimensions. The change of inertial reference frame is defined as a change of values of the constants of motion that leaves the walk operator unchanged. Remarkably, if one considers the Dirac walk with a fixed value of the mass parameter, the group of allowed changes of reference frame reduces to the trivial group. We then introduce a unique walk encompassing all values of mass as an extra degree of freedom, that is the value of the mass parameter. After deriving the graph corresponding to the new walk, we proceed to the analysis of the symmetry group, and we find that it consists in a realization of the Poincaré group in 2+1 dimensions. Since one space-like dimension does not correspond to an actual spatial degree of freedom, representing instead the mass, the group is interpreted as a 2+1 dimensional version of the de-Sitter group.

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I. INTRODUCTION

The reconciliation of quantum theory with general relativity is one of the most ambitious goals of contemporary physics, and counts a wealth of approaches based on radically different standpoints. One of the ideas behind some of the relatively recent approaches is the proposal that space-time might be a derived notion instead of a primitive one, thus emerging from some non-geometric underlying structure [1–5].

The reconstruction of free quantum field theory from quantum cellular automata[6], through principles controlling the processing of information carried by elementary quantum systems [7–11], constitutes one of the promising approaches to emergent physical laws. A characteristic trait of this approach is that the starting structure is a discrete array of memory cells, governed by an update rule that acts in a discrete sequence of evolution steps. A similar model for elementary physical processes was the subject of Feynman's pioneering proposal of a universal quantum simulator [12].

Some of the approaches to the dynamics of quantum fields based on quantum cellular automata and their simplified description through quantum walks implicitly assume a pre-defined geometry, translated into the properties of the quantum gates producing the evolution of the cellular automaton [13–17]

One of the remarkable features of the approach ini-

tiated by some of the present authors is the fact that space-time is not a primitive notion in this framework, while geometry emerges only in the presence of systems evolving in it—quantum fields—and its very essence cannot be disentangled from the dynamical equations derived within space-time itself, such as Weyl's, Dirac's or Maxwell's equations.

The intimate discreteness of cellular automata appears at odds with the symmetries of known physical laws, in particular the Poincaré group of special relativity. It was already proved in Refs. [18–20] that for the Weyl automata the Poincaré symmetry can be recovered by generalizing the relativity principle, defining changes of inertial frames as those changes of representation of the cellular automaton, in terms of the values of its constants of motion, that preserve the update rule. Such a notion is suitable to the study of dynamical symmetries, without the need of resorting to a space-time background.

The above mentioned result represents a proof of principle that a discrete quantum dynamics is consistent with the symmetries of classical space-time. The Poincaré group acts on the space of wave vectors through a realization, i.e. a group of diffeomorphisms, instead of the usual unitary representation of quantum field theory. The non-linearity of the group of diffeomorphisms gives rise to a peculiar behaviour in the position-time representation, that is typical of models of doubly-special relativity (DSR) [21].

A partial classification of the full symmetry group of the Weyl automaton in 3+1 dimensions was derived in Ref. [22]. In the present paper we provide an extension of the analysis to the case of the Dirac automaton in 1+1 dimensions, namely an automaton where the extra parameter represented by a mass term plays an important

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dynamical role. Remarkably, if one considers a Dirac automaton with a fixed value of the mass parameter, one finds a trivial symmetry group. However, treating the mass as an extra degree of freedom, on the same footing as wave-vector, one obtains a cellular automaton with a symmetry group that is a realization of the group $SO(1, 2)$. This group is interpreted as a variation of the de Sitter group $SO(1, 4)$, which occurs in 3+1 space-time dimensions. The reason for this is that the extra dimension emerging in our case is not a spatial one, but is associated with the variable mass parameter.

This important result introduces a very inspiring relation between degrees of freedom giving rise to the behaviour of massive quantum fields in the emerging physics and the symmetry group of the emerging space-time geometry.

