

Quantum Information Becomes Classical When Distributed to Many Users

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Any physical transformation that equally distributes quantum information over a large number M of users can be approximated by a classical broadcasting of measurement outcomes. The accuracy of the approximation is at least of the order $O(M^{-1})$. In particular, quantum cloning of pure and mixed states can be approximated via quantum state estimation. As an example, for optimal qubit cloning with 10 output copies, a single user has an error probability $p_{\text{err}} \geq 0.45$ in distinguishing classical from quantum output, a value close to the error probability of the random guess.

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Differently from classical information, which can be perfectly read out and copied, quantum information cannot, because nonorthogonal quantum states can be neither perfectly distinguished [1], nor perfectly copied [2]. Because ideal distribution of quantum information is impossible, one is then interested in the performance limits of optimal distribution, and such interest has focused much attention in the literature to the problem of optimal cloning [3]. Optimal cloning consists in finding the physical transformation that converts N copies of a pure state, randomly drawn from a given set, into the best possible approximation of $M \geq N$ copies of the same state. More recently, the analogous problem for mixed states (optimal broadcasting) has been considered [4]. In both cases of pure and mixed states, the optimal transformation requires a coherent interaction of the input systems with a set of ancillae. On the other hand, classical incoherent schemes, such as the measure-and-prepare, where the N initial copies are measured and M copies of an estimated state are prepared, are suboptimal for any finite M .

When cloning pure states, the measure-and-prepare scheme becomes optimal in the asymptotic limit $M \rightarrow \infty$ in all known kinds of cloning. This leads to conjecture that pure state cloning is asymptotically equivalent to quantum state estimation [5,6], a conjecture recently proved in Ref. [7]. Essentially, the line of proof is that a symmetric cloning transformation with $M = \infty$, when restricted to single clones, must be an entanglement breaking channel, whence it can be realized by the measure-and-prepare scheme [8]. Such an argument, however, does not provide any estimate of the goodness of the classical scheme for finite number M of output copies, the situation of interest for applications and experiments.

In this Letter we analyze the general class of quantum channels that equally distribute quantum information to M users, producing output states that are invariant under permutations. This class contains cloning as a special case. We will show that for M sufficiently large any channel of the class can be efficiently approximated by a classical measure-and-prepare channel. Indeed, we will show that from the point of view of single users the states

produced by the quantum and by the classical channels are almost indistinguishable, with probability of error approaching the random guess value $1/2$ at a rate at least α/M , α constant. More generally, for any group of k users, the coherent and the incoherent schemes produce the same reduced state within an accuracy $k\alpha/M$. This also implies that entanglement between the output copies asymptotically disappears at any given order k : for large M only the k -partite entanglement with $k = O(M)$ can survive. The scaling M^{-1} is a general upper bound holding for all physical transformations that equally distribute quantum information among M users, including pure state cloning and mixed state broadcasting. Of course for specific transformations the actual scaling can be even faster.

The mathematical description of a quantum channel that transforms states on the Hilbert space \mathcal{H}_{in} into states on the Hilbert space \mathcal{H}_{out} is provided by a completely positive trace-preserving map \mathcal{E} . Because here we focus on channels that distribute quantum information to M users, we have $\mathcal{H}_{\text{out}} = \mathcal{H}^{\otimes M}$, with \mathcal{H} denoting the single user's Hilbert space. Moreover, because we require the information to be equally distributed among all users, for any input state ρ on \mathcal{H}_{in} the state $\mathcal{E}(\rho)$ must be invariant under permutations of the M output spaces. Invariance under permutations implies that any group of k users will receive the same state

