$SU(1,1)$ Symmetry of the Balanced Homodyne and Statistics of the Output.

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Summary. — An operator approach, allowing the evaluation of the output statistics of a balanced homodyne detector with an input beam in a squeezed state, is presented. The case of one squeezed input is connected to the one with a coherent input through a novel $SU(1,1)$ symmetry transformation. This symmetry is peculiar of the 50-50 balanced scheme and corresponds to a symmetrical squeezing of the two inputs. The squeezing of the local oscillator and the associated effects on the statistics are analysed.

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1. – Introduction.

The ultimate performance in the measurement of very small optical phase shifts is presently of great interest for highly sensitive interferometric detectors. After reduction of all the technical sources of noise (acoustical, thermal, etc.), the sensitivity in the phase shift ϕ remains affected by the quantum fluctuations of the light injected into the interferometer. For coherent laser light, it turns out that the sensitivity cannot be improved beyond the so-called «shot noise limit» (SNL)

$$(\Delta\phi)_{\text{SNL}} = 1/\sqrt{N}, \tag{1}$$

N being the number of photons detected during the integration time.

As discussed by many authors [1-4], the sensitivity can be enhanced beyond the SNL by using squeezed light instead of the conventional laser light. As a matter of fact, the coherent noise does not depend on the phase of the field, whereas the squeezed one, being phase dependent, has a minimum noise below the coherent value. Therefore, improvement of the signal-to-noise ratio can be obtained by selecting the amplitude component of the field at the phase with minimum noise.

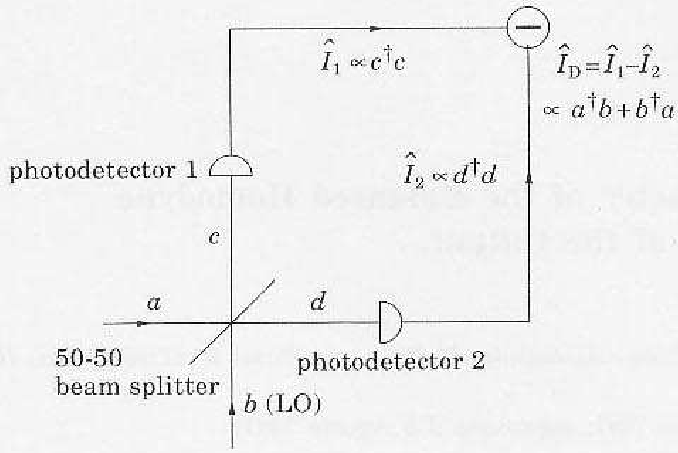

Fig. 1. – Scheme of a balanced homodyne detector.

The detection of an amplitude component of the field can be implemented by means of a homodyne detector. The signal beam is combined by means of a beam splitter with an intense «local-oscillator» (LO) field operating at the same frequency. The combined field is then directed to a photodetector and the amplitude component of the field is revealed as the beating between the two input fields.

To avoid noise from intensity fluctuations of the LO, the balanced configuration is usually chosen (see fig. 1), by using two photodetectors with equal gains and a 50-50 beam splitter. The difference photocurrent I_D between the two photodetectors is not influenced by the fluctuations of the LO intensity. On the contrary, I_D measures the interference between the signal beam and the LO, the interference being constructive at one photodetector and destructive at the second one.

In the theoretical description of the balanced homodyne detector one usually assumes a classical (highly excited) LO. This assumption is certainly realistic for most of the actual set-up, as, for example, in recent highly sensitive interferometers [5, 6]. In the limit of infinite LO amplitude, the statistics of I_D reproduce exactly the statistics of an amplitude component of the input field at a fixed phase

$$\hat{x} = \exp[-i\phi]\, a^\dagger + \exp[i\phi]\, a\,, \tag{2}$$

and the homodyne detector behaves like an ideal detector of $\hat{x}$. However, the quantum nature of the LO affects the measure of $\hat{x}$. As pointed out by S. L. Braunstein [7], even the most obvious condition—that the number of photons in the LO be much larger than the number in the signal—is not always sufficient to ensure an ideal behaviour for the balanced homodyne detector.

