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PRL **101** 180504 (2008)
PRL **102** 010404 (2009)
EL **83** 30004 (2008)

The quantum comb: theory and applications to quantum networks

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QUIT group, University of Pavia

Institute for Theoretical Physics, University of Innsbruck,
"Seminar/Theory Colloquium", March 19 2009

QUIT Group in Pavia

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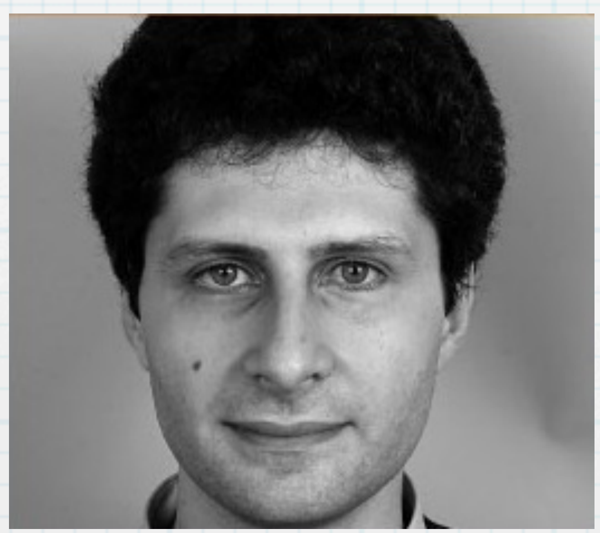
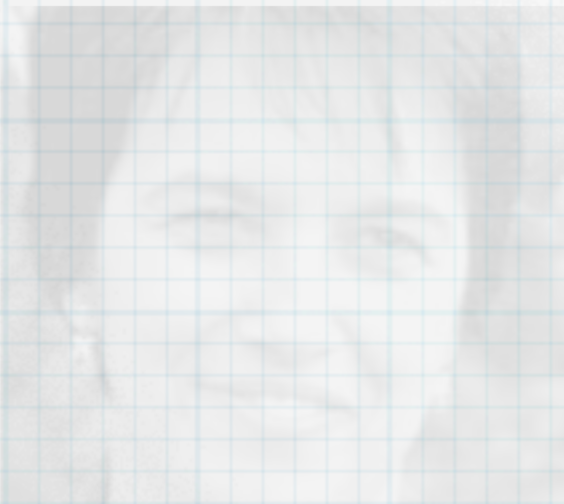


Daniele Magnani

Stefano Facchini

Alessandro Bisio

Theory of Quantum Combs in collaboration with



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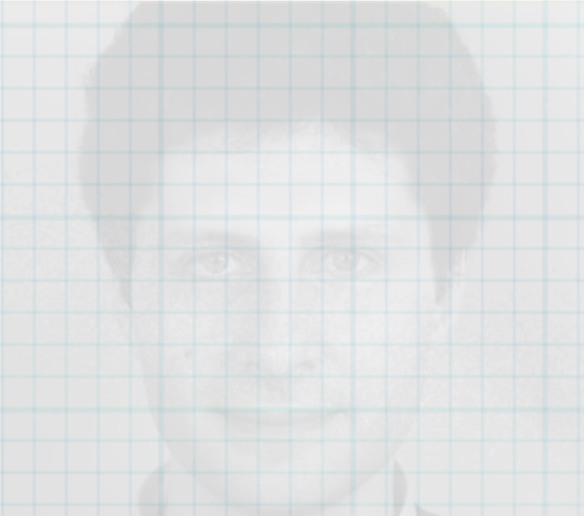

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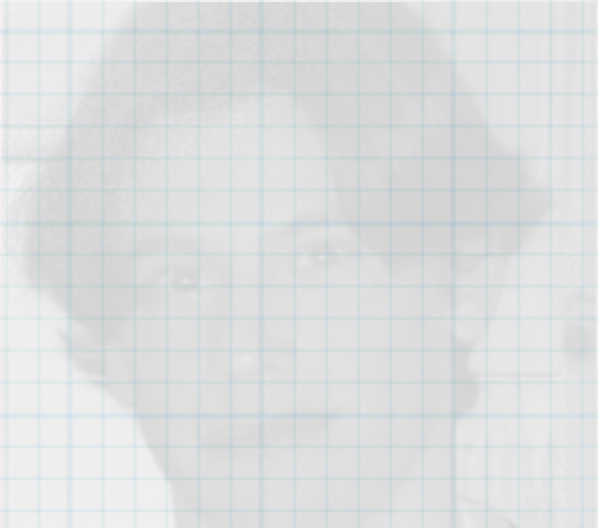

Alessandro Bisio

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Applications to Optimal Q-Tomography and Q-Learning

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Outline

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  - Convex optimization method based on the new notions of quantum comb and quantum tester

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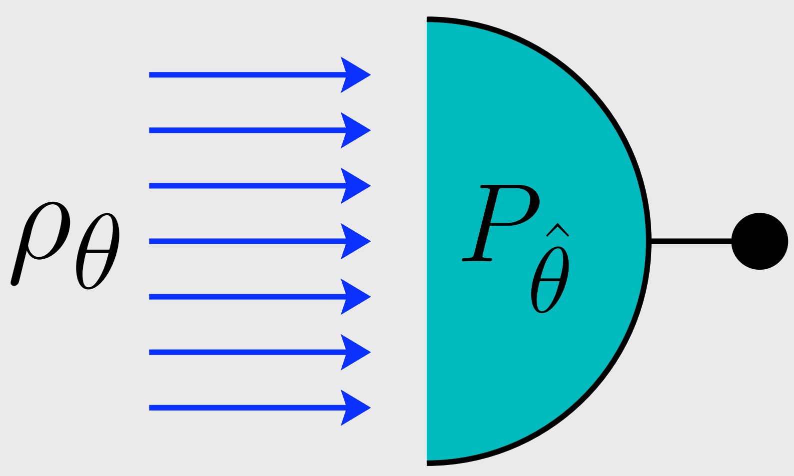
Helstrom-Holevo Quantum Estimation Theory

Quantum state ρ_θ parameterized by θ

Problem: estimate θ optimally according to the cost function $C(\theta, \hat{\theta})$



Helstrom-Holevo Quantum Estimation Theory

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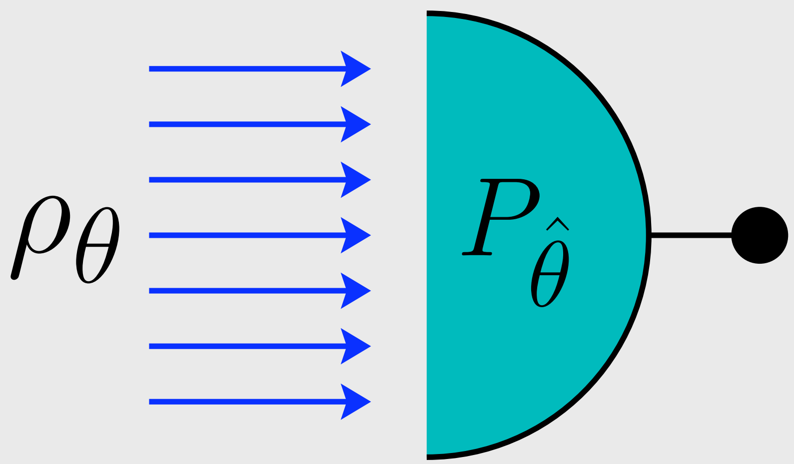
Mathematical formulation:

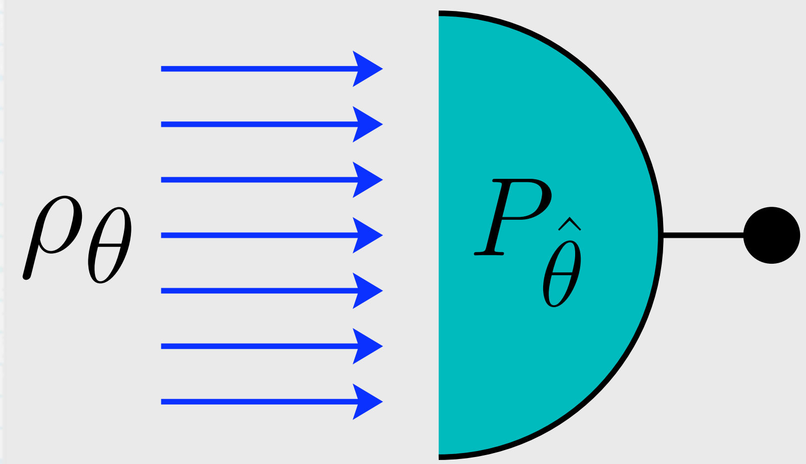
find the optimal POVM $P_{\hat{\theta}}$ minimizing the cost



Helstrom-Holevo Quantum Estimation Theory

Practically interesting situation (e.g. for the phase of an e.m. mode):

$$\boxed{\theta \Longrightarrow \rho_\theta = U_\theta \rho U_\theta^\dagger}$$



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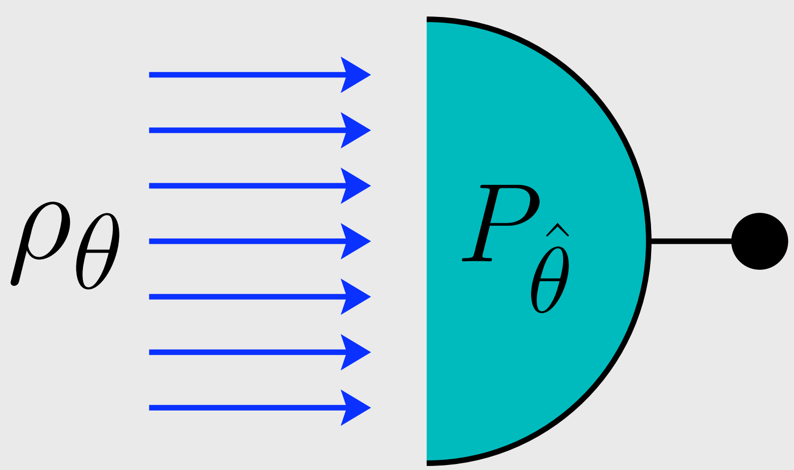

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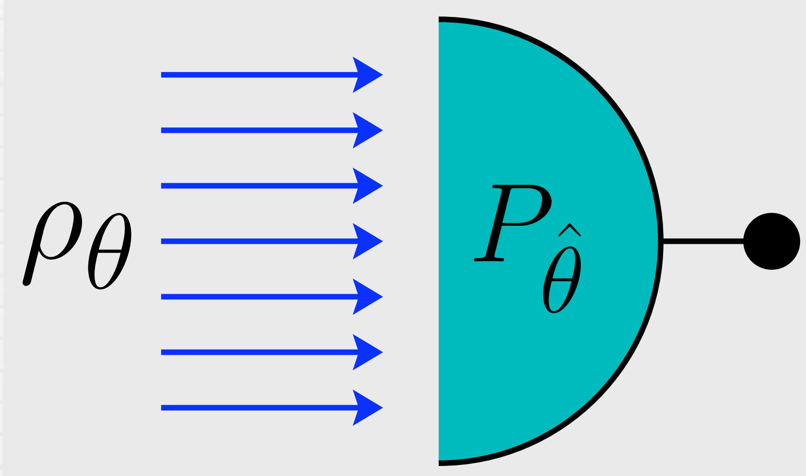

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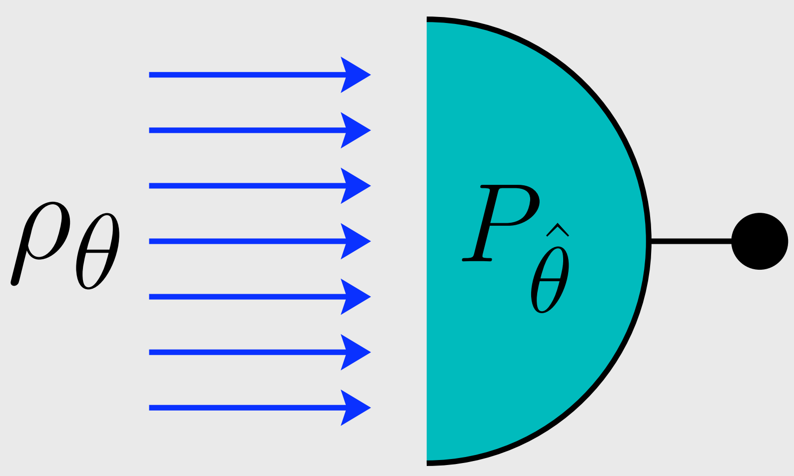

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The optimal POVM for
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Interesting situation: the parameter to be estimated
is encoded on a transformation---not on the state!