The paper is organised as follows: Section II begins with a review of basic notions of quantum walk on Cayley graphs and of the one dimensional Dirac quantum walk. Then, in section II A we introduce the one dimensional Dirac quantum walk with variable mass, whose eigenvalue equation is studied in Section II B. In Section III we define a notion of change of inertial frame which does not rely on a symmetry of a background spacetime. We then characterize the group of changes of inertial frames of the Dirac walk with variable mass and we show that it consists as a non-linear realiation of a semidirect product of the Poincaré group and the group of dilations.

II. THE ONE DIMENSIONAL DIRAC QUANTUM WALK

A discrete time quantum walk[23, 24] describe the unitary evolution of a particle with s internal degrees of freedom (usually called *coin space*) on a lattice Γ . The lattice Γ is usually the *Cayley graph* of a finitely generated group G , i.e. $\Gamma(G, S)$ is edge-colored directed graph having vertex set G , and edge set $\{(x, xh), x \in G, h \in S\}$, and a color assigned to each generator $h \in S$ (S is set of generators). Usually, an edge which corresponds to a generator g such that $h^2 = e$ (e is the identity of G) is represented as undirected (as the green arrow of the first graph in Fig. (1)). Clearly, each Cayley graph corresponds to a presentation of the group G , where relators are just closed paths over the graph. Within this framework, a discrete-time quantum walk on a Cayley graph $\Gamma(G, S)$ with an s -dimensional coin system ($s \geq 1$) is a unitary evolution on the Hilbert space $\mathbb{C}^s \otimes \ell^2(G)$ of the following kind

$$\begin{aligned} A &:= \sum_{h \in S} A_h \otimes T_h \\ 0 &\neq A_h \in M_s(\mathbb{C}) \\ T_h |x\rangle &:= |xh^{-1}\rangle \end{aligned}$$

where, for any $x \in G$, T_x is the right regular representation of G on $\ell^2(G)$ and $\{|x\rangle, x \in G\}$ is an orthonormal

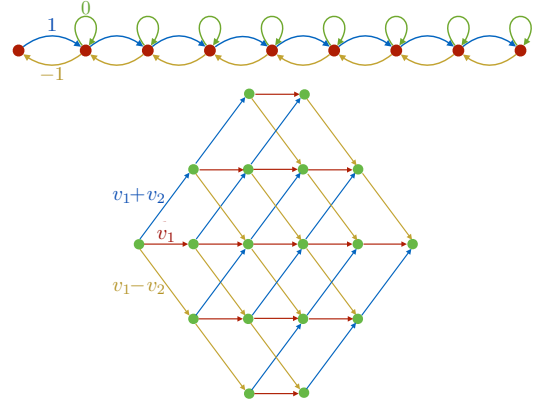


FIG. 1. *Top*: Cayley graph of the group $\mathbb{Z}$, where the green loop arrow represents the identity $\{0\}$, and the blue/green arrow refer to left/right translation, namely $\{1, -1\}$. *Bottom*: Cayley graph of the group $\mathbb{Z}_2$, where the red arrow is associated to the generator v_1 , conversely the blue and yellow arrows refer to the generators $v_1 + v_2$, $v_1 - v_2$ respectively. For the sake of simplicity we have omitted the inverse directions of the generator arrows, so the in the pictures is shown only the up/down right translation.

basis of $\ell^2(G)$.

The one dimensional Dirac quantum walk is a quantum walk on the Cayley graph $\Gamma(\mathbb{Z}, \{0, 1, -1\})$ (see Fig. (1)) of the group $\mathbb{Z}$ with $\mathbb{C}^2$ coin space (the particle as two internal degrees of freedom). The evolution is the following unitary operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^s$:

$$\begin{aligned} A(\mu) &= \begin{pmatrix} \cos(\mu)T & -i \sin(\mu) \\ -i \sin(\mu) & \cos(\mu)T^\dagger \end{pmatrix}, \\ |\psi\rangle &= \sum_{s=L,R} \sum_{x \in \mathbb{Z}} \psi(s, x, \mu) |s\rangle |x\rangle \\ |R\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad (1)$$

