Informationally complete quantum measurements

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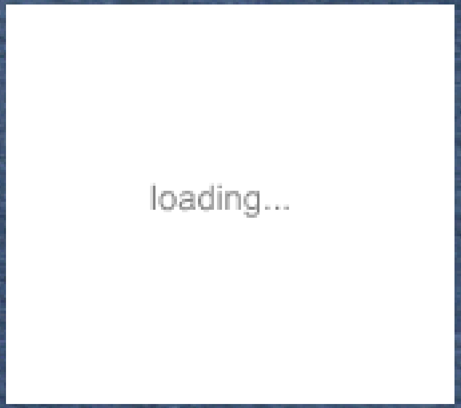

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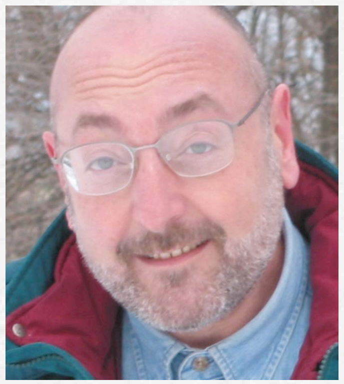

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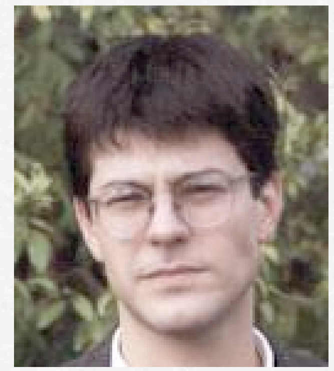

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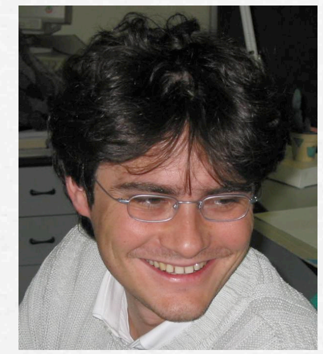

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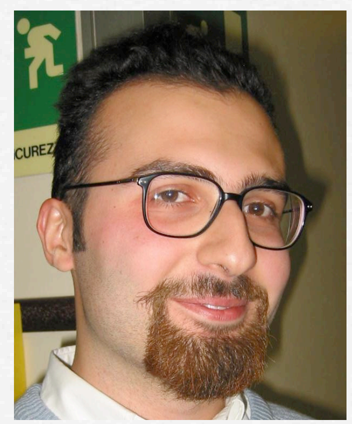

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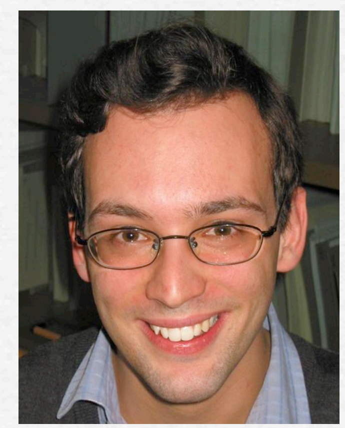
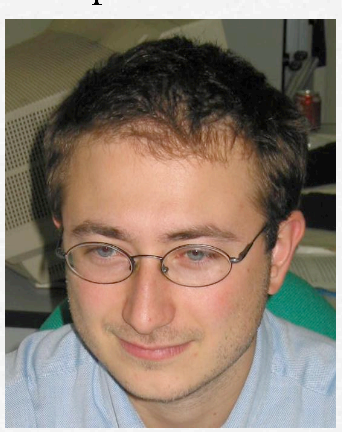


Paoloplacido Lo Presti



Michael Keyl

Marcella Medici (secretary)

Research program

Designing a completely new generation of measuring apparatuses

- Universal Quantum Detectors (this talk)
- Programmable Quantum Detectors

Research program

Establish the minimal set of resources in terms of:

- special quantum states
- special measurements
- special unitary transformations

In this talk

- Informationally complete POVM's
  - *Frames* of operators
  - Bell POVM's
  - Separable POVM's
- Transmission of reference frames
- Quantum Calibration

Essential literature

- *Informationally complete measurements*
    - G. M. D'Ariano, P. Perinotti, and M. F. Sacchi *Informationally complete measurements and groups representation* J. Opt. B: Quantum Semiclass. Opt. 6 S487-S491 (2004)
    - G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *Optimization of Quantum Universal Detectors,* in *Squeezed States and Uncertainty Relations* ed. by H. Moya-Cessa, R. Jauregui, S. Hacyan, and O. Castanos, Rinton Press (Princeton 2003) pag. 86
- *Transmitting frames using entanglement with the multiplicity space*
    - G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *Covariant quantum measurements which maximize the likelihood* Phys. Rev A (in pres)
    - G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *Efficient use of quantum resources for the transmission of a reference frame,* Phys. Rev. Lett. (in press)
- *Quantum calibration*
    - G. M. D'Ariano, P. Lo Presti, and L. Maccone, Quantum Calibration of Measuring Apparatuses, Phys. Rev. Lett. (quant-ph0408116)