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Quantum Estimation Theory

Problem: estimate x parameterizing the (unitary) transformation U_x optimally according to the cost function

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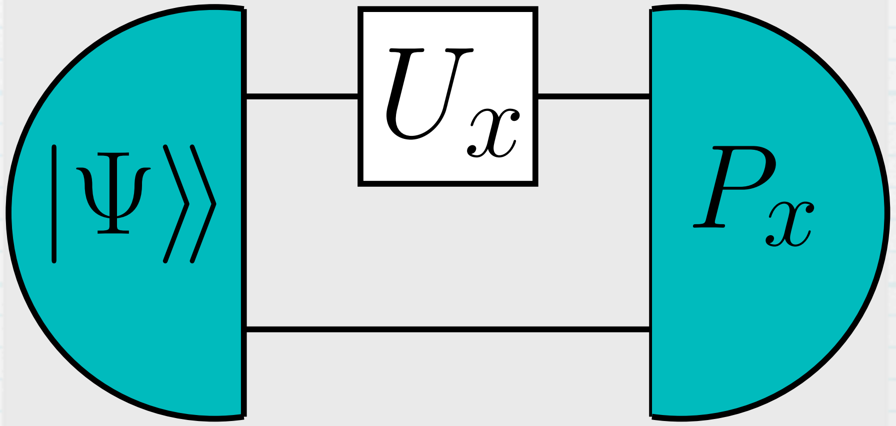

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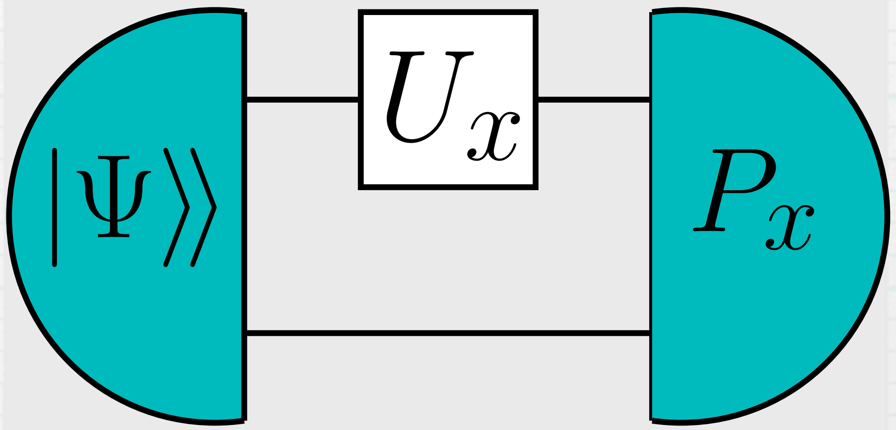

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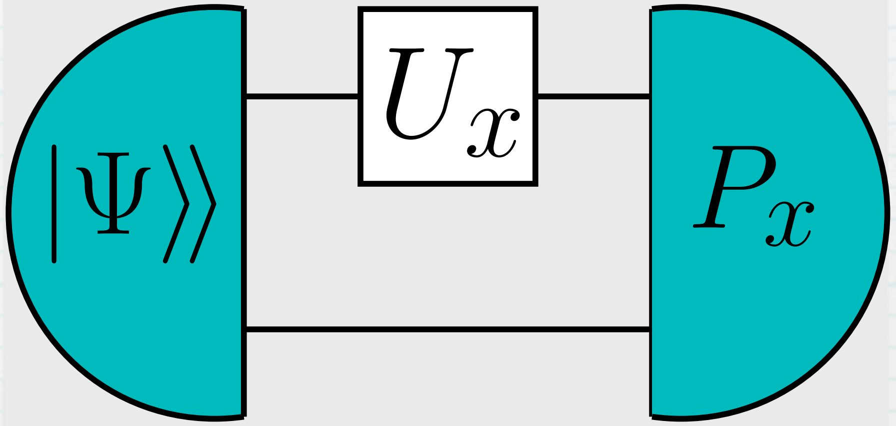
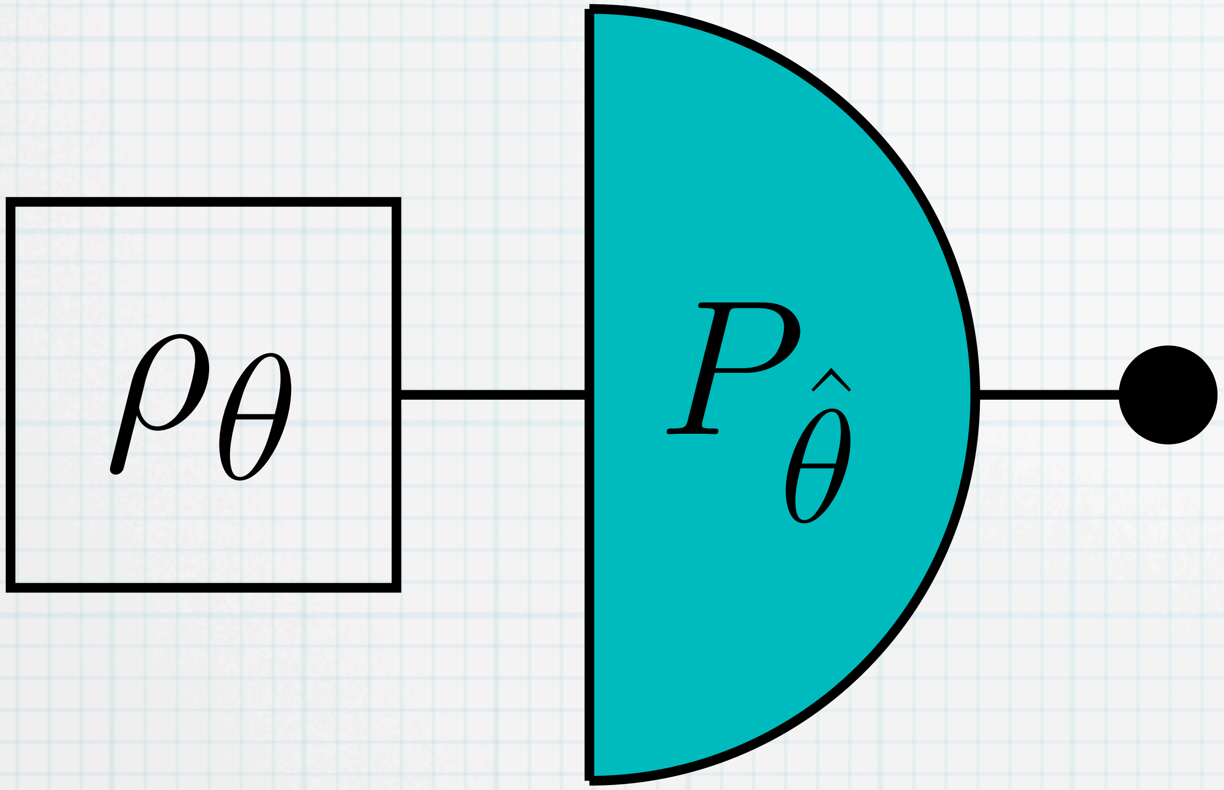


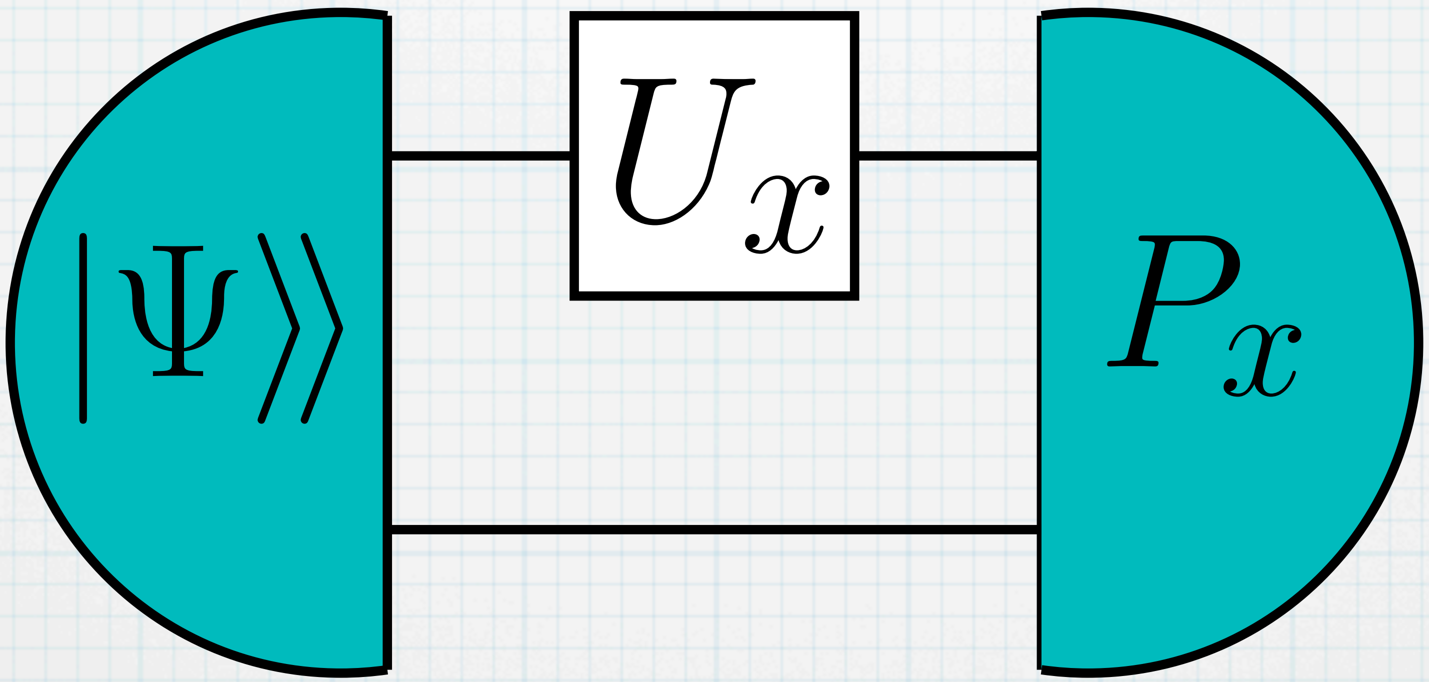
For the phase no need of entanglement (we were lucky!)