where $T := T_1$, $T_1 |x\rangle = |x+1\rangle$ and $T^\dagger = T_{-1}$. Since $A(\mu)$ commutes with the translation operator T , we may represent $A(\mu)$ in the Fourier basis $|k\rangle = \frac{1}{\sqrt{2\pi}} \sum_{x \in \mathbb{Z}} e^{ikx} |x\rangle$ and we obtain

$$\begin{aligned} A(\mu) &= \int_{-\pi}^{\pi} dk \tilde{A}(\mu, k) \otimes |k\rangle \langle k| \\ \tilde{A}(\mu, k) &= \begin{pmatrix} \cos \mu e^{-ik} & i \sin \mu \\ i \sin \mu & \cos \mu e^{ik} \end{pmatrix}. \end{aligned} \quad (2)$$

In the limit $k, \mu \rightarrow 0$ the Dirac Quantum Walk recovers the dynamics of the one dimensional Dirac equation, where k and μ are interpreted as momentum and mass of the particle respectively.

A. Variable mass

As we already discussed in the introduction, and will be shown in the following section, the symmetry group

of the Dirac walk cannot recover the relativistic Lorentz symmetry. This obstruction can be overcome by considering the mass no longer as a fixed parameter, but rather as an additional degree of freedom, as follows

$$A := \int_{-\pi}^{\pi} d\mu A(\mu) \otimes |\mu\rangle\langle\mu| \quad (3)$$

$$|\mu\rangle := \frac{1}{\sqrt{2\pi}} \sum_{\tau \in \mathbb{Z}} e^{i\mu\tau} |\tau\rangle$$

where $|\tau\rangle$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. It is easy to realize that A is a Quantum walk on a Cayley graph of $\mathbb{Z}^2$. Indeed, from Eq. (3) we have

$$A := \int_B dk d\mu \tilde{A}(\mu, k) \otimes |\mu, k\rangle\langle\mu, k|$$

$$\tilde{A}(\mu, k) = \frac{1}{2} \begin{pmatrix} (e^{i\mu} + e^{-i\mu})e^{-ik} & (e^{i\mu} - e^{-i\mu}) \\ e^{i\mu} - e^{-i\mu} & (e^{i\mu} + e^{-i\mu})e^{ik} \end{pmatrix} \quad (4)$$

$$|\mu, k\rangle := |\mu\rangle|k\rangle,$$

$$B := (-\pi, \pi] \times (-\pi, \pi].$$

which in the $|x\rangle|\tau\rangle$ basis reads

$$A = \frac{1}{2} \begin{pmatrix} (T^\dagger + T)S & T^\dagger - T \\ T^\dagger - T & (T^\dagger + T)S^\dagger \end{pmatrix}, \quad (5)$$

$$T|\tau\rangle = |\tau + 1\rangle.$$

In the right regular representation of $\mathbb{Z}$, with basis $|x\rangle|\tau\rangle$, T and S represents the generator $v_1 := (0, -1)$ and $v_2 := (-1, 0)$ respectively. Therefore A is a quantum walk on the Cayley graph $\Gamma(\mathbb{Z}^2, \{\pm v_1, \pm(v_1 + v_2), \pm(v_1 - v_2)\})$ (see Fig. (1))

It is worth noticing that the previous construction depends on the choice of parametrisation for the mass term in Equation (1). For example, the change of variables $\mu' = \sin(\mu)$ would not have led to a Quantum walk in the conjugate variables.

