

Imprinting Complete Information about a Quantum Channel on its Output State

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We introduce a novel property of bipartite quantum states, which we call *faithfulness*, and we say that a state is faithful when acting with a channel on one of the two quantum systems; the output state carries complete information about the channel. The concept of faithfulness can also be extended to sets of states, when the output states altogether carry a complete imprinting of the channel. Measures of degrees of faithfulness are proposed.

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When a quantum system enters a quantum channel, its state transforms according to a linear, trace-preserving, and completely positive (CP) map \mathcal{E} [1]. The output corresponding to an input state ρ can be written in a so-called Kraus form [2]

$$\rho \rightarrow \mathcal{E}(\rho) = \sum_n K_n \rho K_n^\dagger, \quad (1)$$

where K_n are operators on the Hilbert space \mathbf{H} of the quantum system, and satisfy the completeness relation $\sum_n K_n^\dagger K_n = I$, in order to preserve the trace of ρ .

It is natural now to pose the question: is it possible to recover the channel \mathcal{E} from the output state $\mathcal{E}(\rho)$? Moreover, which sets of input states support a complete imprinting of the channel \mathcal{E} on their respective outputs? By using such inputs, and then performing quantum tomography of their outputs, one will be able to recover the full channel. Therefore, these states will be the key element of any possible characterization or diagnostic tool for quantum devices.

The easiest answer to the previous question is represented by the “tomographically complete” set of input states proposed for the *quantum process tomography* [3,4]. A more efficient solution is provided by just a single maximally entangled state [5,6], which supports a complete imprinting of the channel, playing the role of all possible input states in a *quantum parallel* fashion. This also overcomes the practical infeasibility of a tomographically complete input set in quantum optics when states with many photons are involved.

The aim of this Letter is to fill the gap between the two above possibilities, characterizing any set of input states—either bipartite or not—that can be used to extract all the information about a quantum channel. In other words, we want to establish which sets of input states have outputs in one-to-one correspondence with the channel. This property of the set of states will be named *faithfulness*. As we will see in the following, a surprising result is that even a mixed separable state can be faithful, or in other terms, that even a classically correlated state can retain a perfect imprinting of a quantum channel. Clearly, a single maximally entangled

state will be much more efficient in recovering the channel than those mixed states—i.e., fewer measurements will be needed—and for this reason a quantification of *faithfulness* is in order.

In the following we will make an extensive use of the notation $|A\rangle\rangle = \sum_{ij} A_{ij} |i\rangle \otimes |j\rangle$ that sets a correspondence between the vector $|A\rangle\rangle \in \mathbf{H} \otimes \mathbf{H}$ and the operator $A \in \mathbf{B}(\mathbf{H})$, for a fixed orthonormal basis $\{|i\rangle \otimes |j\rangle\}$. With this notation and the Kraus decomposition in Eq. (1), it is easy to show [7] the following well-known mathematical result [8]: the operator $S_{\mathcal{E}}$ on $\mathbf{H} \otimes \mathbf{H}$ defined as

$$S_{\mathcal{E}} \doteq \mathcal{E} \otimes I(|I\rangle\rangle\langle\langle I|) \quad (2)$$

is in one-to-one correspondence with the quantum channel \mathcal{E} , the inverse relation being

$$\mathcal{E}(\rho) = \text{Tr}_2[(I \otimes \rho^\tau) S_{\mathcal{E}}], \quad (3)$$

where I is the identity map, and ρ^τ denotes transposition of the operator ρ with respect to the same basis used before to introduce our notation.

This result was the key feature of the method presented in [5]. In fact, by using a bipartite pure input state $|A\rangle\rangle = A \otimes |I\rangle\rangle = I \otimes A^\tau |I\rangle\rangle$ and letting the channel act only on the first subsystem, the output state is

$$\mathcal{E} \otimes I(|A\rangle\rangle\langle\langle A|) = (I \otimes A^\tau) S_{\mathcal{E}} (I \otimes A^*), \quad (4)$$

so that it is possible to recover $S_{\mathcal{E}}$, whence \mathcal{E} , from this output, given that A is invertible. Therefore, $|A\rangle\rangle$ is faithful iff A is full rank, i.e., iff $|A\rangle\rangle$ has maximal Schmidt number.