Quantum Estimation Theory

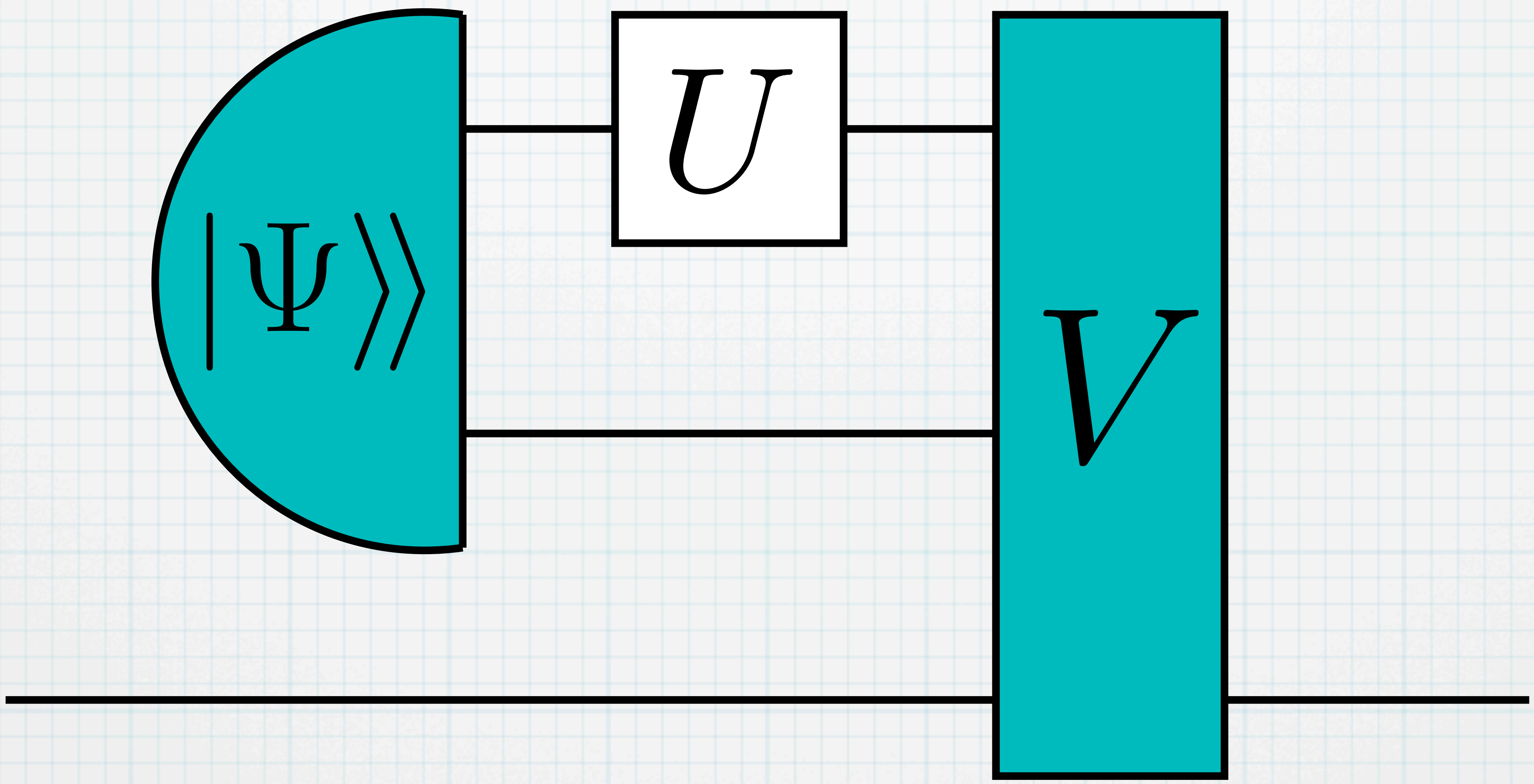


Quantum Estimation Theory

New scheme

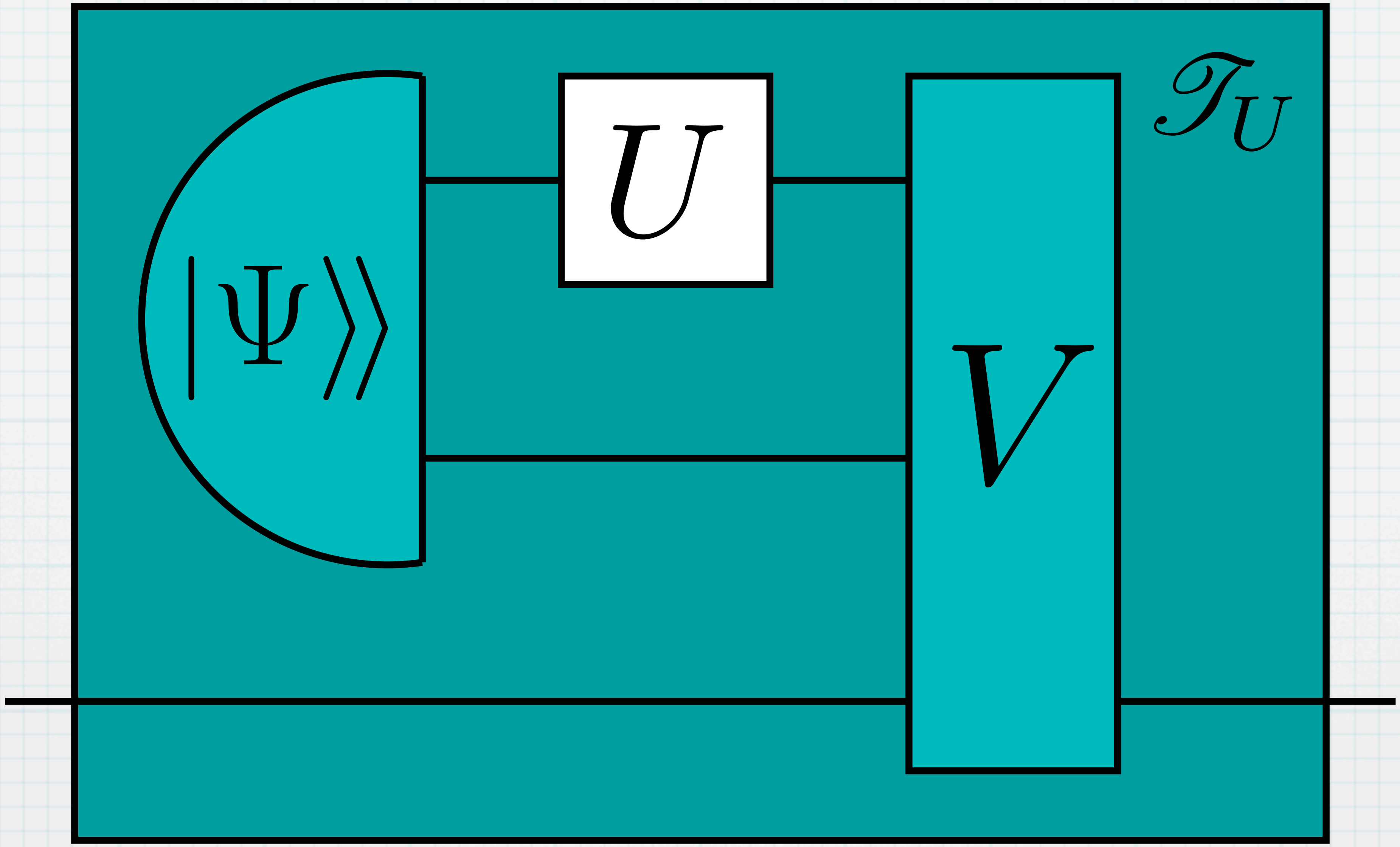


Quantum Feedback



- quantum feedback: perform a transformation $\mathscr{T}_U$ on a system which depends on an unknown unitary transformation U occurring on a (generally different) system (e.g. reference-frames realignment).

Quantum Feedback



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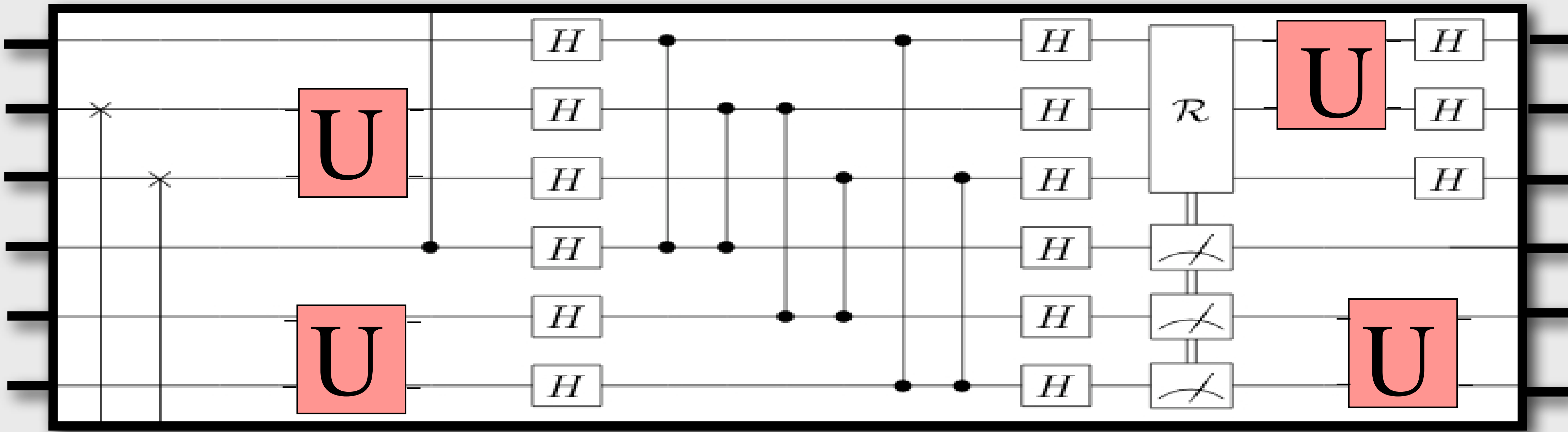
- However, you better use a coherent strategy, in which you perform a joint POVM
- and you want to do the same for the quantum feedback

What is the best that you can do?

Use a Quantum Board!



General scheme: put the copies of the unknown unitary in a suitable quantum circuit which performs the desired transformation/estimation.