B. Study of the eigenvalue equation

Let us consider the eigenvalue equation for the Dirac Quantum Walk with variable mass. From Equation (4) we have

$$A(\mu, k)\psi(k, \mu) = e^{i\omega(k, \mu)}\psi(k, \mu), \quad (6)$$

$$\psi(k, \mu) = \begin{pmatrix} \psi(R, k, \mu) \\ \psi(L, k, \mu) \end{pmatrix}.$$

which can be rewritten as

$$(\cos(\mu)\cos(k) - \cos(\omega))\psi(k, \mu) = 0 \quad (7)$$

$$(\cos(\mu)\sin(k)\sigma_3 - \sin(\mu)\sigma_1 + \sin(\omega)I)\psi(k, \mu) = 0. \quad (8)$$

From the first equation we get the expression for the eigenvalue, namely $\omega = \arccos(\cos(\mu)\cos(k))$, while multiplying the second equation for σ_2 we obtain

$$(\cos(\mu)\sin(k)i\sigma_1 + \sin(\mu)i\sigma_3 + \sin(\omega)\sigma_2)\psi(k, \mu) = 0.$$

It is worth noting that the set $\{\sigma_2, i\sigma_1, i\sigma_3\}$ represents the generators of the Clifford algebra $\mathcal{Cl}_{1,2}(\mathbb{R})$. Indeed, by renaming the elements of the set as $\{\tau_1, \tau_2, \tau_3\}$, the following relation is satisfied

$$\{\tau_i, \tau_j\} = 2\eta_{ij},$$

where η_{ij} denotes the Minkowski metric tensor. Hence, it is natural to rewrite the eigenvalue equation in the relativistic notation

$$n_\mu(k, \mu)\tau^\mu\psi(k, \mu) = 0,$$

$$n := (\sin(\omega), \cos(\mu)\sin(k), \sin(\mu)), \quad (9)$$

$$\tau := (\sigma_2, -i\sigma_1, -i\sigma_3).$$

Furthermore, it is easy to ascertain that if the eigenvalue equation is verified, then we have

$$n_\nu(k, \mu)n^\nu(k, \mu) = 0, \quad (10)$$

and consequently Eq. (7) is trivially satisfied, i.e. $\omega(k, \mu) = \arccos(\cos(\mu)\cos(k))$. Now, let us analyze the map

$$\bar{n}(k, \mu) : B \rightarrow \mathbb{R}^2$$

$$(k, \mu) \mapsto (\cos(\mu)\sin(k), \sin(\mu)), \quad (11)$$

if we compute the norm of the considered map, we have

$$\|\bar{n}_\nu\|^2 = \sin^2(k)\cos^2(\mu) + \sin^2(\mu) \leq 1, \quad (12)$$

which implies that the Brillouin zone is mapped in the open unit disc in $\mathbb{R}^2$. Clearly, $\bar{n}$ is smooth and analytic. The Jacobian of $\bar{n}$ is

$$J_{\bar{n}}(k, \mu) = \det(\partial_i n_j) = \cos^2(\mu)\cos(k) \neq 0,$$

and the map results singular for $k = \pi/2 + m\pi$ and $\mu = \pi/2 + m\pi$, with $m \in \mathbb{N}$. Let us define the following regions $B_i \subset B$

$$B_0 := \{(k, \mu) | k \in (-\frac{\pi}{2}, \frac{\pi}{2}), \mu \in (-\frac{\pi}{2}, \frac{\pi}{2})\},$$

$$B_1 := B_0 + (\frac{\pi}{2}, 0),$$

$$B_2 := B_0 + (0, \frac{\pi}{2}),$$

$$B_3 := B_0 + (\frac{\pi}{2}, \frac{\pi}{2}), \quad (13)$$

where $B_0 + (a, b)$ denotes the translation of the set B_0 by the vector (a, b) (see Fig. (2)). Denoting by $\bar{n}|_{B_0}$ the map $\bar{n}$ restricted to the region B_0 , and referring Eq. (12), it is easy to note that $\bar{n}|_{B_0}$ is an analytic diffeomorphism between B_0 and the open unit disc in $\mathbb{R}^2$. Then thanks to the periodicity of the map $\bar{n}$, the property of being an analytic diffeomorphism is extended to $\bar{n}|_{B_i}$, $\forall i \in \{0, 1, 2, 3\}$, where $\bar{n}|_{B_i}$ denotes the restriction of $\bar{n}$ to the region B_i .

