Optimal time reversal of multiphase equatorial states

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Even though the time reversal is unphysical (it corresponds to the complex conjugation of the density matrix), for some restricted set of states it can be achieved unitarily, typically when there is a common dephasing in a n-level system. However, in the presence of multiple phases (i.e., a different dephasing for each element of an orthogonal basis occurs) the time reversal is no longer physically possible. In this paper we derive the channel which optimally approaches in fidelity the time reversal of multiphase equatorial states in arbitrary (finite) dimension. We show that, in contrast to the customary case of the universal-NOT on qubits (or the universal conjugation in arbitrary dimension), the optimal phase covariant time reversal for equatorial states is a nonclassical channel, which cannot be achieved via a measurement-and-preparation procedure. Unitary realizations of the optimal time reversal channel are given with minimal ancillary dimension, exploiting the simplex structure of the optimal maps.

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I. INTRODUCTION

Time reversal is not a physically achievable transformation on arbitrary quantum states, since it corresponds to a positive, but not completely positive map [1]. However, if restricted to special sets of states, time reversal can be easily achieved by a unitary transformation, e.g., when generating NMR spin echoes by turning over the fixed magnetic field along the rotation axis, with all the spins rotating in the equatorial plane. For a general n-level system, this is possible only when the levels are equally spaced, corresponding to a common dephasing. However, more generally, when the levels are not equally spaced, we have different dephasing for each level, and in the presence of multiple phases the time reversal is no longer physically possible.

The ideal time reversal of a generic state corresponds to the complex conjugation (or, equivalently, transposition) of the corresponding density matrix. Such transformation has also recently attracted much interest in relation to the problem of entanglement, in regard to the so-called PPT (positive partial transpose) criterion [2,3]. Since complex conjugation cannot be achieved unitarily, one can try to approximate the transformation with a physical channel, optimizing the fidelity of the output state with the complex-conjugated input. For the set of all pure states the resulting optimal channel is "classical" [4,5], in the sense that it can be achieved by state estimation followed by state preparation. In this paper we show that for multiphase equatorial states the optimal phase covariant time reversal for equatorial states is a nonclassical channel, namely it cannot be achieved via the measurement-and-preparation procedure. We will see that the optimal channels form a simplex (i.e., a convex set which is generated by convex combination of a finite set of extremal points, e.g., a tetrahedron). Such a structure simplifies the search for unitary realizations of the channels, which will be derived in the following for minimal ancillary dimension.

The paper is organized as follows. In Sec. II we introduce the notation, and derive the optimal multiphase conjugation maps and the corresponding fidelity. In Sec. III we compare the present optimal phase-covariant maps with the universally covariant ones, and discuss their relation with optimal state estimation and phase estimation. In Sec. IV we analyze the simplex structure of the set of optimal multiphase conjugation maps, and explicitly construct their unitary realizations with minimal ancilla dimension. Section V closes the paper with some concluding remarks.

II. OPTIMAL MULTIPHASE CONJUGATION MAPS

In the following we will restrict attention to equatorial states of a d-dimensional quantum system, defined as

$$|\psi(\{\phi_j\})\rangle = \frac{1}{\sqrt{d}}(|0\rangle + e^{i\phi_1}|1\rangle + e^{i\phi_2}|2\rangle + \cdots + e^{i\phi_{d-1}}|d-1\rangle), \tag{1}$$

expanded with respect to the fixed orthonormal basis

$$\mathrm{B} \doteq \{|0\rangle,|1\rangle, \ldots ,|d-1\rangle\} \tag{2}$$

of the Hilbert space $\mathcal{H}$ of the quantum system. We consider transformations that treat all input states (1) in the same way, namely that are covariant under the group $\mathbb{U}(1)^{\times(d-1)}$ of rotations of $d-1$ independent phases $\{\phi_j\}$. The state (1) can be equivalently written as

$$|\psi(\{\phi_j\})\rangle = U(\{\phi_j\})|\psi_0\rangle, \tag{3}$$

where

$$|\psi_0\rangle = d^{-1/2}\sum_{i=0}^{d-1} |i\rangle \tag{4}$$

is a fixed real state and

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$$U(\{\phi_j\}) = |0\rangle\langle 0| + \sum_{j=1}^{d-1} e^{i\phi_j}|j\rangle\langle j| \quad (5)$$

is the generic element of $\mathbb{U}(1)^{\times(d-1)}$.