Is it possible to keep the correspondence between output state $R_{\mathcal{E}}$ and channel \mathcal{E} one to one by using a generic bipartite input state R on $\mathbf{H} \otimes \mathbf{H}$, as described in Fig. 1?

Using the spectral decomposition $R = \sum_l |A_l\rangle\rangle\langle\langle A_l|$ for the input state R , we can write

$$\begin{aligned} R_{\mathcal{E}} &= \mathcal{E} \otimes I(R) = \sum_l (I \otimes A_l^\tau) S_{\mathcal{E}} (I \otimes A_l^*) \\ &= \mathcal{E} \otimes \mathcal{R}(|I\rangle\rangle\langle\langle I|) \equiv I \otimes \mathcal{R}(S_{\mathcal{E}}), \end{aligned} \quad (5)$$

where \mathcal{R} is the completely positive map whose action on

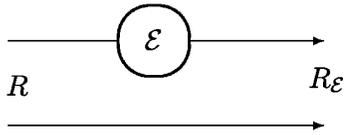


FIG. 1. The input state R is called *faithful* when the correspondence between the output state $R_E \doteq \mathcal{E} \otimes I(R)$ and the quantum channel \mathcal{E} is one to one, namely, the output state R_E carries a complete imprinting of the quantum channel \mathcal{E} .

an operator M on H is defined by

$$\mathcal{R}(M) = \sum_l A_l^\tau M A_l^* \tag{6}$$

and A^* denotes the complex conjugated of the operator A with respect to the chosen basis. As illustrated in Fig. 2, from Eqs. (2) and (5) it follows that whenever the map \mathcal{R} is invertible the output state R_E will be in one-to-one correspondence with S_E , and thus with the channel \mathcal{E} , so that the *faithfulness of R is equivalent to the invertibility of the map \mathcal{R}* .

The invertibility of the CP map \mathcal{R} resorts to the invertibility of a customary operator, by considering the following equation involving vectors in $H \otimes H$:

$$|\mathcal{R}(M)\rangle\rangle = \left| \sum_l A_l^\tau M A_l^* \right\rangle\rangle = \left(\sum_l A_l^\tau \otimes A_l^\dagger \right) |M\rangle\rangle. \tag{7}$$

In fact, the map \mathcal{R} is invertible iff the relation $|\mathcal{R}(M)\rangle\rangle \leftrightarrow |M\rangle\rangle$ is invertible, and looking at the above equation it is clear that this happens iff the operator $\check{R} \doteq \sum_l A_l^\tau \otimes A_l^\dagger$ on $H \otimes H$ is invertible. In fact, in this case the action of the inverse map \mathcal{R}^{-1} on an operator M can be defined through the relation

$$|\mathcal{R}^{-1}(M)\rangle\rangle \doteq \check{R}^{-1} |M\rangle\rangle, \tag{8}$$

so that $|\mathcal{R}^{-1}(\mathcal{R}(M))\rangle\rangle = \check{R}^{-1} \check{R} |M\rangle\rangle = |M\rangle\rangle$. The operator \check{R} can be calculated directly from R as

$$\check{R} = (ER)^{\tau_2} E = (R^{\tau_2} E)^{\tau_1}, \tag{9}$$

where $E = \sum_{ij} |ij\rangle\langle ji|$ is the swap operator, and O^{τ_l} denotes the partial transposition of the operator O on the l th Hilbert space. Notice that the correspondence $\mathcal{R} \leftrightarrow \check{R}$ preserves the multiplication of maps, as $\mathcal{A}\mathcal{B} \leftrightarrow \check{A}\check{B}$. Briefly, we have found that R is faithful iff \check{R} is invertible. In this case the relation $R_E = \mathcal{E} \otimes I(R) \leftrightarrow S_E$ is one to