Therefore, for any $i \in \{0, 1, 2, 3\}$ and $(k, \mu) \in B_i$, if $n_\mu(k, \mu)\tau^\mu|\psi(k, \mu)\rangle = 0$, there exists $(k', \mu') \in B_0$ such that $n_\mu(k, \mu) = n_\mu(k', \mu')$ and $|\psi(k, \mu)\rangle = |\psi(k', \mu')\rangle$. We may understand the B_i regions as kinematically equivalent sets, and the quantum walk dynamics is completely

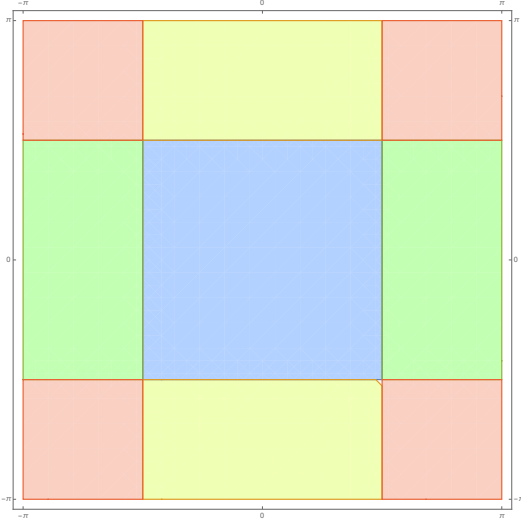


FIG. 2. Brillouin zone: the blue region refers to B_0 , while the other colored regions correspond to B_i , and are obtained simply translating by the vector (a, b) with $a, b \in \{0, \pi/2\}$.

specified by the solution of the eigenvalue equation (9) in any of the regions B_i . Let us now consider the map $n|_{B_0}$, defined as the restriction of Eqs. (9) and (10), to the region B_0 , which acts as follows:

$$n|_{B_0} : \begin{pmatrix} \omega \\ \mathbf{k} \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \sin(\omega) \\ \bar{n}|_{B_0}(\mathbf{k}, \mu) \end{pmatrix} = \begin{pmatrix} \sin(\omega) \\ \cos(\mu) \sin(\mathbf{k}) \\ \sin(\mu) \end{pmatrix}$$

The map $n|_{B_0}$ defines a diffeomorphism between the mass-shell

$$V = \{(\omega, \mathbf{k}, \mu) | (\mathbf{k}, \mu) \in B_0, \omega = \arccos(\cos(\mathbf{k}) \cos(\mu))\},$$

defined by condition (10) and represented in Fig. IIB, and the truncated cone

$$K := \{(x, y, z) | x^2 + y^2 = z, 0 \leq z \leq 1\},$$

represented in Fig (IIB). From now on, unless otherwise specified we will consider $(\mathbf{k}, \mu) \in B_0$, and consequently we remove the restriction symbol $\cdot|_{B_0}$ from all the maps.

III. CHANGE OF INERTIAL FRAME

As we saw in the previous section, the solution of the of the eigenvalue equation (9) in one of the regions B_i , which were defined in Eq. (13), completely characterizes the quantum walk dynamics. We then require that a change of reference frame leaves invariant the eigenvalue equation (9) restricted to the domain B_0 .

Definition 1 (Change of inertial reference frame for the Dirac walk) A change of inertial reference frame for the Dirac walk is a triple (k', a, M) where

$$k' : V \rightarrow V, \quad k \mapsto k'(k),$$

is a diffeomorphism, $a : V \rightarrow \mathbb{R}$, $k \mapsto a(k)$ is a smooth map, and $M \in GL(2, \mathbb{C})$ such that:

$$\begin{aligned} n_\mu(k) \tau^\mu \psi(k) = 0 &\Leftrightarrow n_\mu(k') \tau^\mu \psi'(k') = 0, \\ \psi'(k') &= e^{ia(k)} M^{-1} \psi(k) \end{aligned} \quad (14)$$

for any $k \in V$.