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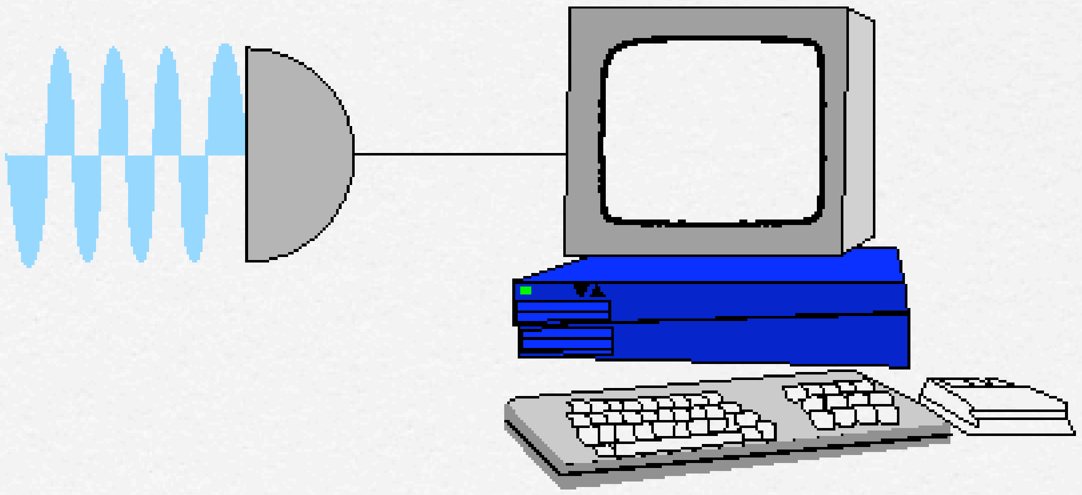
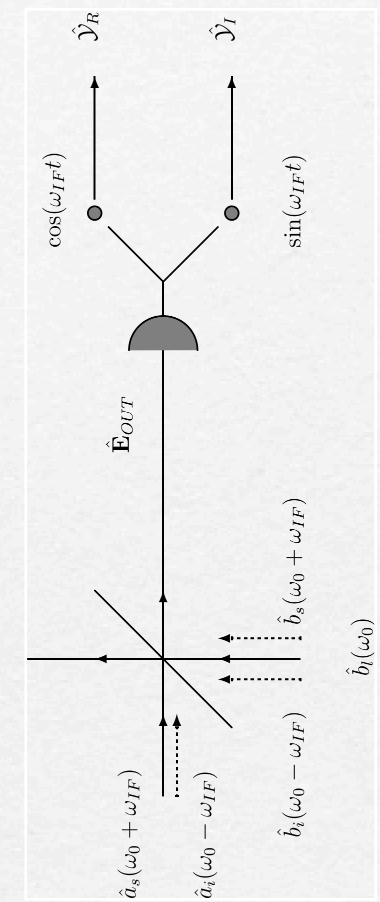
Info-complete measurements

- We call informationally complete a measurement by which we can determine the expectation $\langle O \rangle$ of an arbitrary operator O by using a different data-processing for each O .



Info-complete measurements

- state estimation
- measurement of the spin direction
- covariant measurements
- extremal measurements with the largest number of outcomes
- applications: eavesdropping, tomography, ...



heterodyne detector

Info-complete POVM's

- For an informationally complete POVM $\{\Xi_i\}$ we must have

$$\mathrm{Tr}[\rho O] = \sum_i f_i(O)\mathrm{Tr}[\rho \Xi_i],$$

$f_i(O)$ **data-processing** for O.

Universal detectors

quantum information theory group

Dilation point of view

Couple the quantum system (Hilbert space H) with an ancilla (Hilbert space K).

- A POVM $\{\Pi_i\}$, $\Pi_i \geq 0$ on $\mathsf{H} \otimes \mathsf{K}$ is **universal** for the system iff there exists a state of the ancilla ν such that for any operator O on H one has

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\mathrm{Tr}[(\rho \otimes \nu)\Pi_i]\,,$$

for a suitable ***data-processing*** $f_i(\nu, O)$ of the outcome i.

- Relation with **informationally complete POVM**

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\mathrm{Tr}[\rho \Xi_i[\nu]], \qquad \Xi_i[\nu] \doteq \mathrm{Tr}_2[(I \otimes \nu)\Pi_i].$$

Problems

- Classify universal and info-complete POVM's
- Find the “optimal” POVM’s and the optimal ancilla states
- Establish the optimal data processing

Notation

- Bipartite states $|\Psi\rangle\rangle \in \mathsf{H} \otimes \mathsf{K} \Longleftrightarrow$ operators $\Psi \in \mathsf{HS}(\mathsf{K}, \mathsf{H})$

$$|\Psi\rangle\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle.$$

- Matrix notation (for fixed reference basis in theHilbert spaces)

$$A \otimes B|C\rangle\rangle = |AC\,B^{\intercal}\rangle\rangle,$$

$$\langle\langle A|B\rangle\rangle \equiv \mathrm{Tr}[A^{\dagger}B].$$

Frames of operators

- $\{\Xi_i\}$ is a frame for a (Banach) space of operators if $\exists 0 < a \leq b < +\infty$ s.t. for all operators A one has

$$a\|A\|^2 \leq \underbrace{\sum_i |\langle A, \Xi_i\rangle|^2}_{\text{Bessel series}} \leq b\|A\|^2.$$