$$\rho_{\text{out}}^{(k)} = \text{Tr}_{M-k}[\mathcal{E}(\rho)], \quad (1)$$

Tr_n denoting partial trace over n output spaces, no matter which ones. In particular, each single user receives the same state $\rho_{\text{out}}^{(1)} = \text{Tr}_{M-1}[\mathcal{E}(\rho)]$. In the following, we will name a channel with the above properties a channel for symmetric distribution of information (SDI channel, for short). Our goal will be to approximate any SDI channel \mathcal{E} with a classical channel $\tilde{\mathcal{E}}$, corresponding to measure the input and broadcast the measurement outcome, with each user preparing locally the same state accordingly. Such channels have the special form

$$\tilde{\mathcal{E}}(\rho) = \sum_I \text{Tr}[P_I \rho] \rho_i^{\otimes M}, \quad (2)$$

where the operators $\{P_i\}$ represent the quantum measurement performed on the input ($P_i \geq 0$, $\sum_i P_i = \mathbb{1}_{\text{in}}$), and ρ_i is the state prepared conditionally to the outcome i . The accuracy of the approximation is given by the trace-norm distance $\|\rho_{\text{out}}^{(1)} - \tilde{\rho}_{\text{out}}^{(1)}\| = \text{Tr}|\rho_{\text{out}}^{(1)} - \tilde{\rho}_{\text{out}}^{(1)}|$ between the single user output states. The trace-norm distance governs the distinguishability of states [1], namely, the minimum error probability p_{err} in distinguishing between two equally probable states ρ_1 and ρ_2 is given by

$$p_{\text{err}} = \frac{1}{2} - \frac{1}{4}\|\rho_1 - \rho_2\|_1, \quad (3)$$

and for small distances it approaches the random guess value $p_{\text{err}} = 1/2$. In our case, a small distance $\|\rho_{\text{out}}^{(1)} - \tilde{\rho}_{\text{out}}^{(1)}\|_1$ means that a single user has little chance of distinguishing between the outputs of the two channels \mathcal{E} and $\tilde{\mathcal{E}}$ by any measurement on his local state. In addition, to discuss the multipartite entanglement in the state $\rho_{\text{out}}^{(k)}$, we will consider the distance $\|\rho_{\text{out}}^{(k)} - \tilde{\rho}_{\text{out}}^{(k)}\|_1$. Since the state $\tilde{\rho}_{\text{out}}^{(k)}$ coming from $\tilde{\mathcal{E}}$ in Eq. (2) is separable, a small distance means that any group of k users has little chance of detecting entanglement.

The key idea of this Letter is to get the approximation of SDI channels exploiting the invariance of their output states under permutations. In fact, permutationally invariant states have been thoroughly studied in the research about quantum de Finetti theorem [9], where the goal is to approximate any such state ρ on $\mathcal{H}^{\otimes M}$ with a mixture of identically prepared states $\tilde{\rho} = \sum_i p_i \rho_i^{\otimes M}$. In particular, as we will see in the following, the recent techniques of Ref. [10] provide a very useful tool to prove our results. For simplicity, we will first start by considering the special case of SDI channel with output states in the totally symmetric subspace $\mathcal{H}_+^{\otimes M} \subset \mathcal{H}^{\otimes M}$, which is the case, for example, of the optimal cloning of pure states. Then, all results will be extended to the general case of arbitrary SDI channels.

In order to approximate channels we use the following finite version of quantum de Finetti theorem, which is proved with the same techniques of Ref. [10], with a slight improvement of the bound given therein [11]:

Lemma 1: For any state ρ on $\mathcal{H}_+^{\otimes M} \subset \mathcal{H}^{\otimes M}$, consider the separable state

$$\tilde{\rho} = \int d\psi p(\psi) |\psi\rangle\langle\psi|^{\otimes M}, \quad (4)$$

the probability distribution $p(\psi)$ being given by

$$p(\psi) = \text{Tr}[\Pi_\psi \rho], \quad \Pi_\psi = d_M^+ |\psi\rangle\langle\psi|^{\otimes M}, \quad (5)$$

where $d\psi$ denotes the normalized Haar measure over the pure states $|\psi\rangle \in \mathcal{H}$, and $d_M^+ = \dim(\mathcal{H}_+^{\otimes M})$. Then, one has

$$\|\rho^{(k)} - \tilde{\rho}^{(k)}\|_1 \leq 4s_{M,k}, \quad s_{M,k} \doteq 1 - \sqrt{\frac{d_{M-k}^+}{d_M^+}}, \quad (6)$$

$\rho^{(k)}$ denoting the reduced state $\rho^{(k)} = \text{Tr}_{M-k}[\rho]$.