After the earlier studies of the quantum nature of the LO [8-10], new interest arose in the fully quantum-mechanical treatment of the homodyne statistics. The $SU(2)$ symmetry of the beam splitter has been exploited in order to evaluate the photon statistics [11], whereas the matching between the statistics of I_D and those of the quadrature phase amplitude $\hat{x}$ has been analysed [7] in the framework of the positive-P representation [12].

In this paper an alternative operator approach is presented, allowing one to evaluate the statistics of I_D in the case of quantum uncorrelated input beams, with one input prepared in a squeezed state, while the other one is general. Such a case pertains to most of the experimental situations presently of interest.

In sect. 2 the case of one squeezed input beam is connected to the case of one coherent input by means of a novel $SU(1,1)$ symmetry transformation, which is peculiar of the 50-50 balanced scheme and corresponds to a symmetrical squeezing of the inputs.

The generating function of the moments of I_D is evaluated in sect. 3, for one coherent input. The generating function is factorized into the product of two averages, each of them involving observables of a single input field. The trace on the coherent beam is carried out and the generating function is written in terms of a single average on the other input beam.

The asymptotic behaviour of these expressions in the limit of a highly excited field (classical LO) is obtained in sect. 4, where both the cases of coherent and squeezed beams are analysed, comparing the statistics of $\hat{I}_D$ and $\hat{x}$.

Section 5 gives a few conclusive comments regarding the use of a squeezed LO.

2. – $SU(1,1)$ symmetry of the difference photocurrent.

In the following, a single-mode analysis is given, in the assumption of a lossless beam splitter and of photodetectors having unit quantum efficiency. The input fields a and b combine at the 50-50 beam splitter giving the sum and the difference fields c and d in the output arms. After tuning the overall phases (by adjusting the path lengths), one has

$$c = \frac{a+b}{\sqrt{2}}\,, \qquad d = \frac{a-b}{\sqrt{2}}\,. \tag{3}$$

The output photocurrent $\hat{I}_1$ and $\hat{I}_2$ are proportional to the number operators $c^\dagger c$ and $d^\dagger d$ and the difference photocurrent $\hat{I}_D$ (for photodetectors with equal gains) has the form

$$\hat{I}_D = \hat{I}_1 - \hat{I}_2 \propto a^\dagger b + b^\dagger a\,. \tag{4}$$

The difference photocurrent $\hat{I}_D$ can be identified with the operator $a^\dagger b + b^\dagger a$ by omitting the proportionality constant.

The particular form of $\hat{I}_D$ in eq. (4) is highly symmetrical, as a consequence of the 50-50 balanced scheme. Besides the trivial symmetry under permutation of the input fields, $\hat{I}_D$ is invariant under the unitary transformation

$$\hat{I}_D = \hat{U}(\mu,\nu)\,\hat{I}_D\,\hat{U}^\dagger(\mu,\nu)\,, \tag{5}$$

where

$$\hat{U}(\mu,\nu) = \hat{S}_a^\dagger(\bar{\mu},\nu)\,\hat{S}_b(\mu,\nu)\,. \tag{6}$$

Here $\hat{S}_{a,b}(\mu,\nu)$ are the squeezing operators of Yuen [13] acting on the input fields as follows:

$$\begin{cases} \hat{S}_a(\mu,\nu)\,a\hat{S}_a^\dagger(\mu,\nu) = \mu a + \nu a^\dagger\,, \\ \hat{S}_b(\mu,\nu)\,b\hat{S}_b^\dagger(\mu,\nu) = \mu b + \nu b^\dagger\,. \end{cases} \tag{7}$$

The complex numbers μ and ν satisfy the identity

$$|\mu|^2 - |\nu|^2 = 1 . \tag{8}$$

Invariance (5) can be checked by using eq. (8) and the identity $\hat{S}^\dagger_{a,b}(\mu, \nu) = \hat{S}_{a,b}(\bar{\mu}, -\nu)$. Notice that the symmetry (6) is a $SU(1,1)$ symmetry, hence it does not belong to the usual $SU(2)$ symmetry class [11]. However, operator (6) provides a symmetry only for the balanced scheme, since it does not commute with $\hat{I}_1$ and $\hat{I}_2$ separately (for a general treatment of squeezing symmetries in linear interferometers see ref. [14]).