We now derive the channel—e.g., completely positive trace-preserving map—$\mathcal{T}$ which optimally approximates the (antilinear) multiphase conjugation on equatorial states, namely

$$|\psi(\{\phi_j\})\rangle = \frac{1}{\sqrt{d}}(|0\rangle + e^{i\phi_1}|1\rangle + e^{i\phi_2}|2\rangle + \cdots) \mapsto |\psi^*(\{\phi_j\})\rangle$$
$$= \frac{1}{\sqrt{d}}(|0\rangle + e^{-i\phi_1}|1\rangle + e^{-i\phi_2}|2\rangle + \cdots). \quad (6)$$

The map $\mathcal{T}$ is covariant under the multiphase $\mathbb{U}(1)^{\times(d-1)}$ group, namely

$$\mathcal{T}(U(\{\phi_j\})\rho U^\dagger(\{\phi_j\})) = U^*(\{\phi_j\})\mathcal{T}(\rho)U^T(\{\phi_j\}). \quad (7)$$

In the above equation O^* (O^T) denotes the complex conjugation (transposition) of the operator O with respect to the orthonormal basis (2) kept as real. Among all completely positive trace-preserving (CPT) maps satisfying the covariance condition (7), we single out those maps which maximize the fidelity between the output state and the ideally transformed state in Eq. (6),

$$F[|\psi^*(\{\phi_j\})\rangle, \mathcal{T}(|\psi(\{\phi_j\})\rangle\langle\psi(\{\phi_j\})|)]$$
$$= \mathrm{Tr}[|\psi^*(\{\phi_j\})\rangle\langle\psi^*(\{\phi_j\})|\mathcal{T}(|\psi(\{\phi_j\})\rangle\langle\psi(\{\phi_j\})|)]. \quad (8)$$

Following Ref. [6], we solve the optimization problem under the covariance condition (7), using the positive operator R on $\mathcal{H}\otimes\mathcal{H}$ defined as

$$R = (\mathcal{T}\otimes\mathcal{I})|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|, \quad (9)$$

where $\mathcal{I}$ is the identity channel, and $|\mathbb{1}\rangle\rangle = \Sigma_{i=0}^{d-1}|i\rangle|i\rangle$ is the maximally entangled vector on $\mathcal{H}\otimes\mathcal{H}$ relative to the orthonormal basis B in Eq. (2). The correspondence $\mathcal{T}\leftrightarrow R$ is one to one and can be inverted as

$$\mathcal{T}(\rho) = \mathrm{Tr}_{\mathcal{H}_2}[(\mathbb{1}\otimes\rho^T)R]. \quad (10)$$

In terms of the operator R the trace-preservation condition for $\mathcal{T}$ reads

$$\mathrm{Tr}_{\mathcal{H}_1}[R] = \mathbb{1} \quad (11)$$

and the covariance property of the map (7) becomes the invariance for R,

$$[R, U^*(\{\phi_j\})^{\otimes 2}] = 0. \quad (12)$$

This, in turn, via the Schur lemma implies the following block form for R,

$$R = \bigoplus_{\alpha:\text{equiv. classes}} R_\alpha. \quad (13)$$

The index α runs over the equivalence classes of irreducible one-dimensional representations of $U(\{\phi_j\})^{\otimes 2}$ (without loss of generality we suppressed the complex conjugation). The equivalence classes with the respective characters are listed in Table I.

TABLE I. Equivalence classes and respective characters of irreducible one-dimensional representations of $U(\{\phi_j\})^{\otimes 2}$.

Equivalence classes	Characters
$\|00\rangle$	1
$\|11\rangle$	$e^{2i\phi_1}$
$\vdots$	$\vdots$
$\|ii\rangle$	$e^{2i\phi_i}$
$\vdots$	$\vdots$
$\|01\rangle, \|10\rangle$	$e^{i\phi_1}$
$\vdots$	$\vdots$
$\|ij\rangle, \|ji\rangle, \quad i\neq j$	$e^{i(\phi_i+\phi_j)}, \quad i\neq j$
$\vdots$	$\vdots$

Noticing that $|\psi_0\rangle\langle\psi_0| = |\psi_0\rangle\langle\psi_0|^*$, and using Eqs. (3), (10), and (12), we can rewrite the fidelity in Eq. (8) as

$$F = \mathrm{Tr}[|\psi_0\rangle\langle\psi_0|^{\otimes 2}R]. \quad (14)$$

From Eq. (4), one has $|\psi_0\rangle\langle\psi_0|^{\otimes 2} = d^{-2}\Sigma_{i,j,k,l}|i\rangle\langle j|\otimes|k\rangle\langle l|$, and the fidelity (14) is equal to $F = d^{-2}\Sigma_\alpha\Sigma_{i,j,k,l}r^{(\alpha)}_{jl,ik}$, where $r^{(\alpha)}_{jl,ik} = \langle jl|R_\alpha|ik\rangle$. As argued in Refs. [7,8], the maximum F is then obtained when the off-diagonal terms of the operator R are positive and as large as possible, namely when the blocks R_α are rank-one [9], or in equations