- Then, there exists a dual frame $\{\Theta_i\}$ such that every operator A can be expanded as follows

$$A = \sum_i \langle \Theta_i, A\rangle \Xi_i \ .$$

$\langle \Theta_i, A\rangle \doteq \mathrm{Tr}[\Theta_i^\dagger A].$

- The sequence of operators $\{\Xi_i\}$ is a frame iff the following operator on $\mathsf{H} \otimes \mathsf{K}$ is bounded and invertible

$$F = \sum_i |\Xi_i\rangle\!\rangle\langle\!\langle\Xi_i| \ . \qquad \text{(frame operator)}$$

Frames of operators

- Completeness relation

$$E = \sum_i \Theta_i^\dagger \otimes \Xi_i \qquad E: \text{swap operator on } \mathsf{H} \otimes \mathsf{K}$$

- Alternate dual frames:

$$|\Theta_i\rangle\!\rangle = F^{-1}|\Xi_i\rangle\!\rangle + |Y_i\rangle\!\rangle - \sum_j \langle\!\langle \Xi_j|F^{-1}|\Xi_i\rangle\!\rangle |Y_j\rangle\!\rangle \,,$$

Y_i arbitrary (Bessel), and $F^{-1}|\Xi_i\rangle\!\rangle$ canonical dual frame.

- *Exact* frames: dual $\equiv$ canonical.
- Alternate duals useful for optimization.

Info-complete POVM's

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\mathrm{Tr}[\rho \Xi_i[\nu]], \qquad \Xi_i[\nu] \doteq \mathrm{Tr}_2[(I \otimes \nu)\Pi_i].$$

true independently of ρ iff

$$O = \sum_i f_i(\nu, O)\Xi_i[\nu],$$

- namely $\{\Xi_i[\nu]\}$ is a **positive frame**.
- The dual frame provides the data-processing rule

$$f_i(\nu, O) = \mathrm{Tr}\left[\Theta_i^\dagger[\nu] O\right].$$

Info-complete POVM's

Upon diagonalizing the POVM $\{\Pi_i\}$ on $\mathsf{H} \otimes \mathsf{K}$

$$\Pi_i = \sum_{j=1}^{r_i} |\Psi_j^{(i)}\rangle\!\rangle\langle\!\langle\Psi_j^{(i)}|,$$

one has

$$\Xi_i[\nu] \equiv \sum_{j=1}^{r_i} \Psi_j^{(i)} \nu^\tau \Psi_j^{(i)\dagger}.$$

- It follows that $\{\Pi_i\}$ is universal iff both $\{\Psi_j^{(i)}\}$ and $\{\Xi_i[\nu]\}$ are operator frames.

Bell POVM's: abelian case

$$\text{POVM on } \mathsf{H}\otimes\mathsf{H}: \quad \Pi_\alpha \propto |U_\alpha\rangle\rangle\langle\langle U_\alpha|, \quad U_\alpha \text{ unitary.}$$

- Special case: $\{U_\alpha\}$ UIR of some group $\mathbf{G}$.
- **Example:** **nice error basis** $\{U_\alpha\}$

  e. g. projective UIR of abelian group: $\boxed{U_\alpha U_\beta U_\alpha^\dagger = e^{ic(\alpha,\beta)} U_\beta}$

- One can prove that the Bell POVM is necessarily orthogonal, and is universal for ancilla state ν such that $\mathrm{Tr}[U_\alpha^\dagger \nu^\tau] \neq 0$ for all α.
- Dual set (unique) for data-processing:

$$\Theta_\alpha[\nu] = \frac{1}{d}\sum_{\beta=1}^{d^2} \frac{U_\beta e^{-ic(\beta,\alpha)}}{\mathrm{Tr}\left[U_\beta \nu^*\right]} \,.$$

Bell POVM's: SU(d)

- **Example:** UIR of non abelian group $SU(d)$.

- Frame operator for $\Xi_\alpha[\nu] = U_\alpha\, \nu^\tau\, U_\alpha^\dagger \Longleftrightarrow \Pi_\alpha \propto |U_\alpha\rangle\!\rangle\langle\!\langle U_\alpha|$

$$F = \int \mathrm{d}\alpha\, (U_\alpha \otimes U_\alpha^*)\, |\nu^\tau\rangle\!\rangle\langle\!\langle \nu^\tau|\, (U_\alpha^\dagger \otimes U_\alpha^\tau) = P + \frac{1}{a} P^\perp,$$

$$P \doteq \tfrac{1}{d} |I\rangle\!\rangle\langle\!\langle I|, \qquad a = \frac{d^2 - 1}{d\, \mathrm{Tr}[(\nu^\tau)^2] - 1},$$

$\{\Xi_\alpha[\nu]\}$ is a frame unless $\nu = d^{-1} I$.