Invariance (5) means that the quantum statistics of the difference photocurrent $\hat{I}_{\rm D}$ does not change if the input state $\hat{R}$ is symmetry transformed

$$\hat{R} \to \hat{U}^\dagger(\mu, \nu)\hat{R}\hat{U}(\mu, \nu) . \tag{9}$$

For uncorrelated inputs described by density matrices $\hat{\rho}_a$ and $\hat{\rho}_b$, the symmetry transformation (9) is equivalent to the pair of single-mode transformations

$$\hat{\rho}_a \to \hat{S}_a(\bar{\mu}, \nu)\hat{\rho}_a\hat{S}^\dagger_a(\bar{\mu}, \nu) , \tag{10}$$

$$\hat{\rho}_b \to \hat{S}^\dagger_b(\mu, \nu)\hat{\rho}_b\hat{S}^\dagger_b(\mu, \nu) . \tag{11}$$

If a pure squeezed state $\hat{\rho}_b = |\mu, \nu; z\rangle\langle\mu, \nu; z|$ is considered, such that

$$(\mu b + \nu b^\dagger)|\mu, \nu; z\rangle = z|\mu, \nu; z\rangle , \tag{12}$$

then, eq. (11) can be written

$$|\mu, \nu; z\rangle\langle\mu, \nu; z| \to |z\rangle\langle z| , \tag{13}$$

where $|z\rangle$ is the coherent state

$$b|z\rangle = z|z\rangle . \tag{14}$$

In other words, the same output current $\hat{I}_{\rm D}$ is obtained when the following different inputs are used: i) input b is in the squeezed state $|\mu, \nu; z\rangle\langle\mu, \nu; z|$ and input a is in the state $\hat{\rho}_a$; ii) input b is in the coherent state $|z\rangle\langle z|$ and input a is squeezed from $\hat{\rho}_a$ to $\hat{S}_a(\bar{\mu}, \nu)\hat{\rho}_a\hat{S}^\dagger_a(\bar{\mu}, \nu)$.

The preceding equivalence connects the statistics of $\hat{I}_{\rm D}$ when one input is squeezed to the statistics when one input is coherent. Therefore, one has to evaluate only the statistics for one coherent input and then use the symmetry to rewrite the final results for one squeezed input.

3. – Statistics of the difference photocurrent.

The generating function of the moments for the difference photocurrent $a^\dagger b + b^\dagger a$ is defined as follows:

$$\chi(\lambda) = \langle \exp[i\lambda(a^\dagger b + b^\dagger a)]\rangle_{a,b} , \tag{15}$$

where $\langle \hat{O}\rangle_{a,b} = \mathrm{Tr}\{\hat{\rho}_a\hat{\rho}_b\hat{O}\}$. The oscillator b is assumed to be coherent, *i.e.*

$$\hat{\rho}_b = |z\rangle\langle z| . \tag{16}$$

The trace of the oscillator b can be evaluated after normal ordering the exponential in eq. (15) with respect to b and $b^\dagger$. This can be achieved by exploiting the Schwinger [15] representation of the angular-momentum algebra $su(2)$

$$\begin{cases} \hat{J}_+ = a^\dagger b\,, \quad \hat{J}_- = b^\dagger a\,,\\ \hat{J}_z = \dfrac{1}{2}(a^\dagger a - b^\dagger b)\,,\\ \hat{J} = \dfrac{1}{2}(a^\dagger a - b^\dagger b)\,,\\ \hat{J}^2 = \hat{J}(\hat{J}+1)\,, \end{cases} \tag{17}$$

where $\hat{J}_z$, $\hat{J}_\pm$ denote the usual spin-J angular momentum operators ($[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z, [\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm$), $\hat{J}$ commutes with $\hat{J}_\pm$ and $\hat{J}_z$ and, therefore, is a constant of motion. Using the Baker-Campbell-Hausdorff formula [16]

$$\exp[\eta\hat{J}_+ - \bar{\eta}\hat{J}_-] = \exp[-\bar{\delta}\hat{J}_-]\exp[\omega\hat{J}_z]\exp[\delta\hat{J}_+]\,, \tag{18}$$

where

$$\delta = \frac{\eta}{|\eta|}\,\mathrm{tg}\,|\eta|\,, \qquad \exp\left[\frac{\omega}{2}\right] = \cos|\eta|\,. \tag{19}$$