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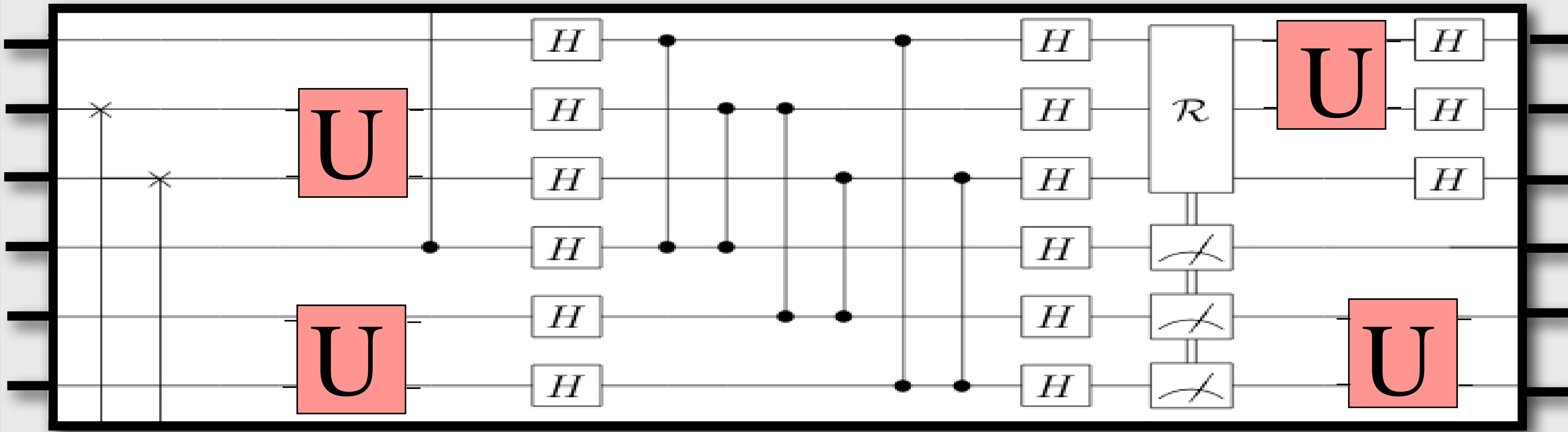
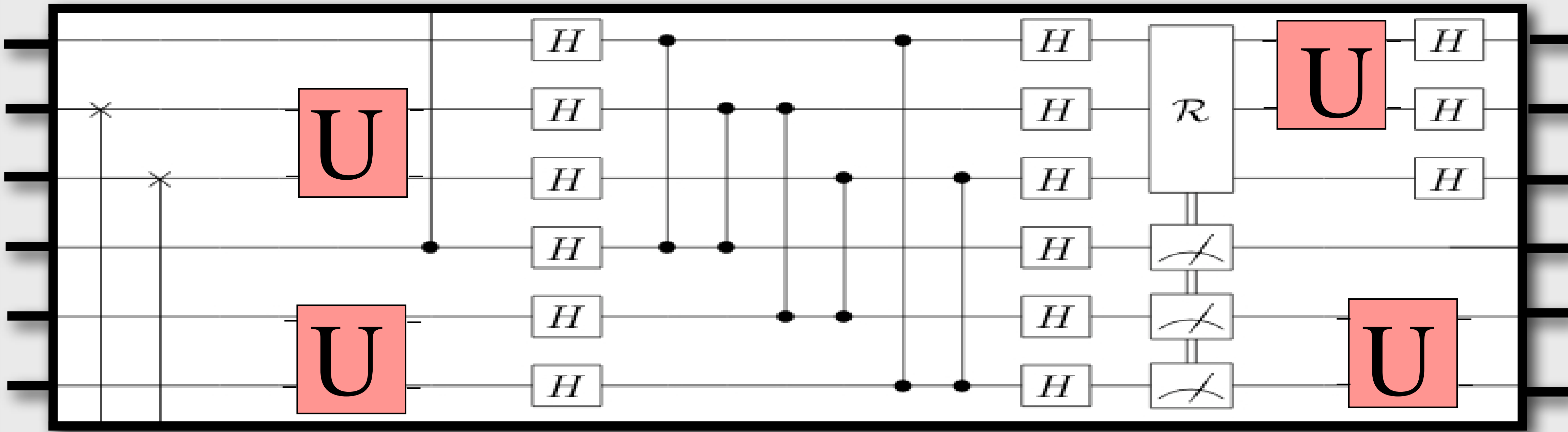


General scheme: put the copies of the unknown unitary in a suitable quantum circuit which performs the desired transformation/estimation.

Quantum circuit board: input and output are themselves circuits that are slotted into the board.

Use a Quantum Board!



Optimize the quantum circuit board for all possible dispositions of the slots

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It looks a difficult problem ...

For example: what is the optimal board for phase estimation?

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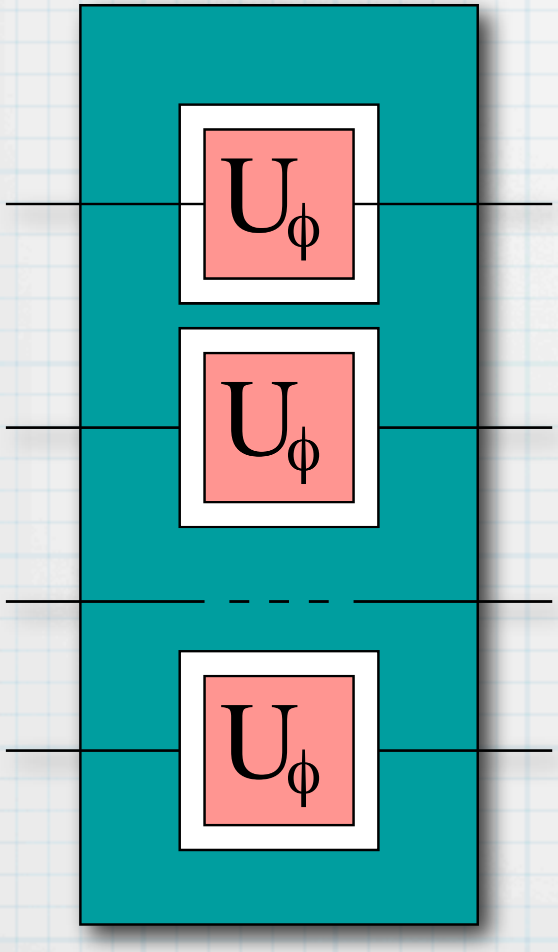

In parallel over a joint entangled state?

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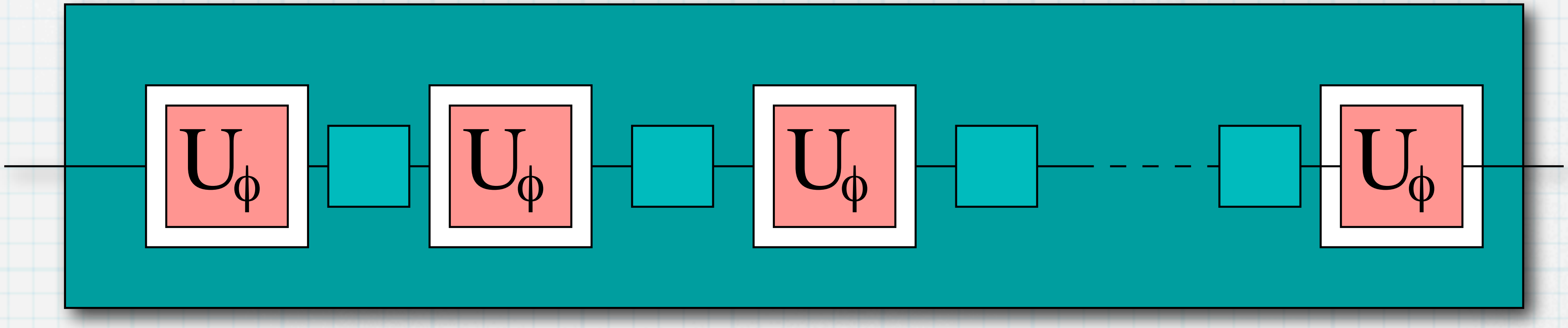
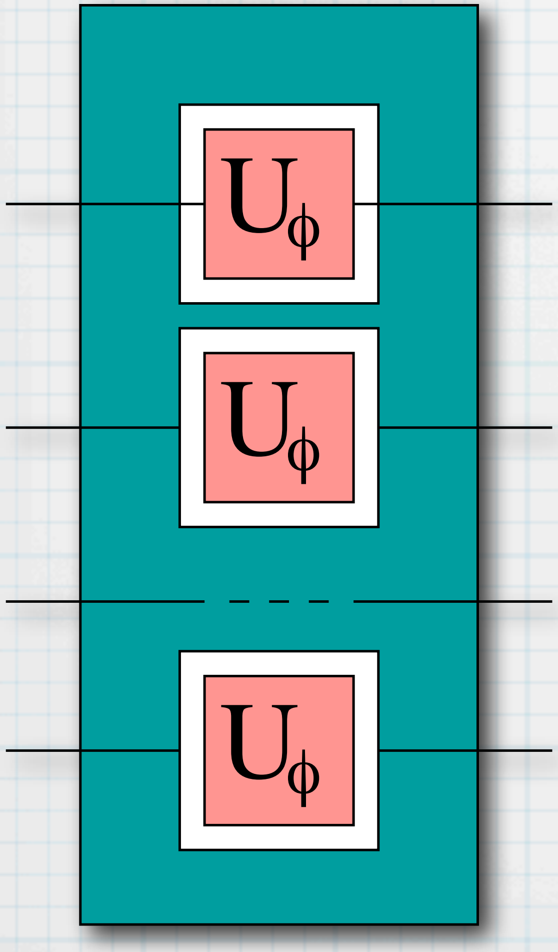


In sequence intercalated by some unitary?



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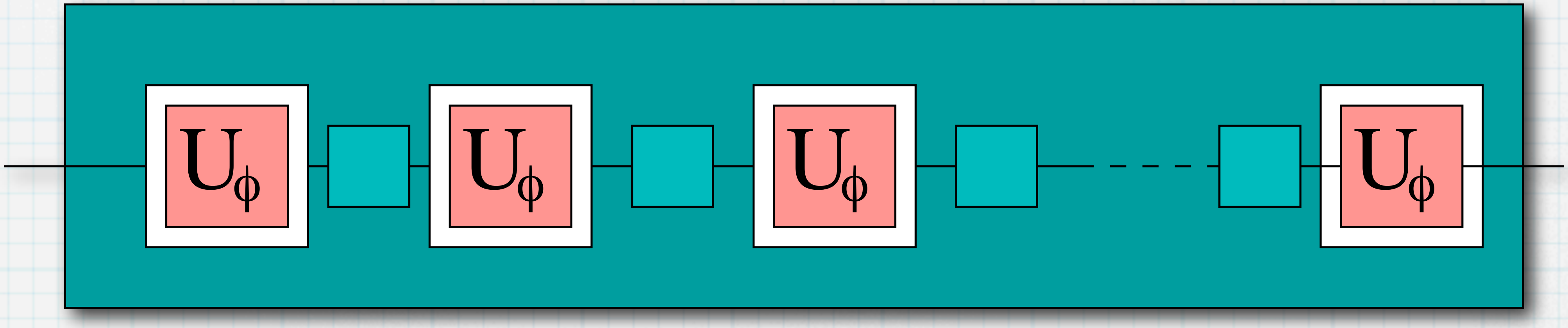

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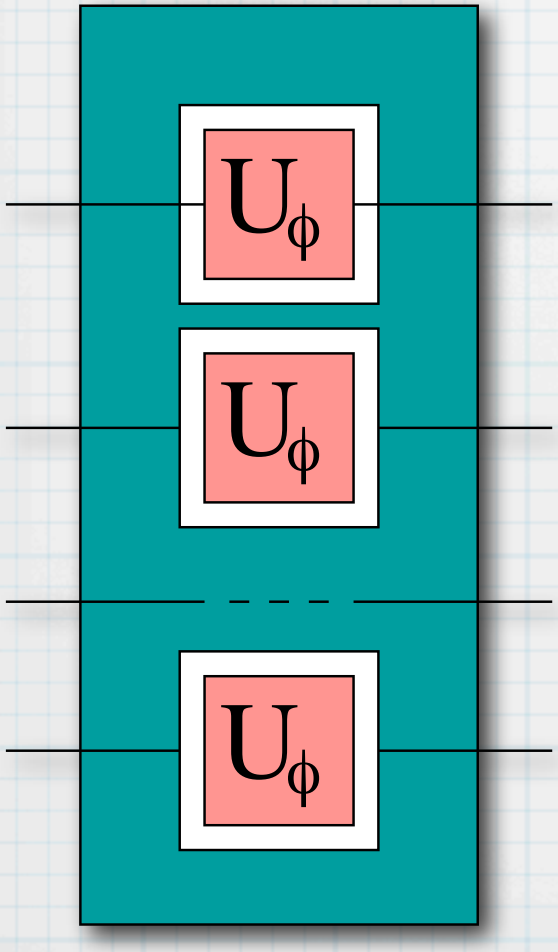

In parallel over a joint entangled state?