- Canonical dual frame

$$\Theta_\alpha^0[\nu] = a\, U_\alpha\, \nu^\tau U_\alpha^\dagger + b\, I\,, \qquad b = \frac{\mathrm{Tr}[(\nu^\tau)^2] - d}{d\mathrm{Tr}[(\nu^\tau)^2] - 1}.$$

Bell POVM's: SU(d)

- Consider alternate dual frames of covariant form

$$\Theta_\alpha[\nu] = U_\alpha \xi U_\alpha^\dagger \,.$$

One must have

$$\mathrm{Tr}[\xi] = 1 \,, \quad \mathrm{Tr}[\nu^\tau \xi] = d \,.$$

- The canonical dual frame minimizes the variance averaged over all pure states.

- The optimal ancilla state ν is pure.

- **Other examples:** $SU(2)$ UIR's on H with $\dim(\mathsf{H}) > 2$, ...

Separable POVM's

For $\dim(\mathsf{K}) \geq \dim(\mathsf{H})^2$ one can obtain "separable" universal POVM's.

- **Example:** observable operator frame on H

$$C(l) = \sum_k c_k(l) |c_k(l)\rangle\langle c_k(l)|, \qquad l = 1, 2, ..., L \geq \dim(\mathsf{H})^2 \ .$$

- By taking $\dim(\mathsf{K}) = L$, one has the following orthogonal POVM for $\mathsf{H} \otimes \mathsf{K}$

$$\begin{aligned} \Pi_{k,l} &= |c_k(l)\rangle\langle c_k(l)| \otimes |l\rangle\langle l|, \qquad \{|l\rangle\} \text{ ONB for } \mathsf{K}. \\ &\Rightarrow \text{tomography + ancillary } \textit{quantum roulette}. \end{aligned}$$

- Data-processing function:

$$f_{k,l}(\nu, O) = \frac{\mathrm{Tr}[C^\dagger(l) O]}{\langle l|\nu|l\rangle} c_k(l), \qquad \langle l|\nu|l\rangle \neq 0 \ \forall l.$$

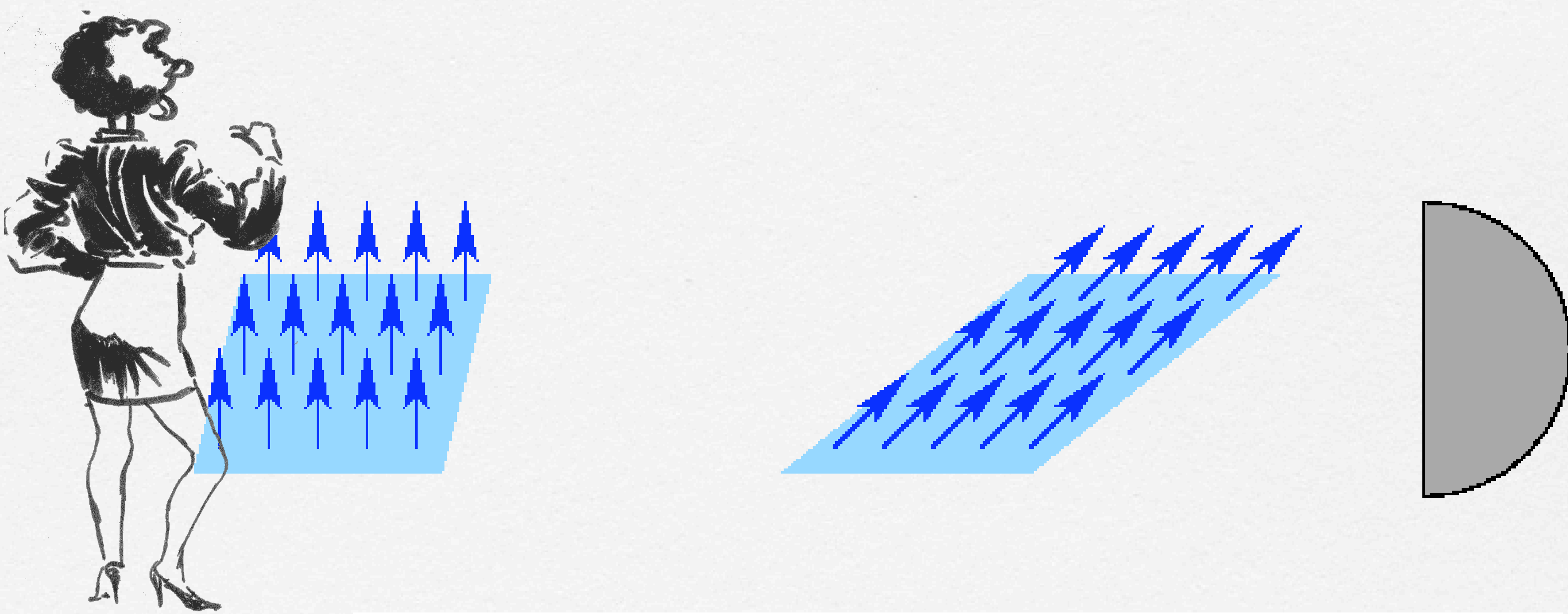
Open problems

- General classification of universal POVM's (with any degree of entanglement)
- Methods for generating positive operator frames from complex operator frames.
- Non group-covariant universal POVMs
- Conjectures:
  - For H≈K every universal POVM is Bell
  - A Bell POVM always ”better” than a separable one
  - The canonical dual frame is optimal (minimizes the rms)
  - There is always an ”optimal” ancillary state that is pure