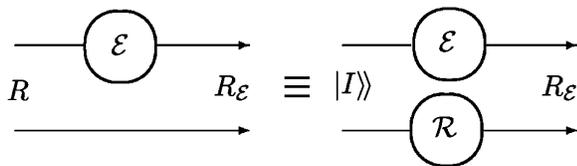


FIG. 2. A generally mixed state $R = \sum_l |A_l\rangle\rangle\langle\langle A_l|$ is faithful when the map $\mathcal{R}(M) = \sum_l A_l^\tau M A_l^*$ is invertible, in order to guarantee the one-to-one correspondence between R_E and \mathcal{E} .

one, and the channel \mathcal{E} can be recovered from R_E as follows:

$$\mathcal{E}(\rho) = \text{Tr}_2[(I \otimes \rho^\tau) I \otimes \mathcal{R}^{-1}(R_E)]. \tag{10}$$

We now discuss the faithfulness of the bipartite state R of two quantum systems described by different Hilbert spaces H and K . We need now to consider vectors in either $H \otimes K$, $H^{\otimes 2}$, or $K^{\otimes 2}$, and in all cases we will keep our notation $|A\rangle\rangle$ with the operator A in $B(K, H)$, $B(H)$, or $B(K)$, respectively, $B(K, H)$ denoting the linear space of bounded operators from K to H and $B(H) \equiv B(H, H)$ and analogously $B(K)$. Similarly to the previous reasoning lines, corresponding to the input state $R = \sum_l |A_l\rangle\rangle\langle\langle A_l|$ on $H \otimes K$ we have now the map $\mathcal{R}(M) = \sum_l A_l^\tau M A_l^*$ from $B(H)$ to $B(K)$. Then, faithfulness of R now is more generally equivalent to *left* invertibility of the map \mathcal{R} . The operator $\check{R} = \sum_l A_l^\tau \otimes A_l^\dagger$ now maps vectors in $H^{\otimes 2}$ to vectors in $K^{\otimes 2}$, and is still such that $\check{R}|M\rangle\rangle = |\mathcal{R}(M)\rangle\rangle$. Therefore, faithfulness of R is equivalent to left invertibility of the operator \check{R} .

We are now in a position to consider the most general situation in which we have a set of bipartite states $\{R^{(n)}\}_1^N$ on $H \otimes K$. When do they support a complete imprinting of the quantum channel that evolves them? The easiest answer is to say that the set $\{R^{(n)}\}_1^N$ on $H \otimes K$ is faithful iff the following state R_{set} on $H \otimes K \otimes \mathbb{C}^N$ is faithful:

$$R_{\text{set}} = \sum_{n=1}^N p_n R^{(n)} \otimes |n\rangle\langle n|, \tag{11}$$

where p_n are nonvanishing probabilities. In fact, by measuring the observable corresponding to the basis $\{|n\rangle\rangle$ on the space \mathbb{C}^N one has $R_E^{(n)}$ as a reduced state on $H \otimes K$, and the output state $\sum_{n=1}^N p_n R_E^{(n)} \otimes |n\rangle\langle n|$ contains the same information of the set of outputs $\{R_E^{(n)}\}_1^N$. In this way the evaluation of faithfulness of the set $\{R^{(n)}\}_1^N$ is reduced to that of a single state R_{set} , according to our previous approach.

Can we also use unfaithful states? An unfaithful state R can still be useful in recovering only some quantum channels or at least in reconstructing their action on some particular states. In fact, when the map \mathcal{R} is not invertible one has $\mathcal{R}(M) = 0$ for all vectors $|M\rangle\rangle \in \text{Ker}(\check{R})$, where $\text{Ker}(O)$ denotes the kernel of the operator O . Now, one can use the Moore-Penrose pseudoinverse \check{R}^\ddagger [9], which allows inversion only in $\text{Ker}^\perp(\check{R})$, with $\check{R}^\ddagger \check{R} = \check{Q}$, \check{Q} being the orthogonal projector on $\text{Ker}^\perp(\check{R})$. Correspondingly, one defines the pseudoinverse map \mathcal{R}^\ddagger through the identity $|\mathcal{R}^\ddagger(M)\rangle\rangle \doteq \check{R}^\ddagger |M\rangle\rangle$. It is clear that pseudoinversion, instead of the full S_E , will give its projection