Proof: The identity in the totally symmetric subspace $\mathcal{H}_+^{\otimes n} \subset \mathcal{H}^{\otimes n}$ can be written as

$$\mathbb{1}_n^+ = d_n^+ \int d\psi P_n(\psi), \quad (7)$$

where $P_n(\psi) = |\psi\rangle\langle\psi|^{\otimes n}$. Using Eq. (7) with $n = M - k$, we can write $\rho^{(k)} = d_+^{M-k} \int d\psi \rho_k(\psi)$, where $\rho_k(\psi) = \text{Tr}_{M-k}[\rho \mathbb{1}^{\otimes k} \otimes P_{M-k}(\psi)]$. On the other hand, the reduced state $\tilde{\rho}^{(k)}$ can be written as $\tilde{\rho}^{(k)} = d_M^+ \int d\psi P_k(\psi) \rho_k(\psi) P_k(\psi)$. Then, the difference between $\rho^{(k)}$ and $\tilde{\rho}^{(k)}$, denoted by $\Delta^{(k)}$, is given by

$$\Delta^{(k)} = d_{M-k}^+ \int d\psi \left[\rho_k(\psi) - \frac{d_M^+}{d_{M-k}^+} P_k(\psi) \rho_k(\psi) P_k(\psi) \right].$$

Notice that the integrand on the right-hand side has the form $A - BAB$, with $A(\psi) = \rho_k(\psi)$ and $B(\psi) = \sqrt{d_M^+/d_{M-k}^+} P_k(\psi)$. Using the relation

$$A - BAB = A(\mathbb{1} - B) + (\mathbb{1} - B)A - (\mathbb{1} - B)A(\mathbb{1} - B) \quad (8)$$

we obtain

$$\Delta^{(k)} = d_{M-k}^+ (C + C^\dagger - D), \quad (9)$$

where

$$C = \int d\psi A(\psi) [\mathbb{1} - B(\psi)], \quad (10)$$

$$D = \int d\psi [\mathbb{1} - B(\psi)] A(\psi) [\mathbb{1} - B(\psi)]. \quad (11)$$

The operator C is easily calculated using the relation

$$\begin{aligned} \int d\psi \rho_k(\psi) P_k(\psi) &= \int d\psi \text{Tr}_{M-k}[\rho P_M(\psi)] = \frac{\text{Tr}_{M-k}[\rho]}{d_M^+} \\ &= \frac{\rho^{(k)}}{d_M^+}, \end{aligned}$$

which follows from Eq. (7) with $n = M$. In this way we obtain $C = s_{M,k}/d_{M-k}^+ \rho^{(k)}$. Because C is nonnegative, we have $\|C\|_1 = \text{Tr}[C] = s_{M,k}/d_{M-k}^+$. Moreover, due to definition (11) also D is nonnegative, then we have $\|D\|_1 = \text{Tr}[D] = \text{Tr}[C + C^\dagger]$, as follows by taking the trace on both sides of Eq. (9). Thus, the norm of D is $\|D\|_1 = 2\|C\|_1$. Finally, taking the norm on both sides of Eq. (9), and using triangular inequality we get $\|\Delta^{(k)}\| \leq 4d_{M-k}^+ \|C\|_1 = 4s_{M,k}$, that is bound (6). ■

Because the dimension of the totally symmetric subspace $\mathcal{H}_+^{\otimes n}$ is given by

$$d_n^+ = \binom{d+n-1}{n},$$

$d \doteq \dim(\mathcal{H})$, for $M \gg kd$ the ratio d_{M-k}^+/d_M^+ tends to $1 - \frac{k(d-1)}{M}$. Therefore, Lemma 1 yields

$$\|\rho^{(k)} - \tilde{\rho}^{(k)}\|_1 \leq \frac{2(d-1)k}{M}, \quad M \gg kd; \quad (12)$$

i.e., the distance between $\rho^{(k)}$ and the separable state $\tilde{\rho}^{(k)}$ vanishes as k/M .