Asymptotically: same sensitivity [Giovannetti, LLoyd, Maccone, PRL 96, 010401 (2006)]

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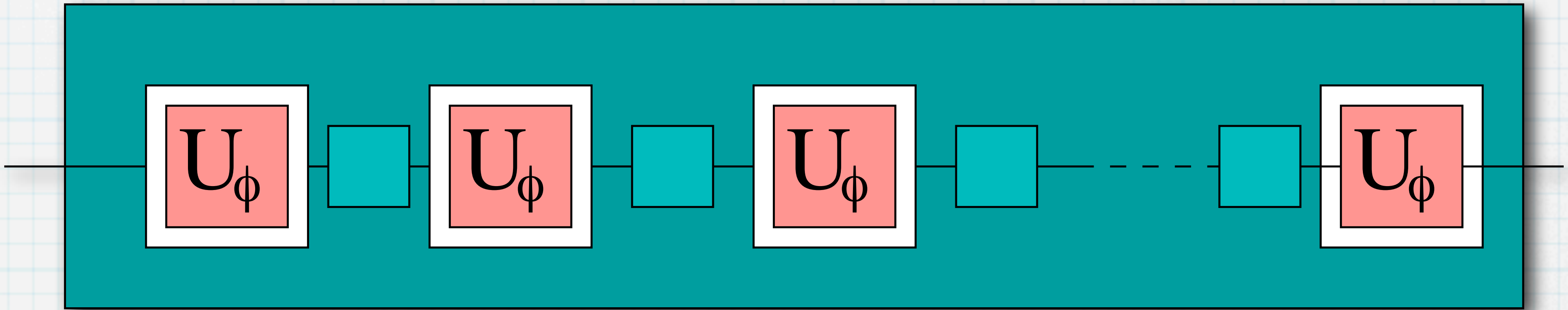
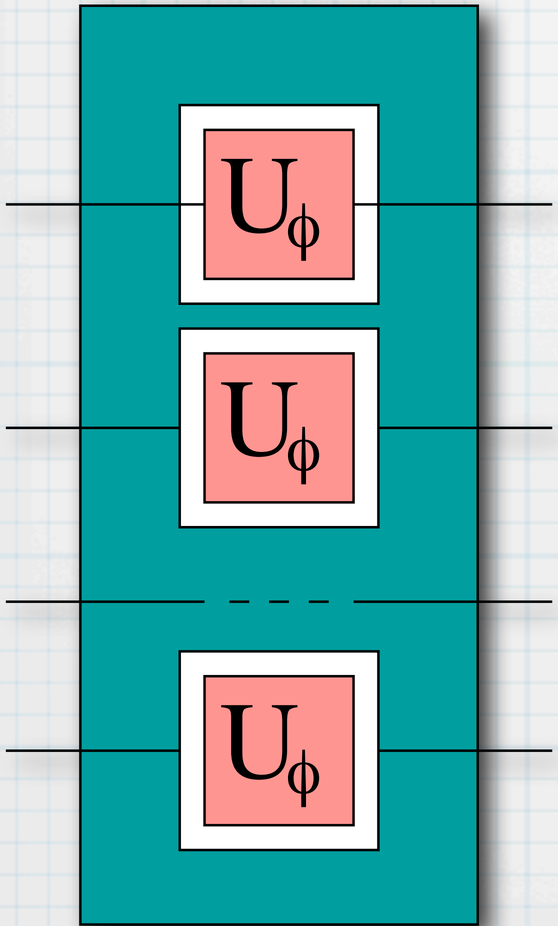


In sequence intercalated by some unitary?

For unitary discrimination: [Duan, Feng, Ying, PRL 98, 100503 (2007)]



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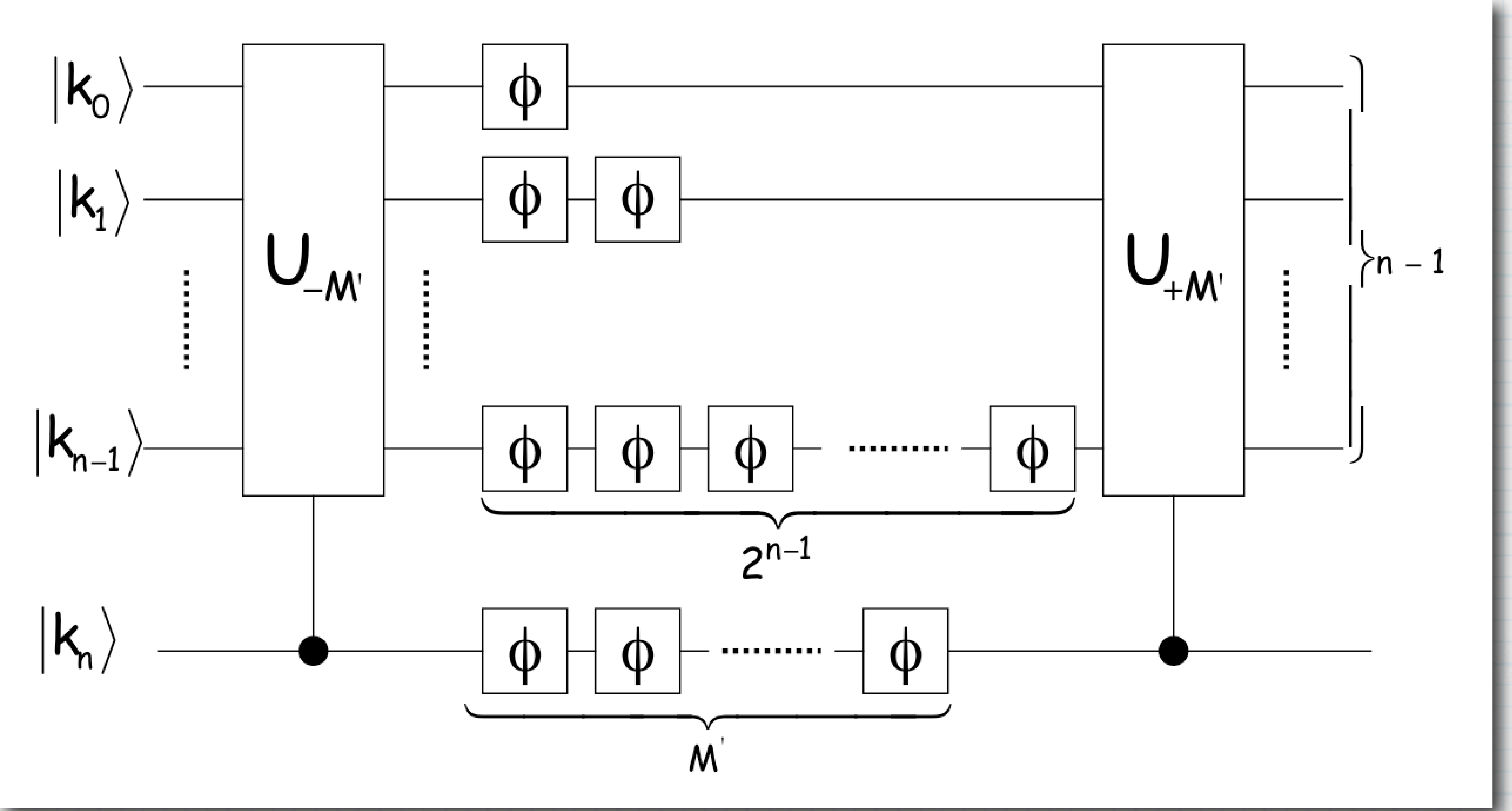
For example: what is the optimal board for phase estimation?

An optimal board architecture [van Dam, D'Ariano, Ekert, Macchiavello, Mosca, PRL 98, 090501 (2007)]