$$\tilde{S}_E = I \otimes \mathcal{R}^\ddagger(R_E) = I \otimes \mathcal{Q}(S_E), \tag{12}$$

where $\mathcal{Q} = \mathcal{R}^\ddagger \mathcal{R} = \mathcal{Q}^2$ is an orthogonal projection map on $B(H)$, also defined as $|\mathcal{Q}(M)\rangle\rangle = \check{Q}|M\rangle\rangle$. The partially recovered map $\tilde{\mathcal{E}}(\rho) = \text{Tr}_2[I \otimes \rho^\tau \tilde{S}_E]$ is generally not CP, and can also be written as $\tilde{\mathcal{E}} = \mathcal{E} \mathcal{Q}^*$, \mathcal{Q}^* being the map

corresponding to the operator \check{Q}^* . Clearly $\check{\mathcal{E}}$ coincides with \mathcal{E} for any ρ such that $\check{Q}^*|\rho\rangle\rangle = |\rho\rangle\rangle$.

For any bipartite R one can define a *number of faithfulness* φ as $\varphi(R) = \text{rank}(\check{R})$, i.e., as the dimension of the space of states for which the reconstruction of the action of \mathcal{E} is possible. Clearly, a state is faithful for $\varphi(R) = \dim(\mathbf{H})^2$. Notice that for $\varphi(R) < \dim(\mathbf{H})^2$ one can have the situation in which $\text{Ker}^\perp(\check{R}) = \text{Span}\{|M\rangle\rangle, M \text{ commuting}\}$, in which case the state R allows one to reconstruct completely only ‘‘classical’’ channels, with the input restricted to an Abelian algebra of states.

The introduction of pseudoinversion provides an alternative yet equivalent way for studying the faithfulness of a set of states $\{R^{(n)}\}$. Suppose they lead to the projection maps $\{Q^{(n)}\}$, then the set will be faithful iff we can recover any operator $M \in \mathbf{B}(\mathbf{H})$ from its projections $Q^{(n)}(M)$, and this is possible iff, given a basis $\{B_j\}$ for $\mathbf{B}(\mathbf{H})$, one has $\text{Span}\{Q^{(n)}(B_i)\}_{i,n} = \mathbf{B}(\mathbf{H})$. In such circumstances, any element of the basis can be expressed as a linear combination of the $Q^{(n)}(B_i)$, i.e., $B_i = \sum_{jn} \lambda_{ij}^n Q^{(n)}(B_j)$, and therefore it is possible to recover $M \equiv \sum_i \text{Tr}[B_i^\dagger M] B_i$ by ‘‘patching’’ the projections $Q^{(n)}(M)$ as

$$M = \sum_{ijn} \lambda_{ij}^{n*} \text{Tr}[B_j^\dagger Q^{(n)}(M)] B_i. \quad (13)$$

Analogously, by patching the partially recovered $\{\check{S}_\mathcal{E}^{(n)}\}$ [see Eq. (12)] we get $S_\mathcal{E}$ as

$$S_\mathcal{E} = \sum_{ijn} \lambda_{ij}^{n*} \text{Tr}_2[(I \otimes B_j^\dagger) \check{S}_\mathcal{E}^{(n)}] \otimes B_i. \quad (14)$$

Of course this patching procedure can also be used with an unfaithful set of states to obtain a partial recovery of the channel.