$$R = \frac{1}{\mathcal{N}}\left[\sum_i c_i|ii\rangle\langle ii| + \sum_{i>j} c_{ij}(\alpha_{ij}|ij\rangle + \beta_{ij}|ji\rangle)(\alpha^*_{ij}\langle ij| + \beta^*_{ij}\langle ji|)\right], \quad (15)$$

where $\mathcal{N} = (\Sigma_i c_i + \Sigma_{i>j}c_{ij})/d$ is a normalization constant, $|\alpha_{ij}|^2 + |\beta_{ij}|^2 = 1$, and $c_i \geqslant 0$, $c_{ij}\geqslant 0$, $\forall i,j$. The fidelity then takes the form

$$F = \mathrm{Tr}[|\psi_0\rangle\langle\psi_0|^{\otimes 2}R]$$
$$= \frac{1}{d^2}\frac{d}{\sum_i c_i + \sum_{i>j} c_{ij}}$$
$$\times\left[\sum_i c_i + \sum_{i>j} c_{ij}[|\alpha_{ij}|^2 + |\beta_{ij}|^2 + 2\,\mathrm{Re}(\alpha_{ij}\beta^*_{ij})]\right]$$
$$= \frac{1}{d}\frac{1}{\sum_i c_i + \sum_{i>j} c_{ij}}\left[\sum_i c_i + \sum_{i>j} c_{ij}[1 + 2\,\mathrm{Re}(\alpha_{ij}\beta^*_{ij})]\right]$$
$$= \frac{1}{d} + 2\frac{\sum_{i>j} c_{ij}\,\mathrm{Re}(\alpha_{ij}\beta^*_{ij})}{d\left(\sum_i c_i + \sum_{i>j} c_{ij}\right)}. \quad (16)$$

Now, the maximum of $\mathrm{Re}(\alpha_{ij}\beta^*_{ij})$ is achieved when $\alpha_{ij} = \beta_{ij}$, implying $|\alpha_{ij}|^2 = 1/2$ for all i,j. Therefore we choose $\alpha_{ij} = 2^{-1/2}$. The fidelity becomes

$$F = \frac{1}{d} + \frac{\sum_{i>j} c_{ij}}{d\left(\sum_i c_i + \sum_{i>j} c_{ij}\right)}. \tag{17}$$

In order to maximize F we put $c_i=0$ for all i. The optimal fidelity takes the following simple form:

$$F = \frac{2}{d}. \tag{18}$$

Then, we impose the trace-preservation condition, obtaining

$$\mathrm{Tr}_1[R] = \frac{d}{2\sum_{i>j} c_{ij}} \sum_{i>j} c_{ij}(|i\rangle\langle i| + |j\rangle\langle j|) \doteq \sum_{i>j} b_{ij}(|i\rangle\langle i| + |j\rangle\langle j|)$$
$$\equiv \mathbb{1}, \tag{19}$$

where

$$b_{ij} = d\frac{c_{ij}}{2\sum_{i>j} c_{ij}}. \tag{20}$$

Since the projector $|0\rangle\langle 0|$, for example, appears in the sum (19) multiplied by $\Sigma_{i=1}^{d-1} b_{i0}$, the coefficients are constrained as follows:

$$\sum_{i=1}^{d-1} b_{i0} = 1, \tag{21}$$

and, similarly, for the $|j\rangle\langle j|$, element one has

$$\sum_{i=0}^{j-1} b_{ji} + \sum_{i=j+1}^{d-1} b_{ij} = 1. \tag{22}$$

Rearranging the positive coefficients $\{b_{ij}\}_{i>j}$ into a square matrix array, they define the lower-trianguar section of a square matrix. Such a matrix can be *uniquely* completed to a null-diagonal symmetric bistochastic matrix, that is, a symmetric null-diagonal matrix with non-negative entries, such that all its rows' and columns' entries sum up to 1, namely all its rows and columns are probability distributions.