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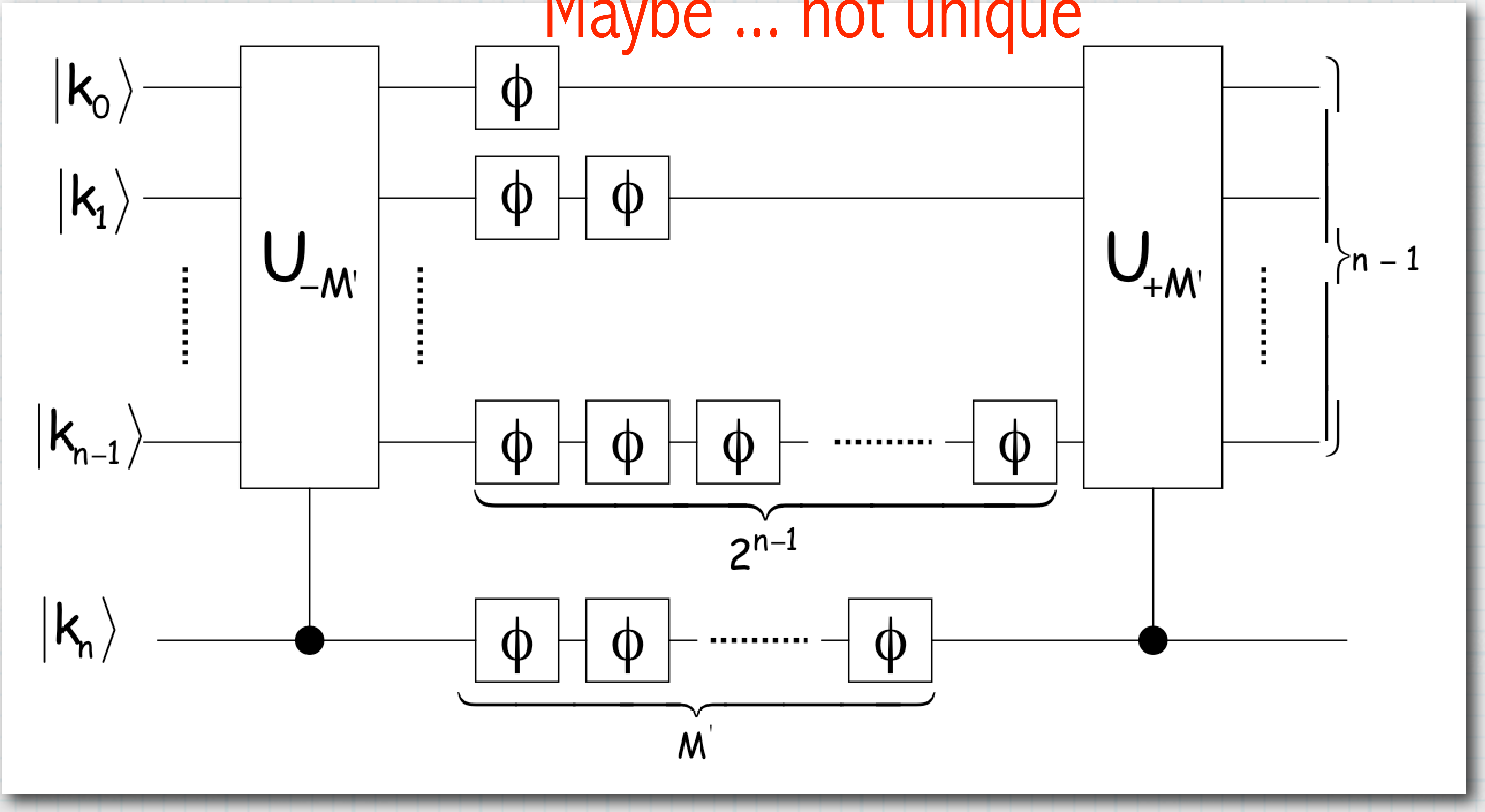
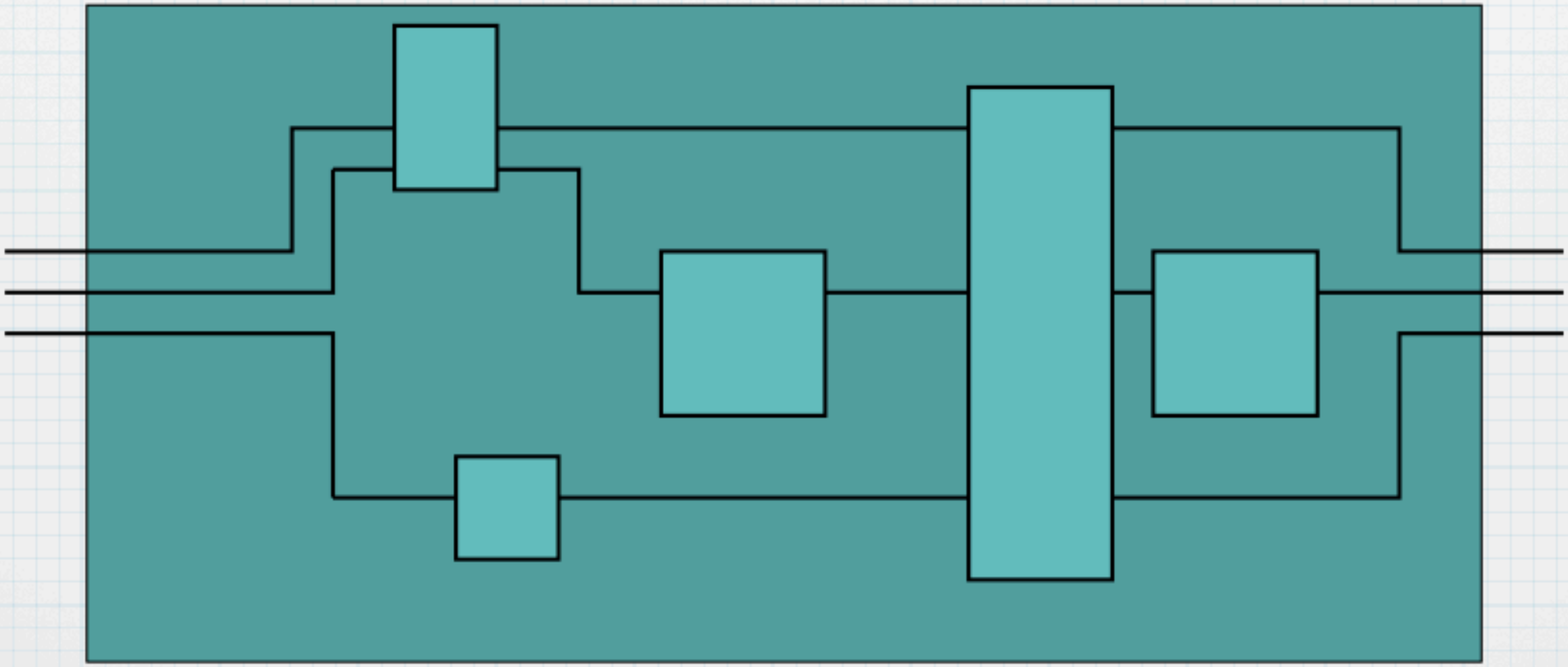
Maybe ... not unique



What is the mathematical formulation of the problem?

Quantum Channel

It can be regarded as an equivalence class of quantum circuits performing the same input-output transformation ...



Quantum Channel

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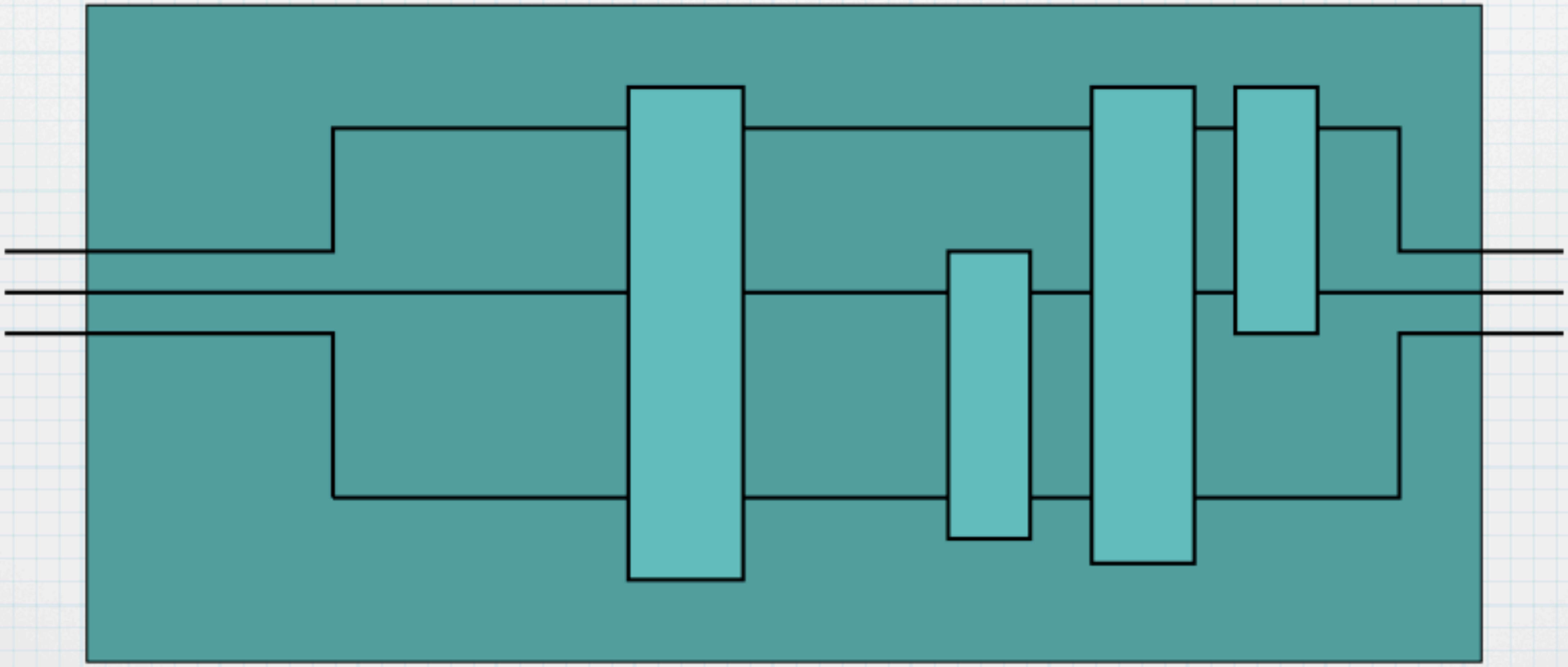
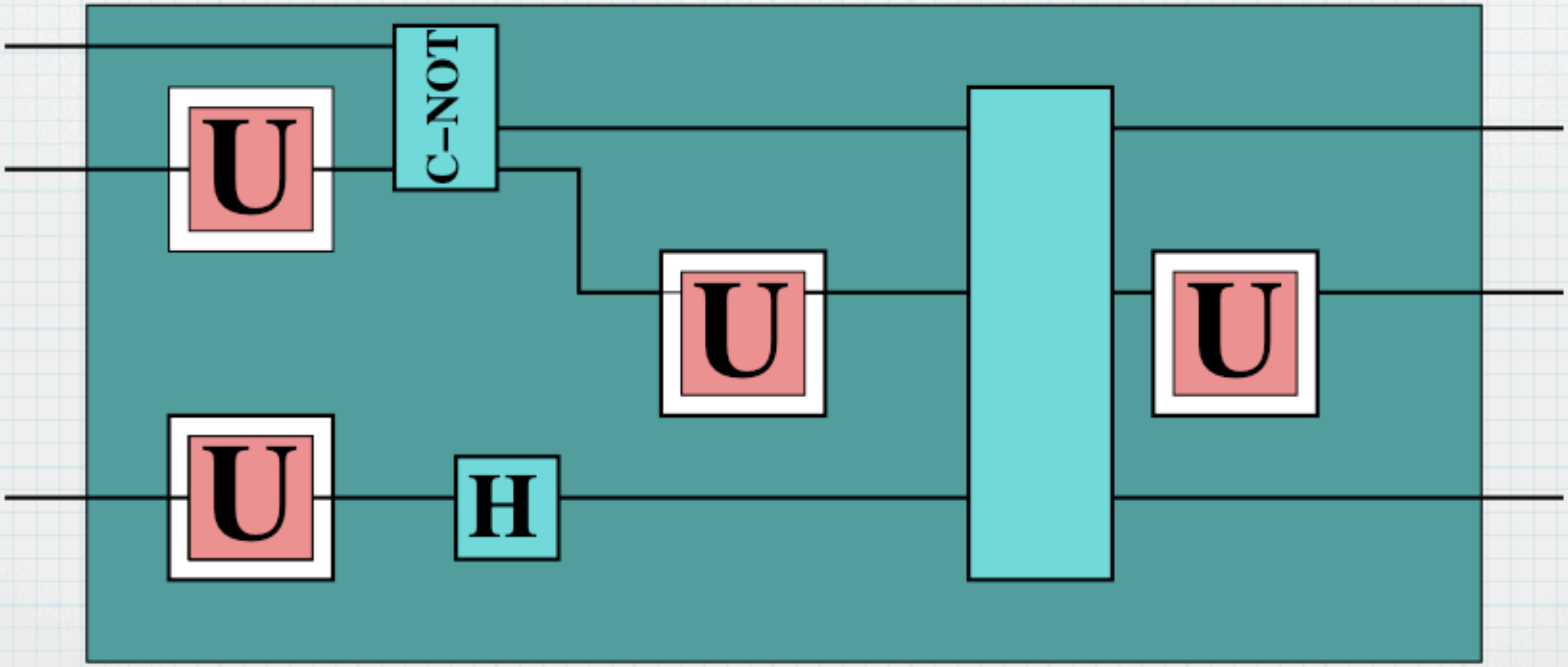
For a channel the input and the output are states