Transmission of frames

Transmission of frames

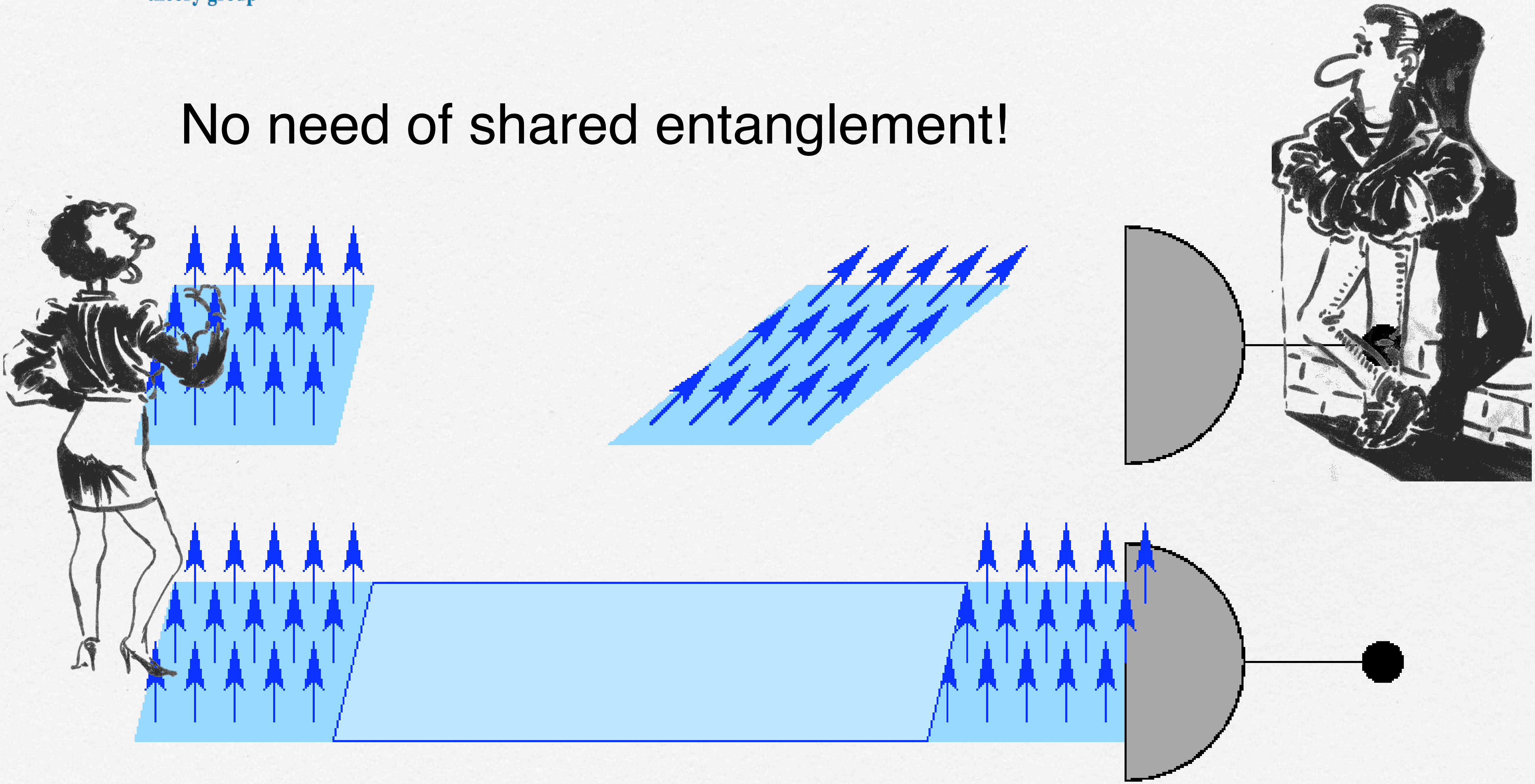
Sensitivity N^{-2} instead of N^{-1}



$$\mathsf{H}^{\otimes N} = \oplus_{\nu}(\mathsf{H}_{\nu} \otimes \mathbb{C}^{m_{\nu}})$$