According to Definition 1, a change of inertial frame is a relabeling $k'(k)$ of the constants of motion of the quantum walk such that the eigenvalue equation is preserved in the region B_0 . The same definition straightforwardly generalises to the other regions B_i . Let us now characterize the group of symmetries which follows from this definition. The first step is the following lemma

Lemma 1 We have

$$n_\mu(k) \tau^\mu \psi(k) = 0 \Leftrightarrow n_\mu(k') \tau^\mu \psi'(k') = 0, \quad \forall k \in V$$

if and only if

$$f(k') n_\mu(k') = L_\mu^\nu n_\nu(k), \quad \forall k \in V, \quad (15)$$

where $L \in O(1, 2)$, and $f(k')$ is a suitable non null real function.

Proof. Clearly we have that $n_\mu(k) \tau^\mu \psi(k) = 0 \Leftrightarrow n_\mu(k') \tau^\mu M^{-1} \psi(k) = 0$ for any $k \in V$ if and only if $\sigma_2 n_\mu(k) \tau^\mu \psi(k) = 0 \Leftrightarrow M^{\dagger-1} \sigma_2 n_\mu(k') \tau^\mu M^{-1} \psi(k) = 0$ because $M \in GL(2, \mathbb{C})$. From Equation (8) we have that $\sigma_2 n_\mu(k) \tau^\mu$ is proportional to a rank one projector and therefore we must have

$$\begin{aligned} g(k') n_\mu(k) \tau^\mu &= \sigma_2 M^{\dagger-1} \sigma_2 n_\mu(k') \tau^\mu M^{-1} = \\ &= T_\nu^\mu n_\mu(k') \tau^\nu, \end{aligned} \quad (16)$$

for some non-null real scalar function $g(k')$, and some linear map T . Moreover, since from Eq. (10) we have $n_\mu(k) n^\mu(k) = 0$, the linear transformation T must preserve the basis K of the null mass shell, and therefore, thanks to the Alexandrov-Zeeman theorem [25, 26], $T = \kappa L$ for $L \in O(1, 2)$ and $\kappa \in \mathbb{R}$. In order to determine κ , let us now consider a general $m \in \mathbb{R}^3$. In this case, since we have

$$\sigma_2 M^{\dagger-1} \sigma_2 m_\mu \tau^\mu M^{-1} = \kappa L_\nu^\mu m_\mu \tau^\nu,$$

taking the determinant on both sides we obtain $\kappa = \det(M^{\dagger-1}) \det(M^{-1}) = |\det(M^{-1})|^2 > 0$. If we now divide Eq. (16) by κ on both sides, the thesis follows with $f := g/\kappa$. ■

Lemma 2 Let (k', a, M) be a change of inertial frame for the Dirac walk. Then $M \in GL(2, \mathbb{R})$ and

$$L_\mu^\nu n_\nu(k) = \mathcal{D}^f n_\mu(k') \quad (17)$$

$$\mathcal{D}^f : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \mathbf{v} \mapsto f(\mathbf{v}) \mathbf{v} \quad (18)$$

where $L \in O(1, 2)$ and $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a smooth function such that $\mathcal{D}^f$ is injective.