Quantum Board

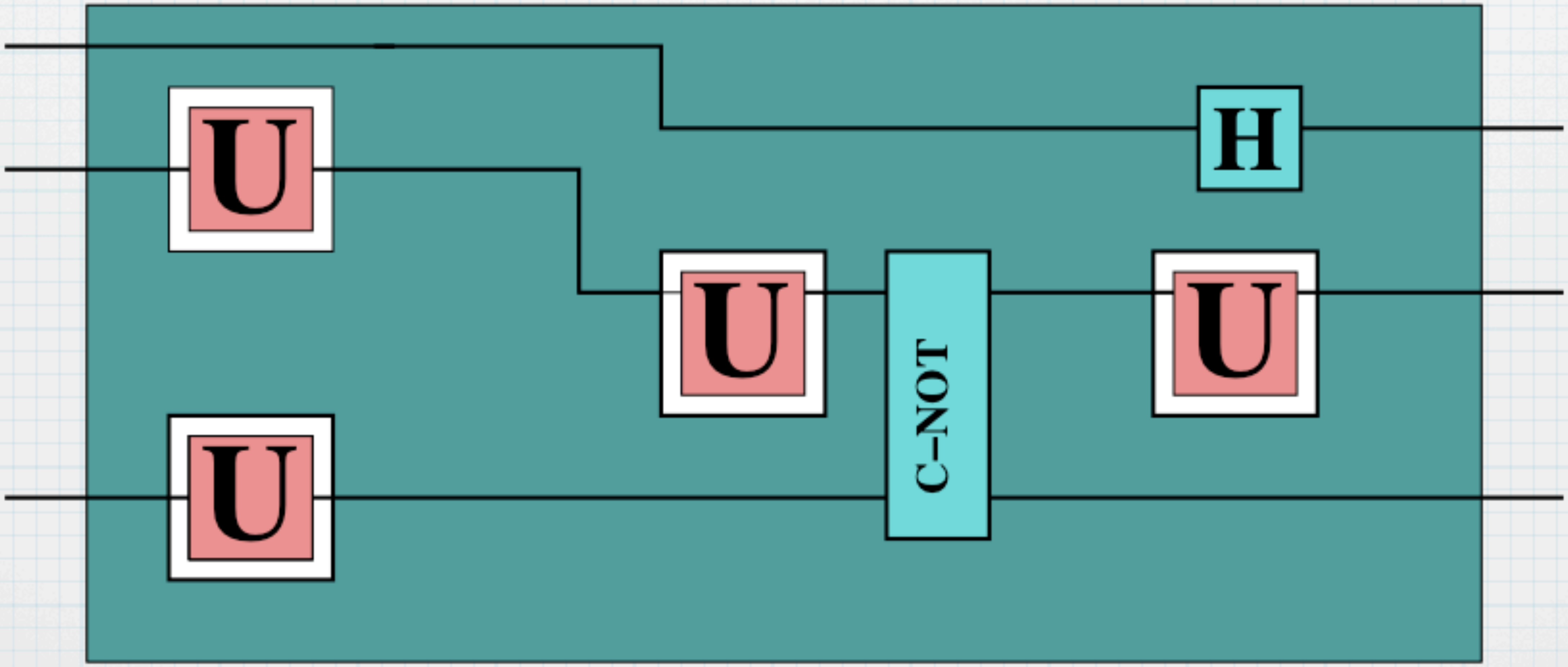
Equivalence class of quantum circuits boards performing the same overall input-output transformation ...



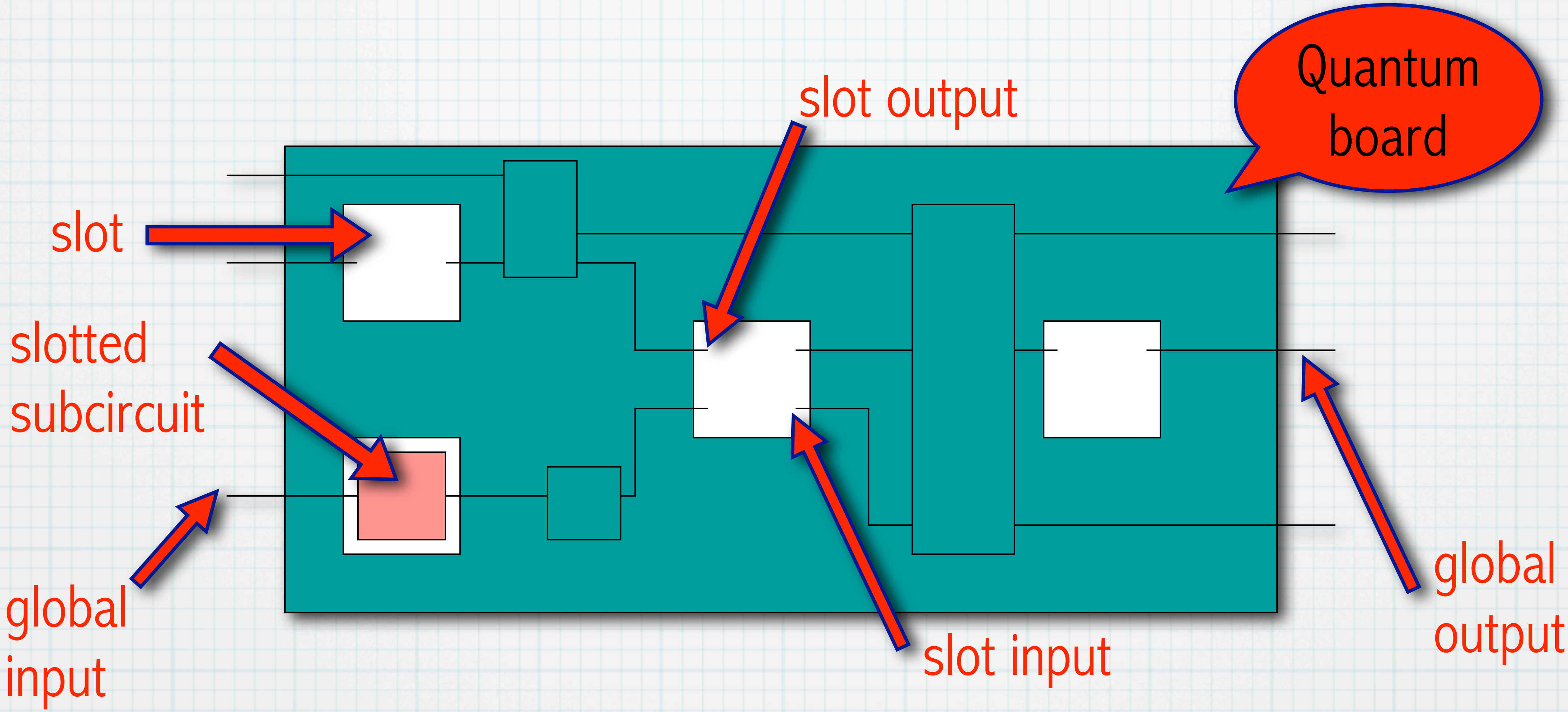
Quantum Board

Equivalence class of quantum circuits boards performing the same overall input-output transformation ...

But now, the input and the output are transformations



Quantum Board

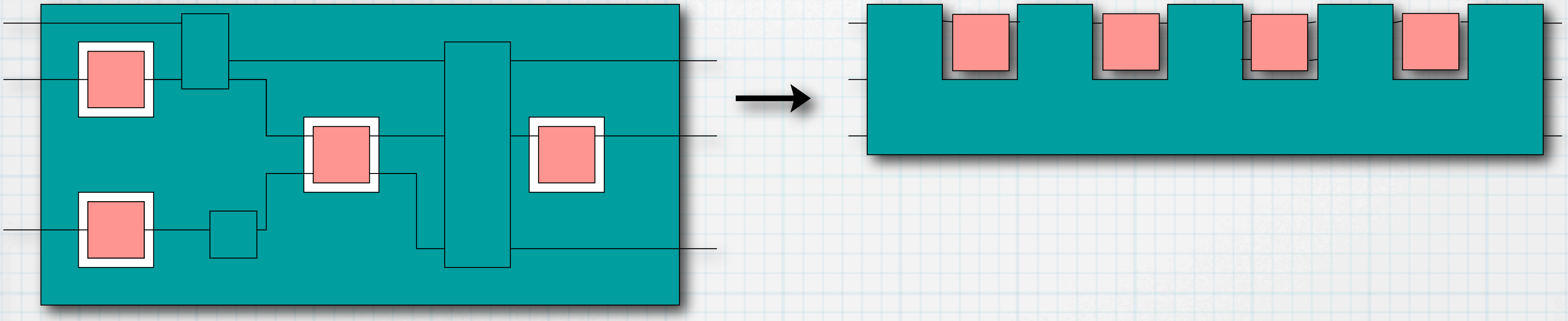


Problem: what is the optimal board for given slots achieving a global input/output transformation optimally according to a given cost function?

Quantum Combs

G.Chiribella, G.M.D'Ariano, P.Perinotti, PRL 101 060401 (2008)

All circuits-boards can be reshaped in form of "combs", with an ordered sequence of slots, each between two successive teeth