Transmission of frames

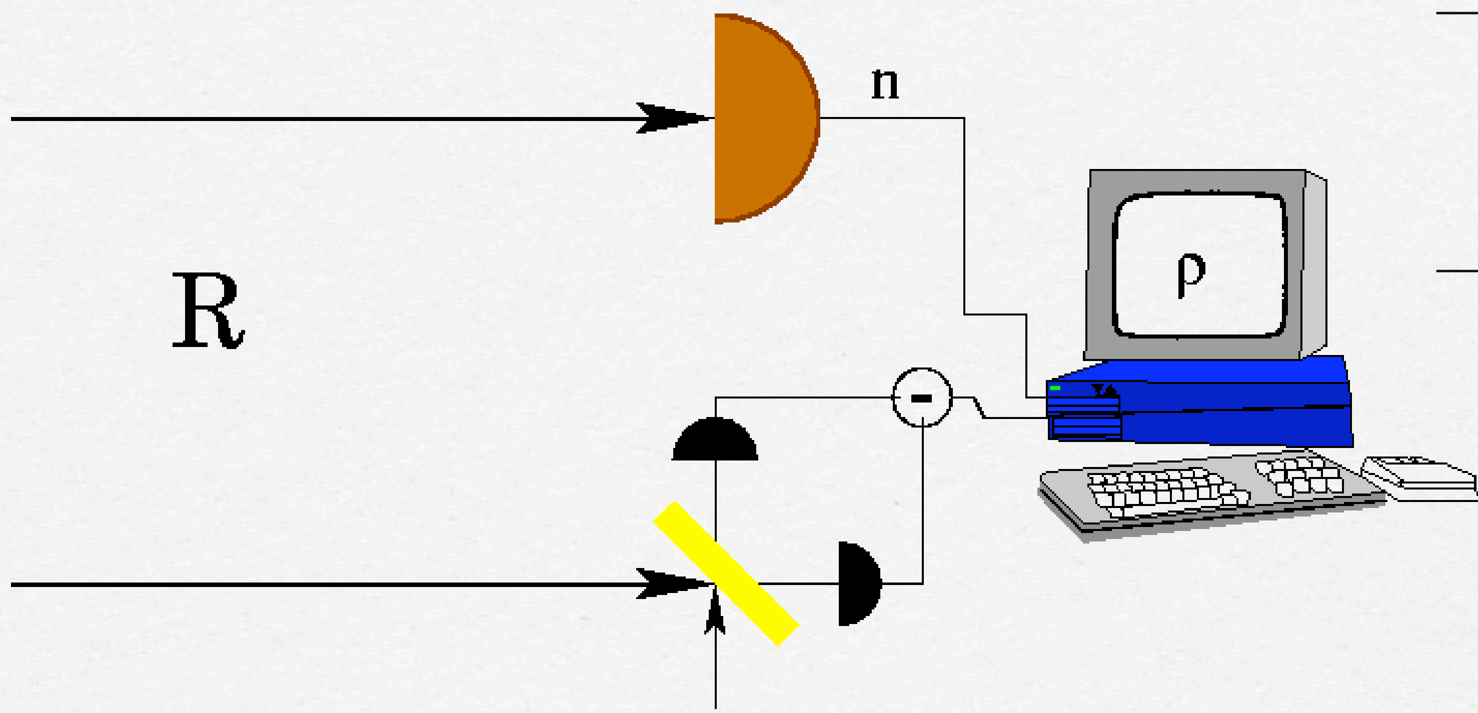
No need of shared entanglement!

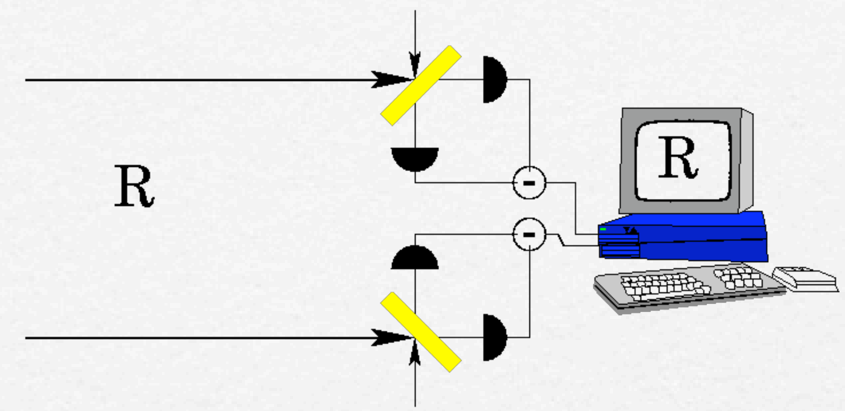


Quantum calibration

We can calibrate a measuring apparatus without knowing its functioning, by determining experimentally its POVM

Quantum calibration





pre calibration

$$P_n = p(n)[\mathscr{R}^{-1}(\rho_n)]^{\intercal}, \quad \mathscr{R}(\rho) \doteq \mathrm{Tr}_1[(\rho^{\intercal} \otimes I)R].$$

- $p(n)$ probability of the outcome n,
- ρ_n conditioned state, to be determined by quantum tomography,
- $\mathscr{R}$ associated map of the faithful state R.