With the above lemma, we are ready to prove the approximation theorem for SDI channels with output in the totally symmetric subspace:

Theorem 1: Any SDI channel \mathcal{E} with output states in the totally symmetric subspace $\mathcal{H}_+^{\otimes M} \subset \mathcal{H}^{\otimes M}$ can be approximated by a classical channel

$$\tilde{\mathcal{E}}(\rho) = \int d\psi \text{Tr}[P_\psi \rho] |\psi\rangle\langle\psi|^{\otimes M}, \quad (13)$$

where P_ψ is a quantum measurement ($P_\psi \geq 0$ and $\int d\psi P_\psi = \mathbb{1}_{\text{in}}$). For large M , the accuracy of the approximation is

$$\|\rho_{\text{out}}^{(k)} - \tilde{\rho}_{\text{out}}^{(k)}\|_1 \leq \frac{2(d-1)k}{M}, \quad M \gg kd. \quad (14)$$

Proof: Consider the channel \mathcal{E}^* in the Heisenberg picture, defined by the relation $\text{Tr}[O\mathcal{E}(\rho)] = \text{Tr}[\mathcal{E}^*(O)\rho]$ for any state ρ on \mathcal{H}_{in} and for any operator O on \mathcal{H}_{out} . Because the channel \mathcal{E} is trace preserving, \mathcal{E}^* is identity preserving, namely $\mathcal{E}^*(\mathbb{1}_{\text{out}}) = \mathbb{1}_{\text{in}}$. Applying Lemma 1 to the output state $\rho_{\text{out}} = \mathcal{E}(\rho)$, we get $\tilde{\rho}_{\text{out}} = \int d\psi \text{Tr}[\Pi_\psi \mathcal{E}(\rho)] |\psi\rangle\langle\psi|^{\otimes M}$. Because $\text{Tr}[\Pi_\psi \mathcal{E}(\rho)] = \text{Tr}[\mathcal{E}^*(\Pi_\psi)\rho]$, by defining $P_\psi \doteq \mathcal{E}^*(\Pi_\psi)$, we immediately obtain that $\tilde{\rho}_{\text{out}} = \tilde{\mathcal{E}}(\rho)$, with $\tilde{\mathcal{E}}$ as in Eq. (13). The operators $\{P_\psi\}$ represent a quantum measurement on \mathcal{H}_{in} , because they are obtained by applying a completely positive identity-preserving map to Π_ψ , which is a measurement on \mathcal{H}_{out} . Finally, the bound (14) then follows from Eq. (12). ■

The above theorem proves that for large M the quantum information distributed to a single user can be efficiently replaced by the classical information about the measurement outcome ψ . In fact, the single user output states of the channels \mathcal{E} and $\tilde{\mathcal{E}}$ become closer and closer, and therefore less distinguishable, as M increases. For large M , the error probability in distinguishing between $\rho_{\text{out}}^{(1)}$ and $\tilde{\rho}_{\text{out}}^{(1)}$ has to satisfy the bound

$$p_{\text{err}} \geq \frac{1}{2} - \frac{d-1}{2M}; \quad (15)$$

namely, it approaches $1/2$ at rate M^{-1} . For example, for qubits Eq. (15) gives already with $M = 10$ an error probability $p_{\text{err}} \geq 0.45$, quite close to the error probability of a purely random guess. More generally, the bound (14) implies that for any group of k users there is almost no entanglement in the state $\rho_{\text{out}}^{(k)}$, because it is close to a completely separable state. As the number of users grows, multipartite entanglement vanishes at any finite order: only k -partite entanglement with $k = O(M)$ can survive.