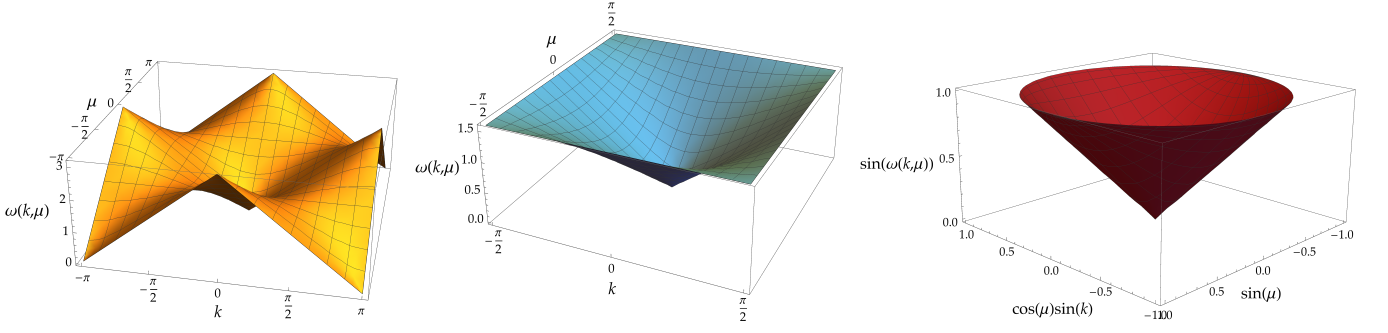


FIG. 3. *Left:* The domain V of the map n_ν . *Middle:* The restriction V_0 of the surface V to the Brillouin region B_0 . *Right:* The image $K := n(V)$, where the truncation is due to the condition $\bar{n}_\nu \bar{n}^\nu < 1$.

Proof. From Equation (16) we have that M is a two dimensional representation of the semi-direct product of the dilation group $(\mathbb{R}_+, \times)$ by $O(1, 2)$. Let now $f(k')$ be as in Lemma 1. Since $n(k')$ is a diffeomorphism, we may consider f as a function of n , namely $f(n) := f(k'(n))$. Let us now assume that $\mathcal{D}^f$ is not injective. Then we would have, $\mathcal{D}^f \circ n(k'_1) = \mathcal{D}^f \circ n(k'_2)$ for some $k'_1 \neq k'_2$. From Eq. (15) we then have $L'_\mu n_\nu(k_1) = L'_\mu n_\nu(k_2)$. However, since both the map $k'(k)$ and L are invertible, this would imply $k_1 = k_2$. ■

We can finally prove the characterization of the symmetry group of the Dirac walk.

Proposition 1 *Let (k', a, M) be a change of inertial frame for the Dirac walk. Then $M \in GL(2, \mathbb{R})$ and*

$$k'(k) = [n^{-1} \circ \mathcal{D}^{f^{-1}} \circ L \circ \mathcal{D}^g \circ n](k) \quad (19)$$

where $\mathcal{D}^f$ and $\mathcal{D}^g$ are two diffeomorphisms between K and the null mass shell, of the form of Eq. (18).

Proof. From Lemma 2 we have that

$$k'(k) = [n^{-1} \circ \mathcal{D}^{f^{-1}} \circ M \circ n](k),$$

where $M \in O(1, 2)$, and $\mathcal{D}^f$ is of the form of Eq. (18). Let now $\mathcal{D}^g$ be any diffeomorphism of the same form between K and the complete null mass shell. Since K is star shaped, such a $\mathcal{D}^g$ exists (see Appendix B for a proof). Then

$$k'(k) = [n^{-1} \circ \mathcal{D}^{\bar{f}^{-1}} \circ L \circ \mathcal{D}^g \circ n](k) \\ \mathcal{D}^{\bar{f}} := \mathcal{D}^{f^{-1}} \circ L \circ \mathcal{D}^{g^{-1}} \circ L^{-1}$$

where also $\mathcal{D}^{\bar{f}}$ is a diffeomorphism between K and the null mass shell of the required form. ■

Eq. (19), in Proposition (1), leads to a final form of a non-linear representation of the symmetry group, due to the non-linear deformation of a generic element $L \in O(1, 2)$ induced by the action of $\mathcal{D}^f$. It is worth noting that, this representation is comparable with the ones considered in the context of doubly special relativity [21].

The symmetry group of the Dirac walk with fixed mass can be obtained from Proposition 1 with the additional requirement that the mass term is kept unchanged. As we prove in Appendix A, this leads to a trivial symmetry group.