Equation (15) can be written

$$\chi(\lambda) = \left\langle \exp[-\bar{\alpha}b^\dagger a]\exp\left[\frac{1}{2}\beta(a^\dagger a - b^\dagger b)\right]\exp[\alpha a^\dagger b]\right\rangle_{a,\,b}\,, \tag{20}$$

where $\alpha = i\,\mathrm{tg}\,\lambda$ and $\exp[\beta/2] = \cos\lambda$. The invariance of trace under cyclic permutations and eq. (14) leads to

$$\chi(\lambda) = \left\langle \exp[-\overline{\alpha z}\,a]\exp\left[\frac{1}{2}\beta(a^\dagger a - b^\dagger b)\right]\exp[\alpha z a^\dagger]\right\rangle_{a,\,b}\,. \tag{21}$$

The average in eq. (21) can be factorized in the form

$$\chi(\lambda) = \left\langle \exp[-\overline{\alpha z}\,a]\exp\left[\frac{1}{2}\beta a^\dagger a\right]\exp[\alpha z a^\dagger]\right\rangle_a \left\langle \exp\left[-\frac{\beta}{2}b^\dagger b\right]\right\rangle_b\,, \tag{22}$$

$\langle\ldots\rangle_a$ and $\langle\ldots\rangle_b$ denoting the single-mode traces $\mathrm{Tr}_a\{\hat{\rho}_a\ldots\}$ and $\mathrm{Tr}_b\{\hat{\rho}_b\ldots\}$. The average on the input b can be explicitly evaluated, whereas the average on a can be more conveniently put in the normal order form

$$\chi(\lambda) = \left\langle :\exp[iz\sin\lambda a^\dagger]\exp\left[-2\sin^2\frac{\lambda}{2}(a^\dagger a + |z|^2)\right]\exp[i\bar{z}\sin\lambda a]:\right\rangle_a\,, \tag{23}$$

($:\hat{O}:$ denotes the normal ordered form of the operator $\hat{O}$ in terms of a and $a^\dagger$). Equation (23) provides a useful tool for evaluating the homodyne statistics as a function of the input state $\hat{\rho}_a$. One should stress that $\chi(\lambda)$ is a periodic function of λ with period 2π, this fact reflecting the discrete nature of the output I_D, whose probability distribution—*i.e.* the Fourier transform of $\chi(\lambda)$—exhibits δ-function behaviour at integral values of I_D.

If the input state $\hat{\rho}_a$ is given in the positive-P representation [12]

$$\hat{\rho}_a = \int \mathrm{d}^2\alpha_1\, \mathrm{d}^2\alpha_2\, \frac{|\alpha_1\rangle\langle\alpha_2|}{\langle\alpha_2|\alpha_1\rangle} P_a(\alpha_1, \bar{\alpha}_2), \tag{24}$$

then eq. (23) becomes

$$\chi(\lambda) = \int \mathrm{d}^2\alpha_1\, \mathrm{d}^2\alpha_2\, P_a(\alpha_1, \bar{\alpha}_2) \exp\left[i \sin\lambda (z\bar{\alpha}_2 + \bar{z}\alpha_1) - 2\sin^2\frac{\lambda}{2}(\bar{\alpha}_2\alpha_1 + |z|^2)\right]. \tag{25}$$

The last equation compares with the analogous results given by Braunstein [7].

4. – Asymptotic behaviour.