Applying our approximation theorem to the particular case of pure state cloning, we obtain a complete proof of its

asymptotic equivalence with state estimation. In fact, taking \mathcal{E} as an optimal pure state cloning, the channel $\tilde{\mathcal{E}}$ yields an approximation of \mathcal{E} based on state estimation (the measurement outcomes of P_ψ are in one to one correspondence with the pure states on \mathcal{H}). On one hand, when applied to a pure state $|\phi\rangle$, the optimal cloning gives fidelity $F_{\text{clon}} = \langle\phi|\rho_{\text{out}}^{(1)}|\phi\rangle$. On the other hand, because the measurement P_ψ gives a possible estimation strategy, the fidelity of the optimal estimation F_{est} cannot be smaller than $\langle\phi|\tilde{\rho}_{\text{out}}^{(1)}|\phi\rangle$. Therefore, the difference between the two fidelities can be bounded as

$$0 \leq F_{\text{clon}} - F_{\text{est}} \leq \|\rho_{\text{out}}^{(1)} - \tilde{\rho}_{\text{out}}^{(1)}\|_1 \leq \frac{2(d-1)}{M}, \quad M \gg d, \quad (16)$$

namely, it approaches zero at a rate $1/M$. Apart from a constant, this is the optimal rate one can obtain in a general fashion holding for any kind of pure state cloning. In fact, $1/M$ is the exact rate in the case of universal cloning, where $F_{\text{clon}} - F_{\text{est}} = \frac{N(d-1)}{M(N+d)}$ (see [12] for the single-clone fidelity). In addition, from Eq. (16) it immediately follows that any quantum cloning map for large numbers N of input copies is approximated by state estimation, because for cloning one has $M > N$, and M is necessarily large. In this way we proved the asymptotic equivalence between cloning and state estimation for any kind of cloning (see also the following Theorem 2 for the general case of $\mathcal{H}_{\text{out}} \neq \mathcal{H}_+^{\otimes M}$), for either large N or large M (see open problems in Ref. [6]). We emphasize that the $M = \infty$ result of Ref. [7] cannot be used to prove the large N asymptotics.

All results obtained for SDI channels with output in the totally symmetric subspace can be easily extended to arbitrary SDI channels, exploiting the fact that any permutationally invariant state can be purified to a totally symmetric one [10]:

Lemma 2: Any permutationally invariant state ρ on $\mathcal{H}^{\otimes M}$ can be purified to a state $|\Phi\rangle \in \mathcal{K}_+^{\otimes M} \subset \mathcal{K}^{\otimes M}$, where $\mathcal{K} = \mathcal{H}^{\otimes 2}$.

Once the state ρ has been purified, we can apply Lemma 1 to the state $|\Phi\rangle$, thus approximating its reduced states. The reduced states of ρ are then obtained by taking the partial trace over the ancillae used in the purification. This implies the following.

Lemma 3: For any permutationally invariant state ρ on $\mathcal{H}^{\otimes M}$, purified to $|\Phi\rangle \in \mathcal{K}_+^{\otimes M}$, $\mathcal{K} = \mathcal{H}^{\otimes 2}$, consider the separable state

$$\tilde{\rho} = \int d\Psi p(\Psi) \rho(\Psi)^{\otimes M}, \quad (17)$$

where $d\Psi$ is the normalized Haar measure over the pure states, $|\Psi\rangle \in \mathcal{K}$, $\rho(\Psi)$ is the reduced state $\rho(\Psi) = \text{Tr}_{\mathcal{H}}[|\Psi\rangle\langle\Psi|]$, and $p(\Psi)$ is the probability distribution given by $p(\Psi) = \text{Tr}[\Pi_\Psi |\Phi\rangle\langle\Phi|]$, with $\Pi_\Psi = D_M^+ |\Psi\rangle\langle\Psi|^{\otimes M}$, $D_M^+ = \text{dim}(\mathcal{K}_+^{\otimes M})$. Then, one has

$$\|\rho_{k,A} - \tilde{\rho}_{k,A}\|_1 \leq 4S_{M,k}, \quad S_{M,k} \doteq 1 - \sqrt{\frac{D_{M-k}^+}{D_M^+}}. \quad (18)$$