In summary, we have completely characterized any faithful set of states and explained how to recover a channel \mathcal{E} from its outputs. In the following we apply our theoretical framework to some examples. As a general rule, from the above consideration, it is clear that the set of faithful states R is *dense* within the set of all bipartite states. Therefore, there must be faithful states among mixed separable ones. For example, the Werner’s states for dimension d

$$R_f = \frac{1}{d(d^2 - 1)} [(d - f)I + (df - 1)E], \quad -1 \leq f \leq 1, \quad (15)$$

are separable for $f \geq 0$, however, they are faithful for all $f \neq \frac{1}{d}$, since one has $(ER_f)^{\tau_2} = [1/d(d^2 - 1)] \times [(d - f)|I\rangle\rangle\langle\langle I| + (df - 1)]$, and the singular values of \check{R}_f are $[df - 1/d(d^2 - 1)]$ and $\frac{1}{d}$. Similarly, the ‘‘isotropic’’ states

$$R_f = \frac{f}{d}|I\rangle\rangle\langle\langle I| + \frac{1-f}{d^2-1} \left(I - \frac{1}{d}|I\rangle\rangle\langle\langle I| \right), \quad (16)$$

are faithful for $f \neq \frac{1}{d^2}$ and separable for $f \leq \frac{1}{d}$, the sin-

gular values of \check{R}_f being $[d^2f - 1/d(d^2 - 1)]$ and $\frac{f}{d}$. These examples show that classical correlations in mixed bipartite states are sufficient to support the imprinting of the quantum channel.

For infinite dimensions (e.g., for ‘‘continuous variables’’), one needs to restrict $\mathbf{B}(\mathbf{H})$ to the Hilbert space of Hilbert-Schmidt operators on \mathbf{H} , which leads to the problem that the inverse map \mathcal{R}^{-1} is unbounded. The result is that we will recover the channel \mathcal{E} from the measured $R_\mathcal{E}$, however, with unbounded amplification of statistical errors, depending on the chosen complete set of operators $B = \{B_j\}$ in $\mathbf{B}(\mathbf{H})$ used for representing the channel map. As an example, let us consider a twin beam from parametric down-conversion of vacuum

$$|\Psi\rangle\rangle = \Psi \otimes I|I\rangle\rangle, \quad \Psi = (1 - |\xi|^2)^{1/2} \xi^{a^\dagger a}, \quad |\xi| < 1 \quad (17)$$

for a fixed ξ , a^\dagger , and a , with $[a, a^\dagger] = 1$ denoting the creation and annihilation operators of the harmonic oscillator describing the field mode corresponding to the first Hilbert space in the tensor product (in the following we will denote by b^\dagger and b the creation and annihilation operators of the other field mode). The state is faithful, but the operator Ψ^{-1} is unbounded, whence the inverse map \mathcal{R}^{-1} is also unbounded. In a photon number representation $\mathbf{B} = \{|n\rangle\langle m|\}$, the effect will be an amplification of errors for increasing numbers n, m of photons.

Consider now the quantum channel describing the *Gaussian displacement noise* [10]

$$\mathcal{N}_\nu(\rho) = \int_{\mathbb{C}} \frac{d\alpha}{\pi\nu} \exp[-|\alpha|^2/\nu] D(\alpha) \rho D^\dagger(\alpha), \quad (18)$$

where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ denotes the usual displacement operator on the phase space. The Gaussian noise is in a sense the analogous of the depolarizing channel for infinite dimension. The maps \mathcal{N}_ν for varying ν satisfy the multiplication rule $\mathcal{N}_\nu \mathcal{N}_\mu = \mathcal{N}_{\nu+\mu}$, whence the inverse map is formally given by $\mathcal{N}_\nu^{-1} \equiv \mathcal{N}_{-\nu}$. Notice that, since the map \mathcal{N}_ν is compact, the inverse map \mathcal{N}_ν^{-1} is necessarily unbounded. As a faithful state consider now the mixed state given by the twin beam, with one beam spoiled by the Gaussian noise, namely,

$$R = I \otimes \mathcal{N}_\nu(|\Psi\rangle\rangle\langle\langle \Psi|). \quad (19)$$