Equation (22) can be written in a form more suited for a comparison between the statistics of $\hat{I}_{\mathrm{D}}$ and those of the quadrature phase amplitude $\hat{x}$. By means of the usual Baker-Campbell-Hausdorff commuting techniques one has

$$\chi(\lambda) = \exp\left[\frac{1 - \cos\lambda - (1/2)\sin^2\lambda}{\cos\lambda}|z|^2\right] \cdot \langle \exp[i\bar{z}\,\mathrm{tg}\,\lambda a + iz\sin\lambda a^\dagger] \colon \exp[(\cos\lambda - 1)\, a^\dagger a] \colon \rangle_a, \tag{26}$$

where the second exponent in eq. (26) is proportional to $\hat{x}$ in the limit of small λ and for $\arg(z) = \phi$ (see eq. (2)). Strictly speaking, the variable which should match the statistics of $\hat{x}$ is the scaled current

$$\hat{\imath}_{\mathrm{D}} = \frac{\hat{I}_{\mathrm{D}}}{|z|}, \tag{27}$$

whose generating function $\kappa(\lambda)$ is the rescaling of $\chi(\lambda)$

$$\kappa(\lambda) = \chi\left(\frac{\lambda}{|z|}\right). \tag{28}$$

Therefore, in order to obtain the asymptotic behaviour in the limit of the highly excited field b (classical LO), one can expand $\chi(\lambda)$ for small λ. Retaining terms up to the fourth order, one has

$$\chi(\lambda) \simeq \exp\left[\frac{1}{8}\lambda^4|z|^2\right] \cdot \left\langle \exp\left[i\lambda|z|\hat{x} + i\frac{\lambda^3}{3}\left(\bar{z}a - \frac{1}{2}za^\dagger\right)\right] \colon \exp\left[\left(-\frac{1}{2}\lambda^2 - \frac{1}{12}\lambda^4\right)a^\dagger a\right] \colon \right\rangle_a. \tag{29}$$

Rescaling λ and further expanding the slowly oscillating exponentials lead to the asymptotic form for $\kappa(\lambda)$

$$(30)\qquad \kappa(\lambda) = \langle \exp[i\lambda\hat{x}]\rangle_a + \frac{(i\lambda)^2}{2!|z|^2}\langle a^\dagger \exp[i\lambda\hat{x}]\, a\rangle_a +$$

$$+\frac{(i\lambda)^3}{3!|z|^2}\langle \exp[i\phi]\, a^\dagger \exp[i\lambda\hat{x}] + \exp[-i\phi]\exp[i\lambda\hat{x}]\, a\rangle_a + \frac{(i\lambda)^4}{4!|z|^2}\langle \exp[i\lambda\hat{x}]\rangle_a + \mathcal{O}\left(\frac{\lambda^4}{|z|^4}\right),$$

which agrees with the asymptotic expansion of Braunstein [7]. From eq. (30) it follows that the statistics of $\hat{\imath}_\text{D}$ fit those of $\hat{x}$ in the limit of $|z|\to\infty$; moreover, the condition $\langle a^\dagger a\rangle_a \ll |z|^2$ is not sufficient to warrant matching in the general case [7].

Equations (23), (26) and (30) can be used also in the case of one squeezed input beam $\hat{\rho}_b = |\mu,\nu;z\rangle\langle\mu,\nu;z|$. One has simply to perform the symmetry transformation (10) before evaluating the averages on the input a. I give here the explicit result in terms of the unsqueezed input averages $\langle\ldots\rangle_a = \text{Tr}_a\{\hat{\rho}_a\ldots\}$ only for the asymptotic expansion (30)

$$(31)\qquad \kappa(\lambda) = \langle \exp[i\lambda\hat{y}]\rangle_a + \frac{(i\lambda)^2}{2!|z|^2}\langle \hat{O}_2\rangle_a + \frac{(i\lambda)^3}{3!|z|^2}\langle \hat{O}_3\rangle_a + \frac{(i\lambda)^4}{4!|z|^2}\langle \hat{O}_4\rangle_a + \mathcal{O}\left(\frac{\lambda^4}{|z|^4}\right),$$