Proof: Applying Lemma 1 to $\tau = |\Phi\rangle\langle\Phi|$, we get the state $\tilde{\tau} = \int d\Psi p(\Psi)|\Psi\rangle\langle\Psi|^{\otimes M}$. The state $\tilde{\rho}$ is then obtained by tracing out the ancillae used in the purification, namely, it is given by Eq. (17). Because partial traces can only decrease the distance, the bound (18) immediately follows from the bound (6). ■

It is then immediate to obtain the following:

Theorem 2: Any SDI channel \mathcal{E} can be approximated by a classical channel

$$\tilde{\mathcal{E}}(\rho) = \int d\Psi \text{Tr}[P_\Psi \rho] \rho(\Psi)^{\otimes M}, \quad (19)$$

where P_Ψ is a quantum measurement, namely $P_\Psi \geq 0$ and $\int d\Psi P_\Psi = \mathbb{1}_{\text{in}}$. For large M , the accuracy of the approximation is

$$\|\rho_{\text{out}}^{(k)} - \tilde{\rho}_{\text{out}}^{(k)}\|_1 \leq \frac{2(d^2 - 1)k}{M}, \quad M \gg kd^2. \quad (20)$$

This theorem extends Theorem 1 and all its consequences to the case of arbitrary SDI channels. In particular, it proves that asymptotically the optimal cloning of mixed states can be efficiently simulated via mixed state estimation. The results of the measurement P_Ψ are indeed in correspondence with pure states on $\mathcal{H} \otimes \mathcal{H}$, and, therefore, with mixed states on \mathcal{H} . Accordingly, the knowledge of the classical result Ψ is enough to reproduce efficiently the output of the optimal cloning machine.

Notice the dependence on the dimension of the single user's Hilbert space in both Theorems 1 and 2: increasing d makes the bounds (14) and (20) looser, leaving more room to cloning or broadcasting of genuine quantum nature. Rather surprisingly, instead, the efficiency of our approximations does not depend on the dimension of the full input Hilbert space, e.g., it does not depend on the number N of the input copies of a broadcasting channel. No matter how large the physical system carrying the input information, if there are many users at the output there is no advantage of quantum over classical information processing. Accordingly, our results can be applied to channels from $\mathcal{H}^{\otimes N}$ to $\mathcal{H}^{\otimes M}$, even with $M < N$. As long as $M \gg kd^2$ any such channel can be efficiently replaced by a classical one. In particular, this argument holds also for the purification of quantum information [13,14]: if M is large enough, any strategy for quantum purification can be approximated by a classical measure-and-prepare scheme. Only for small M one can have a really quantum purification.

In conclusion, we have considered the general class of quantum channels that equally distribute information among M users, showing that for large M any such channel can be efficiently approximated by a classical one, where

the input system is measured and the measurement outcome is broadcast, and each user prepares locally the same state accordingly. The approximating channel can be regarded as the concatenation of a quantum-to-classical channel (the measurement), followed by a classical-to-quantum channel (the local preparation). Actually, the latter channel is needed only for the sake of comparison with the original quantum transformation to be approximated, because, due to the data processing inequality, this additional stage can only decrease the amount of information contained in the classical probability distribution of measurement outcomes. Therefore, asymptotically, there is no broadcasting of quantum information, but just an announcement of the classical information extracted by a measurement. In synthesis, we cannot distribute more information than what we are able to read out.

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