Up to now the operator R is simplified as follows:

$$R = \sum_{i>j} b_{ij}(|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|), \tag{23}$$

where the coefficients b_{ij}'s, uniquely defining a map $\mathcal{T}$ achieving the optimal fidelity $2/d$, are the entries of a null-diagonal symmetric bistochastic matrix. From Eq. (23) and the reconstruction formula (10), an optimal phase covariant transposition map can then be easily expressed in the Kraus form as follows:

$$\mathcal{T}(\rho) = \sum_{i>j} b_{ij}(|i\rangle\langle j| + |j\rangle\langle i|)\rho(|i\rangle\langle j| + |j\rangle\langle i|). \tag{24}$$

Notice that the constraint (22) over $\{b_{ij}\}$ is indeed very strong: for qubits and qutrits it suffices to completely and univocally determine the map. In the case of qubits the only null-diagonal symmetric bistochastic matrix is

$$\{b_{ij}\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{25}$$

and the optimal transposition map is the unitary transformation

$$\mathcal{T}_2(\rho) = \sigma_x \rho \sigma_x \equiv \rho^*, \tag{26}$$

which clearly achieves $F=1$. For $d=3$ there is again a unique choice for a null-diagonal symmetric bistochastic matrix, which is given by

$$\{b_{ij}\} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}. \tag{27}$$

For $d \geqslant 4$ there exist many optimal maps. For example, for $d=4$ we can parametrize the family of maps by varying two positive parameters p_1 and p_2 such that $0 \leqslant p_1 + p_2 = p_{12} \leqslant 1$:

$$\{b_{ij}\} = \begin{pmatrix} 0 & p_1 & p_2 & 1-p_{12} \\ p_1 & 0 & 1-p_{12} & p_2 \\ p_2 & 1-p_{12} & 0 & p_1 \\ 1-p_{12} & p_2 & p_1 & 0 \end{pmatrix}. \tag{28}$$

III. UNIVERSAL TRANSPOSITION AND MULTIPHASE CONJUGATION

In Refs. [4,5] it was shown that the fidelity of the optimal universal transposition map is equal to the fidelity of the optimal universal pure state estimation [10,11], that for a single input copy is given by

$$F = \frac{2}{d+1}, \tag{29}$$

which is always lower than the fidelity of optimal phase covariant transposition (18), as expected. The equivalence between transposition and state estimation means that an optimal universal transposition can be achieved by optimally estimating the input state and then preparing the transposed state. In this sense the optimal universal transposition is a "classical" map.

In contrast to the universal case, the phase covariant transposition map cannot be achieved by phase estimation-and-preparation. Indeed, the fidelity of optimal multiphase estimation for a single input copy is given by [12]

$$F = \frac{2d-1}{d^2}, \tag{30}$$

which is always smaller than the optimal fidelity of the phase covariant transposition map (18). Hence the optimal phase covariant transposition is a genuinely quantum channel. The situation is particularly striking in the case of qubits, where it is possible to perfectly transpose all equatorial states, while the phase can never be measured exactly with finite resources [13].

IV. CONVEX STRUCTURE AND PHYSICAL REALIZATION OF OPTIMAL MAPS

In Sec. II we have shown that optimal multiphase conjugation maps are in one-to-one correspondence with null-diagonal symmetric bistochastic (NSB) matrices, which form a convex set. On the other hand, every bistochastic matrix is a convex combination of permutation matrices—this is the content of the Birkhoff theorem [14]. The null-diagonal and symmetry constraints, however, force the convex set of NSB matrices to be strictly contained into the convex polyhedron of bistochastic matrices. This fact causes the extremal NSB matrices to eventually lie strictly inside the set of bistochastic matrices, generally preventing them from being permutations.

The geometrical study of the set of NSB matrices and its extremal points can shed some light on the unusual feature that there exist different "equally optimal" maps. The problem arises for dimension at least $d=4$. In this case the decomposition of the matrix $\{b_{ij}\}$ in Eq. (28) into extremal components is

$$\{b_{ij}\} = p_1\begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} + p_2\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} + p_3\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix} = p_1P^{(1)} + p_2P^{(2)} + p_3P^{(3)}, \qquad (31)$$

where $p_1,p_2,p_3 \geqslant 0$ and $p_1+p_2+p_3=1$. A natural question is now which optimal maps can be achieved with minimal resources?

In order to physically realize a given CPT map $\mathcal{E}$, one needs to design a specific unitary interaction U and prepare an ancilla in a specific state, say $|0\rangle\langle 0|_a$, in such a way that

$$\mathcal{E}(\rho) = \mathrm{Tr}_a[U(\rho \otimes |0\rangle\langle 0|_a)U^\dagger]. \qquad (32)$$

This is always possible [1,15]. The existence of equivalently optimal maps allows us to choose between realizations with either a smaller ancilla dimension, or a more flexible ancilla state preparation.