G. M. D'Ariano, P. Lo Presti, and L. Maccone, submitted to Phys. Rev. Lett. (quant-ph0408116)

Quantum calibration

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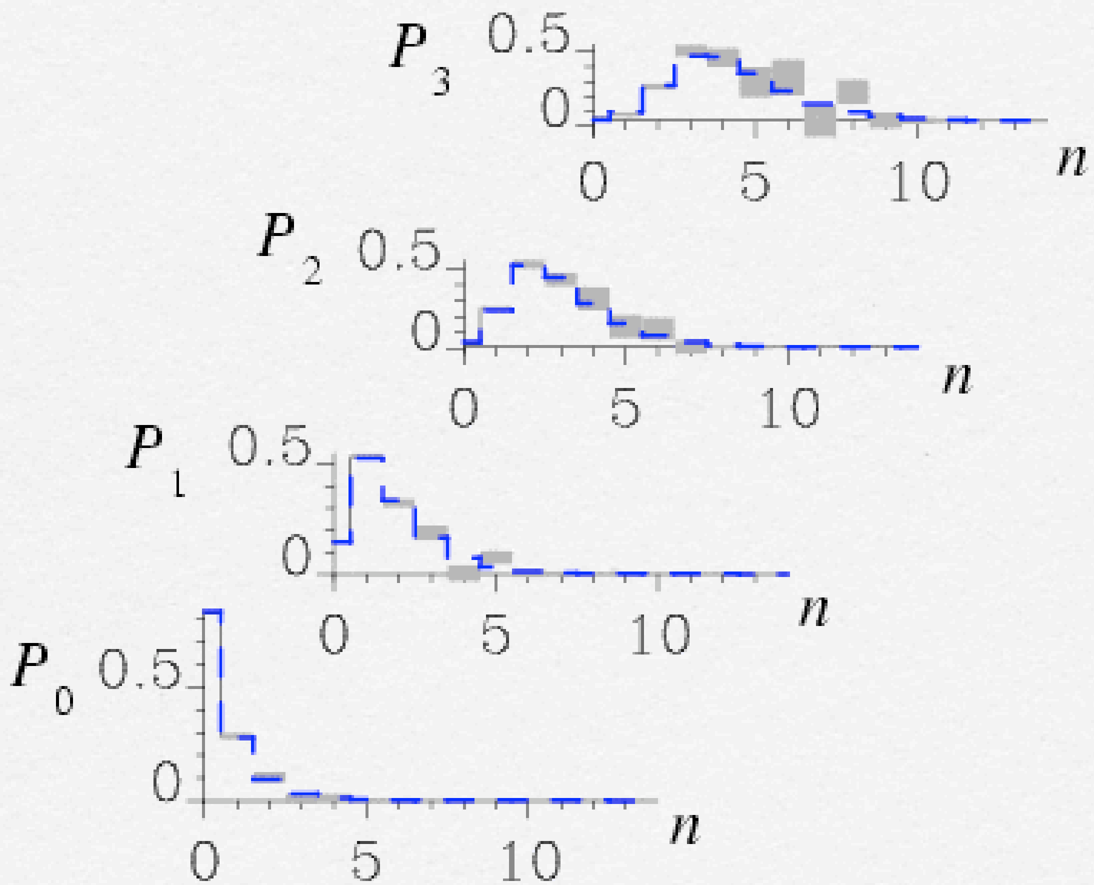
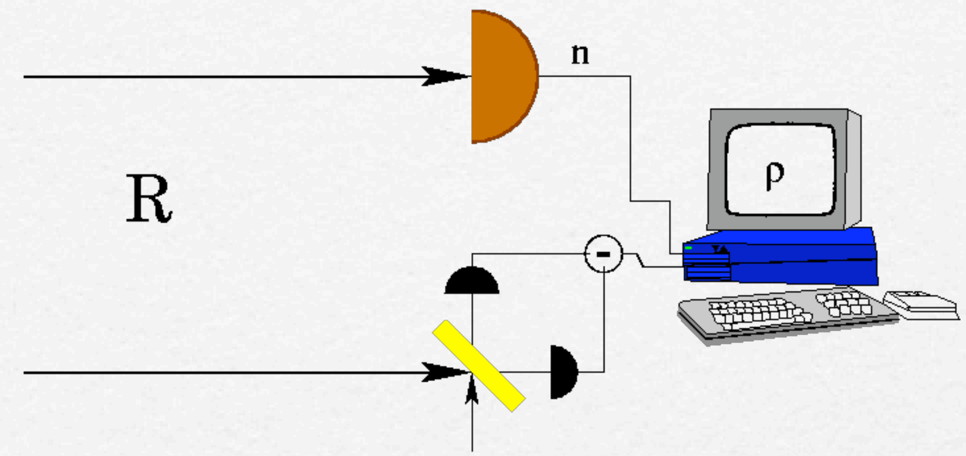
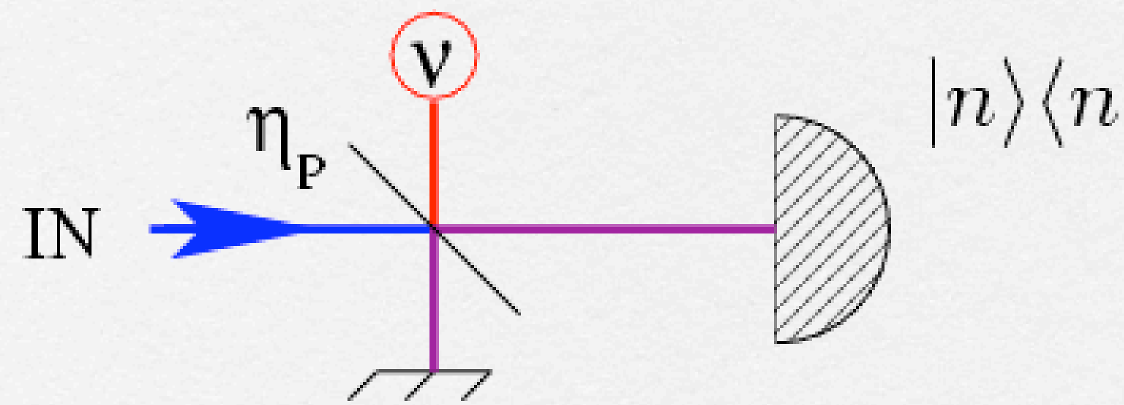