A lengthy straightforward calculation gives the state

$$R = \frac{1}{\nu} (\Psi \otimes I) \exp[-(a - b^\dagger)(a^\dagger - b)/\nu] (\Psi^\dagger \otimes I), \quad (20)$$

and its partial transposed

$$R^{\tau_2} = (\nu + 1)^{-1} (\Psi \otimes I) \left(\frac{\nu - 1}{\nu + 1} \right)^{(1/2)(a-b)^\dagger(a-b)} (\Psi^\dagger \otimes I), \quad (21)$$

where transposition is defined with respect to the basis of

eigenvectors of $a^\dagger a$ and $b^\dagger b$. Since our state is Gaussian, the PPT criterion guarantees separability [11], and for $\nu > 1$ our state (20) is separable (see also Ref. [12]), still it is *formally* faithful, since the operator Ψ and the map N_ν are both invertible. Notice that unboundedness of the map inversion can even wash out completely the information on the channel in some particular chosen representation $\mathbf{B} = \{B_j\}$, e.g., when all operators B_j are out of the boundedness domain of \mathcal{R}^{-1} . This is the case, for example, of the (overcomplete) representation $\mathbf{B} = \{|\alpha\rangle\langle\beta|\}$, with $|\alpha\rangle$ and $|\beta\rangle$ coherent states, since from the identity

$$\mathcal{N}_\nu(|\alpha\rangle\langle\alpha|) = \frac{1}{\nu+1} D(\alpha) \left(\frac{\nu}{\nu+1}\right)^{a^\dagger a} D^\dagger(\alpha), \quad (22)$$

one obtains

$$\mathcal{N}_\nu^{-1}(|\alpha\rangle\langle\alpha|) = \frac{1}{1-\nu} D(\alpha) (1-\nu^{-1})^{-a^\dagger a} D^\dagger(\alpha), \quad (23)$$

which has convergence radius $\nu \leq \frac{1}{2}$, which is the well-known bound for Gaussian noise for the quantum tomographic reconstruction for coherent-state and Fock representations [13]. Therefore, we say that the state is *formally* faithful, however, we are constrained to representations which are analytical for the inverse map \mathcal{R}^{-1} .

Now, let us consider the problem of how to define a measure of faithfulness $F(R)$ of the state R . Even though in principle any faithful state can be used to perform a tomography of the channel \mathcal{E} , the experimental errors on the measured $R_{\mathcal{E}}$ are propagated to \mathcal{E} by the inversion of the map \mathcal{R} . Thus different faithful input states can produce very different errors on the measured channel. It is clear that all the features producing the amplification of errors are contained in the singular values σ_l of \check{R} , since the inversion of this operator involves multiplications by σ_l^{-1} . From one point of view, it is unpractical to have a universal measure for faithfulness, since its actual definition will be dictated by the goodness criterion adopted for the reconstruction of the quantum channel \mathcal{E} . On the other hand, one can give an overall performance indicator for a given faithful state R , such as the quantity $F(R) = \sqrt{\sum_l \sigma_l^2}$. Since $F(R)^2 = \|\check{R}\|_2^2 = \text{Tr}[\check{R}^\dagger \check{R}] = \text{Tr}[R^\dagger R]$, such measure of faithfulness coincides with the purity of the state R , and this shows that the maximally faithful states are pure states with maximal Schmidt number. The definitions of F and φ can be naturally extended to sets of states $\{R^{(n)}\}$ via the introduction of the joint state R_{set} in Eq. (11), the probabilities representing the frequency in using each input.

In conclusion, in this Letter we have introduced a new feature of bipartite quantum states, which we call *faithfulness*, corresponding to the ability of the state of carrying the complete imprinting of a channel acting on one of

the pairs of quantum systems. This property has also been extended to sets of bipartite states, when the channel can be recovered from the corresponding output states patched together. We have seen that there are separable states that are faithful, and the maximally faithful states are the maximally entangled pure states. We want to stress that the property of being faithful is a strictly quantum feature, since a faithful state cannot be written as the mixture of local classical (i.e., commuting) states. This also shows how subtle is the game between the classical and quantum natures in the correlations of a general mixed quantum state.

After the submission of the present Letter, a related experimental paper appeared in Ref. [14].

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- [1] A map \mathcal{E} is completely positive if it preserves positivity when acting locally on a bipartite state. In other words, upon denoting by I the identical map on the Hilbert space K of a second quantum system, the extended map $\mathcal{E} \otimes I$ on $H \otimes K$ must be positive for any extension K .
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