More explicitly, for $d=4$, we define three unitaries U_1, U_2, and U_3 on $\mathbb{C}^4 \otimes \mathbb{C}^2$ as

$$U_1 \doteq \begin{pmatrix} T_{10} & T_{32}\\ T_{32} & T_{10}\end{pmatrix}, \quad U_2 \doteq \begin{pmatrix} T_{20} & T_{31}\\ T_{31} & T_{20}\end{pmatrix}, \quad U_3 \doteq \begin{pmatrix} T_{30} & T_{21}\\ T_{21} & T_{30}\end{pmatrix}, \qquad (33)$$

where $T_{ij}=|i\rangle\langle j|+|j\rangle\langle i|$. Each of them realizes an extremal optimal multiphase conjugation map [corresponding to $p_k=1$ in Eq. (31) for a given k], namely

$$\mathcal{T}_4^{(k)}(\rho) = \sum_{i>j} P_{ij}^{(k)} T_{ij}\rho T_{ij} = \mathrm{Tr}_a[U_k(\rho \otimes |0\rangle\langle 0|_a)U_k^\dagger], \qquad (34)$$

where $|0\rangle\langle 0|_a$ is a fixed qubit ancilla state. Notice that the ancilla must not necessarily be in a pure state, and the optimal map is equivalently achieved for diagonal mixed ancilla state $\alpha|0\rangle\langle 0|_a+\beta|1\rangle\langle 1|_a$. By adding a control qutrit, we can now choose among any of the optimal maps using the controlled-unitary operator on $\mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$,

$$U = U_1 \otimes |0\rangle\langle 0| + U_2 \otimes |1\rangle\langle 1| + U_3 \otimes |2\rangle\langle 2|. \qquad (35)$$

Any optimal multiphase conjugation map can now be written as

$$\mathcal{T}_4(\rho) = \mathrm{Tr}_{a,b}[U(\rho \otimes |0\rangle\langle 0|_a \otimes \sigma_b)U^\dagger], \qquad (36)$$

where σ_b is a generic density matrix on $\mathbb{C}^3$. By superimposing or mixing the three orthogonal states $\{|0\rangle, |1\rangle, |2\rangle\}$ of the qutrit we control the weights p_1,p_2,p_3 in Eq. (31) via the diagonal entries of σ_b.

Equations (33)–(36) can be generalized for higher even dimensions [16], with

$$U_k \doteq \sum_{i,j=0}^{d/2-1} T_{k\oplus 2i\oplus 2j, 2i\oplus 2j} \otimes |i\rangle\langle j|, \quad k=1,\ldots,d-1,$$

$$U = \sum_{k=1}^{d-1} U_k \otimes |k\rangle\langle k|,$$

$$\mathcal{T}_d^{(k)}(\rho) = \mathrm{Tr}_a[U_k(\rho \otimes |0\rangle\langle 0|_a)U_k^\dagger],$$

$$\mathcal{T}_d(\rho) = \mathrm{Tr}_{a,b}[U(\rho \otimes |0\rangle\langle 0|_a \otimes \sigma_b)U^\dagger], \qquad (37)$$

where U_k's are unitary operators acting on $\mathbb{C}^d \otimes \mathbb{C}^{d/2}$, U is a control-unitary operator on $\mathbb{C}^d \otimes \mathbb{C}^{d/2} \otimes \mathbb{C}^{d-1}$, $|0\rangle\langle 0|_a$ is a fixed $(d/2)$-dimensional pure state, and σ_b is a generic $(d-1)$-dimensional density matrix. The minimum dimension of the ancilla space required to unitarily realize an optimal phase covariant transposition map is $d/2$, generalizing the result for $d=4$, for which just a qubit is needed [see Eq. (34)]. Notice that realization of phase covariant transposition generally needs much less resources than realization of universal transposition: the dimension $d/2$ of the ancilla space in the phase covariant case has to be compared with the dimension d^2 required in the universal case [5].

As a final remark, notice that unitary realizations of CPT channels are far from being uniquely determined [15]: here we chose the controlled unitary structure because of its clear geometrical interpretation in connection with the convex structure of the polyhedron of optimal maps.

V. CONCLUSIONS

In this paper we have derived the channel which optimally approaches in fidelity the time reversal of multiphase equatorial states in arbitrary (finite) dimension. We show that, in contrast to the customary case of the universal-NOT on qubits (or the universal conjugation in arbitrary dimension), the optimal phase covariant time reversal for equatorial states is a nonclassical channel, which cannot be achieved via a measurement-and-preparation procedure. We have given unitary realizations of the optimal time reversal channel with minimal ancillary dimension, exploiting the simplex structure of the optimal maps. The optimal channels are related to null-diagonal symmetric bistochastic matrices. For $d \geqslant 4$ this gives a simplex structure of equivalently optimal maps.

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