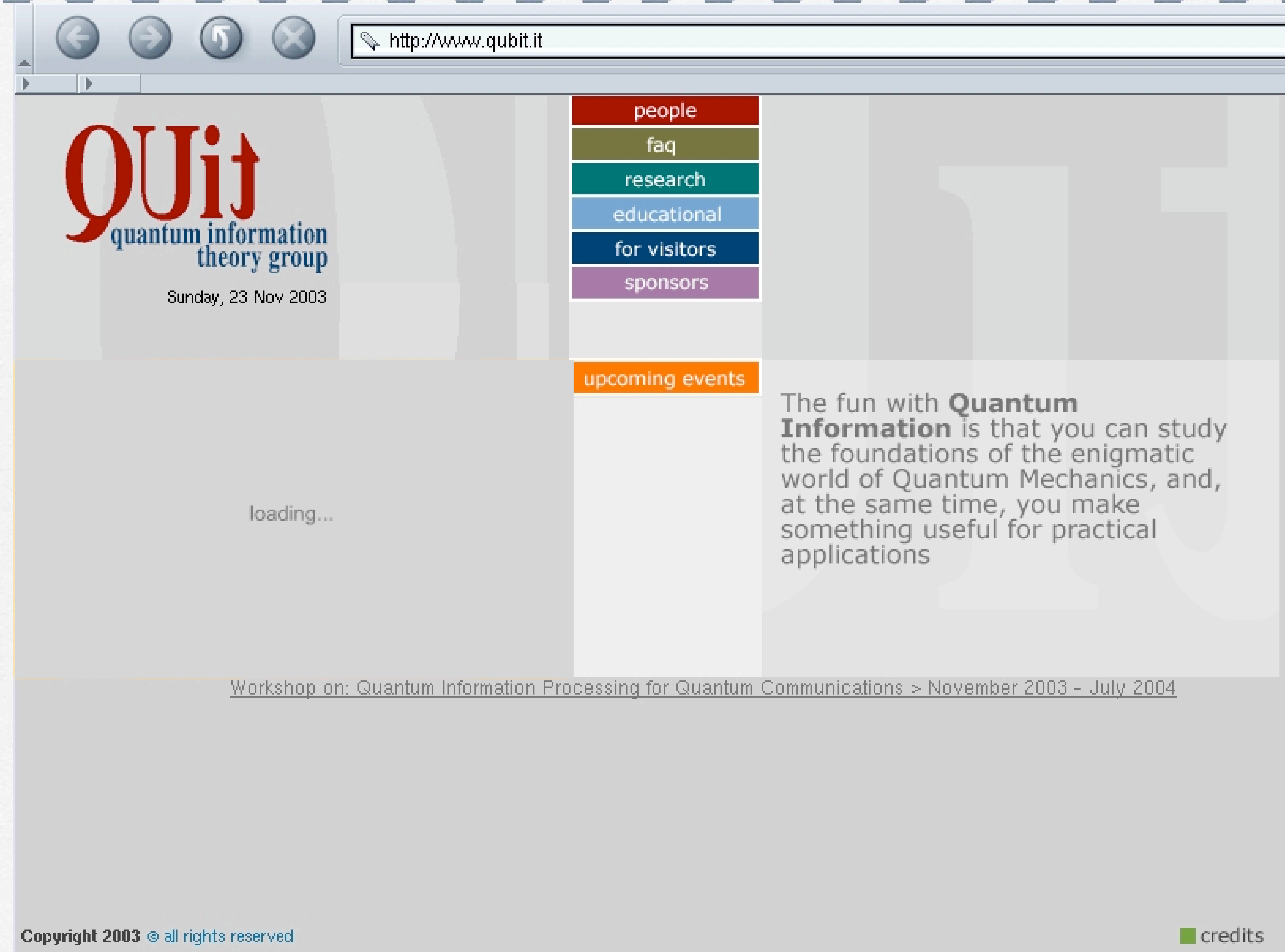






Conclusions

- Informationally complete POVM's
  - Optimization
  - Bell POVM's, separable POVM's, ...
  - Many open problems
- Transmission of reference frames
- Quantum Calibration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