where $\hat{y}$ is the new amplitude component of the field

$$(32)\qquad \hat{y} = \bar{\zeta}a + \zeta a^\dagger$$

and ζ is the amplified phase factor

$$(33)\qquad \zeta = \exp[i\phi]\bar{\mu} - \exp[-i\phi]\,\nu\,.$$

The operators $\hat{O}_n$ are given by

$$(34)\qquad \begin{cases} \hat{O}_2 = |\nu|^2\exp[i\lambda\hat{y}] + (2|\nu|^2+1)\, a^\dagger\exp[i\lambda\hat{y}]\, a - \mu\bar{\nu}\exp[i\lambda\hat{y}]\, a^2 - \bar{\mu}\nu a^{\dagger 2}\exp[i\lambda\hat{y}]\,, \\ \hat{O}_3 = w a^\dagger \exp[i\lambda\hat{y}] + \bar{w}\exp[i\lambda\hat{y}]\, a\,, \\ \qquad\qquad w = \exp[i\phi]\bar{\mu}(1+6|\nu|^2) - \exp[-i\phi]\,\nu(4+6|\nu|^2)\,, \\ \hat{O}_4 = (4|\exp[i\phi]\mu - \exp[-i\phi]\,\nu|^2 - 3)\exp[i\lambda\hat{y}]\,. \end{cases}$$

5. – Conclusions.

Some remarks regarding the use of a squeezed LO in addition to—or in place of—making use of an input beam in a squeezed state, are in order. As a consequence of the symmetry (6), squeezing LO is equivalent to squeezing the signal beam state $\hat{\rho}_\text{in}$. However, no improvement of the signal-to-noise ratio can be achieved, since the squeezing of $\hat{\rho}_\text{in}$ corresponds to a simultaneous squeezing of both the noise and the signal (whereas, in the usual squeezed inputs, the unsqueezed signal is superimposed to the squeezed fluctuations). Thus, in practice, squeezing the LO provides a sort of phase-sensitive parametric amplification [17], which enhances a phase component of the field, while it reduces the conjugated one. It turns out that the homodyne with squeezed LO is essentially equivalent to the homodyne with coherent LO, where the homodyne amplification is shared by both the signal and the squeezing components of the LO power. In order to take advantage of the squeezed LO, a displacement of the input matrix $\hat{\rho}_{in}$ to $D[(1-\mu)\alpha+\nu\bar{\alpha}]\hat{\rho}_{in}D^{\dagger}[(1-\mu)\alpha+\nu\bar{\alpha}]$ is needed ($\alpha=\langle\alpha\rangle_{in}$ is the input signal and $D(w)$ is the usual displacement operator). However, such kind of displacement is equivalent to a sort of amplification with constant noise that would provide the improvement by itself.

Finally one can note that a squeezed LO does not improve the fit of the statistics of $\hat{\gamma}_D$ and $\hat{x}$. In fact, as can be seen from eqs. (31)-(34), the leading corrections to the homodyne statistics would be generally enhanced by squeezing, even in the case that the squeezing transformation of $\hat{\rho}_{in}$ is used to amplify $\hat{x}\to\exp[\rho]\hat{x}$ ($|\mu|=\cosh\rho$). This conclusion also follows from direct inspection of eq. (29), where the asymmetric form of the correcting terms produces amplification of spurious statistics after any squeezing transformation.

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[12] P. D. Drummond and C. W. Gardiner: *Phys. Rev. A*, 13, 2353 (1980).
[13] H. P. Yuen: *Phys. Rev. A*, 13, 2226 (1976) ($\hat{S}_a(\mu,\nu)$ is the evolution operator of a Hamitonian $\hat{H}$ which is a quadratic function of a and $a^{\dagger}$. In the simplest case of μ real and $\hat{H}$ constant as a function of time $\hat{S}_a(\mu,\nu)$ is the usual squeezing operator $\hat{S}_a(\mu,\nu)= =\exp[(1/2)\zeta a^{\dagger 2}-(1/2)\bar{\zeta}a^2]$, where $\mu=\cosh|\zeta|$, $\nu=(\zeta/|\zeta|)\sinh|\zeta|$).
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