We remark that, in the limit of $k \ll 1$ and small mass variation: $\mu = \mu_0 + \epsilon$, with $\epsilon \ll 1$, we recover a linear representation of $O(1, 2)$, simply expanding the rescaling function as follows:

$$f(n(k, \mu)) \approx f(n(0, \mu)) + \partial_k f|_{(0, \mu_0)} k + \partial_\mu f|_{(0, \mu_0)} \epsilon.$$

In conclusion, we want to point out that, although an invertible map $f(n(k, \mu))$ implies the invertibility of the associated $\mathcal{D}^f$ trivially, the reverse implication needs to be discussed. In particular we prove (see Appendix (B)) that injectivity of $\mathcal{D}^f$ leads to a family of invertible radial functions $\tilde{f}_{\hat{n}} : [0, \sqrt{2}) \rightarrow \mathbb{R}_+$, defined as follows:

$$\tilde{f}_{\hat{n}}(\|n\|_E) \hat{n} := \mathcal{D}^f \circ n = f(n)n = f(n)\|n\|_E \hat{n}, \quad (20)$$

where $\hat{n}_\nu$ represents the unit Euclidean norm vector in the direction of n , and $\|n\|_E$ is the Euclidean norm of n .

IV. CONCLUSION

In this paper we derived the group of changes of inertial reference frame for the Dirac walk in 1+1 dimension. If the mass for the walk is fixed, the group of admissible symmetries reduces to the trivial one. As a consequence of this observation, we defined a Dirac walk with variable mass, and studied the symmetry group of the latter. As a result, one finds a group of transformations that, along with ω and k , modify also the variable μ , that defines the mass term. Such a group can be considered as the 1+1-dimensional counterpart of the de Sitter group, that acts on the Brillouin zone of the walk by a realisation in terms of a group of diffeomorphisms. Along with the de Sitter group one is forced to consider a group of non-linear rescaling maps, so that the final group is a semidirect product of the two above components.

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Appendix A: Symmetry of the Dirac Walk with fixed mass

The symmetry transformations of the Dirac Walk with fixed mass can clearly be recovered as the subgroup of the symmetry transformations of the Dirac Walk with variable mass. The condition that the mass term in the Dirac QCA eigenvalue equation is fixed, implies that $f = g$ in equation (19) and that L acts trivially on the third component of $n(k)$.

The above conditions imply that the map $k'(k)$ is such that

$$\begin{pmatrix} n_1(k') \\ n_2(k') \end{pmatrix} = L \begin{pmatrix} n_1(k) \\ n_2(k) \end{pmatrix}.$$

However, this is possible only if $L_\beta = I$, thus proving that the symmetry group of the Dirac Walk with fixed mass is trivial.

Appendix B: An example of a rescaling function

Lemma 3 *The function $\tilde{f}_{\tilde{n}}$ defined in Eq. (20) is strictly monotonic.*

Proof. Let n_1 and n_2 be two parallel vectors, such that $n_1 \propto n_2$, then from Lemma (2) we have

$$\mathcal{D}^f(n_1) \neq \mathcal{D}^f(n_2), \quad (\text{B1})$$

namely

$$\tilde{f}_{\tilde{n}}(\|n_1\|_E) \neq \tilde{f}_{\tilde{n}}(\|n_2\|_E) \quad \forall \|n_1\|_E \neq \|n_2\|_E. \quad \blacksquare$$

We now show that suitable function $\mathcal{D}^f$ exists. In order to have a surjective map, we must impose it to be singular at the superior border of the truncated cone. It is sufficient consider a function f depending only on $\tilde{n}$. This implies that we must have a singularity at the edge of the unit disc. Hence we define

$$f : \tilde{n} \rightarrow \mathbb{R}$$

$$f(\tilde{n}) := \frac{1}{1 - \|\tilde{n}\|_E^2}.$$

The latter function is manifestly singular at the border of the unit disc and also results monotonic respect to $\|\tilde{n}\|_E$.

Now, let $\tilde{f}_{\hat{n}}(\|\bar{n}\|_E) := f(\bar{n})\|n\|_E$, where $\hat{n} := n/\|n\|_E$ and $n := (\|\bar{n}\|_E, \bar{n})$. Clearly, $\tilde{f}_{\hat{n}}$ is monotonic, namely it verifies the invertibility condition on $\mathbb{R}$, defined in Corollary (3). Moreover, it is easy to verify that $\mathcal{D}^f$ is smooth.