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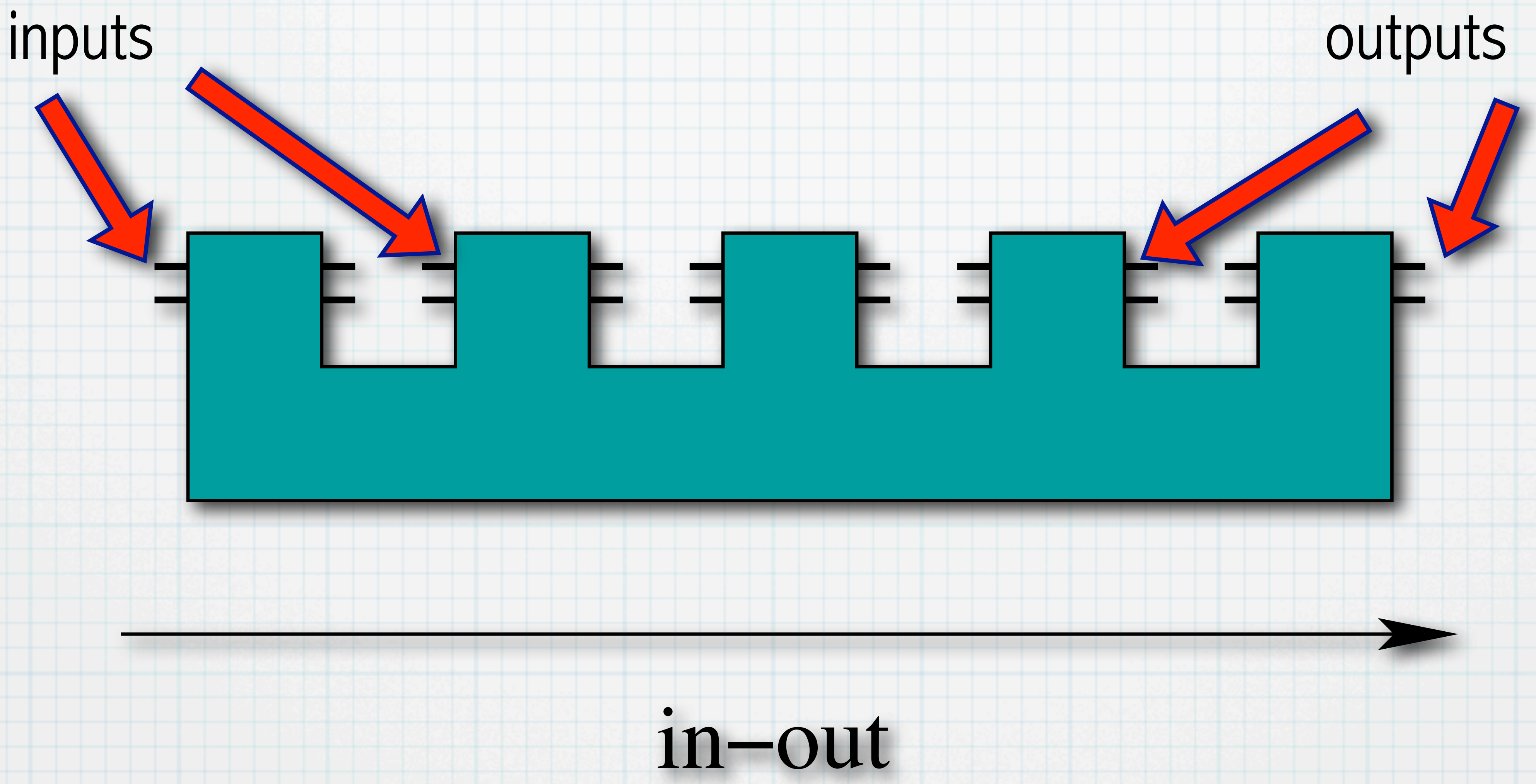
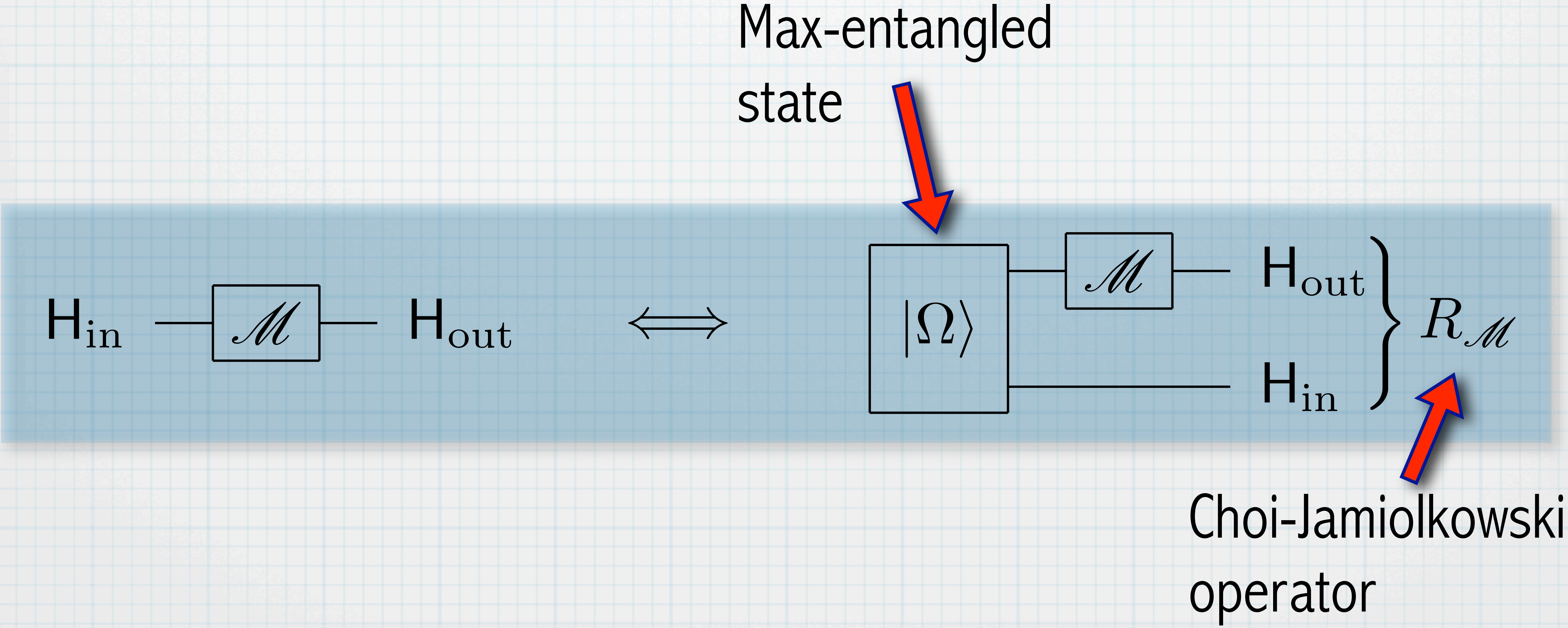


Pins = quantum systems with generally variable dimensions

How do we describe a quantum comb mathematically?

Channel: Choi representation

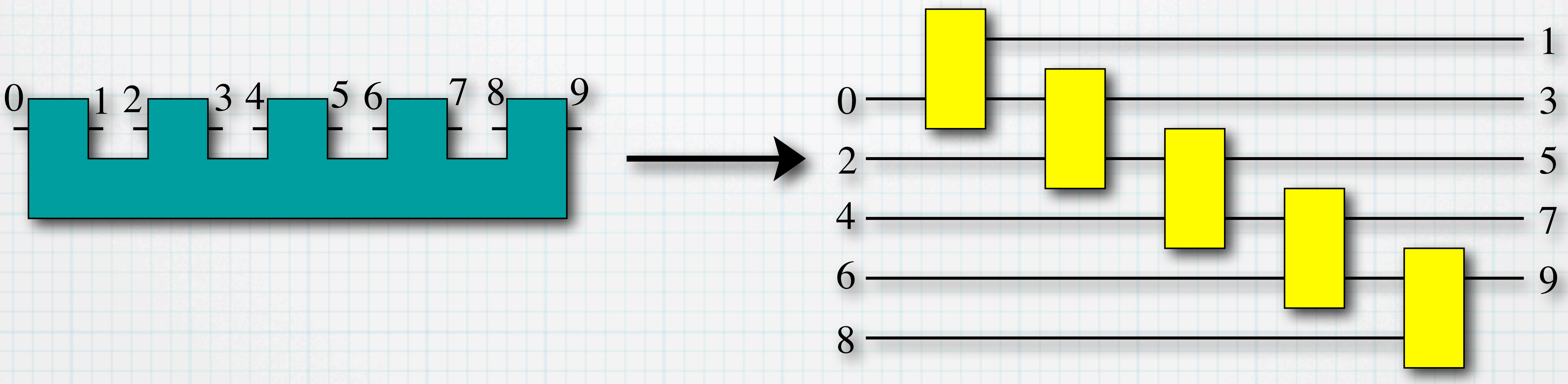
Mathematically the input-output transformation operated by a quantum circuit is a CP map, and is in one-to-one correspondence with a positive operator called "Choi-Jamiolkowski operator"---the output state of the map applied locally to a maximally entangled state.



PRL 101 060401 (2008)

Causal networks

The quantum comb is equivalent to a causal network with all inputs on the left and all outputs on the right



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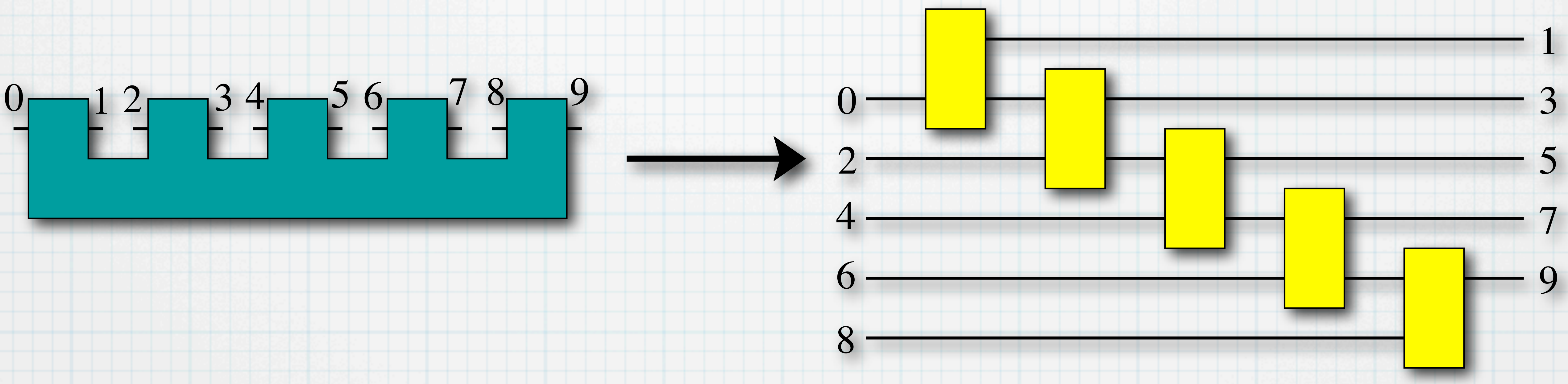
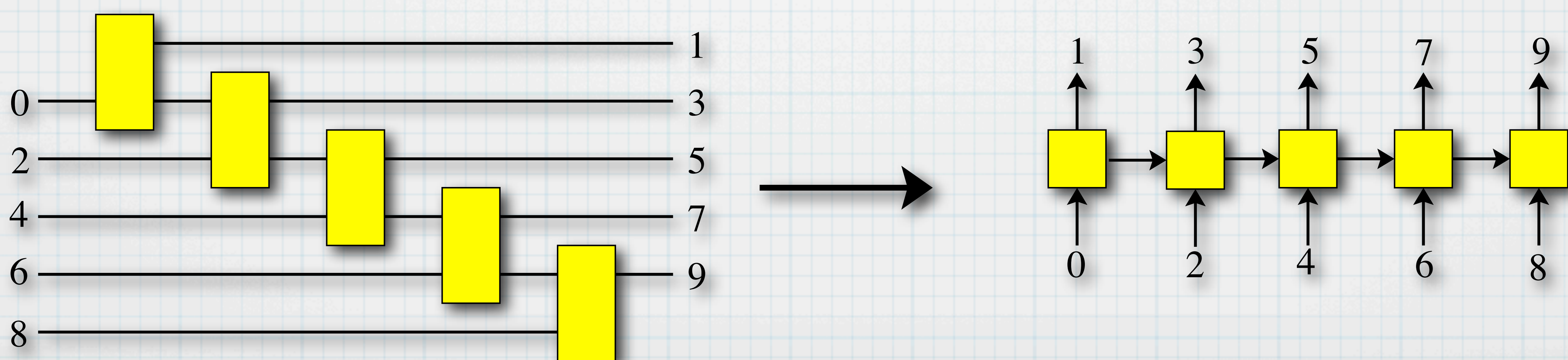


The causal network is also equivalent to the stack of memory channels



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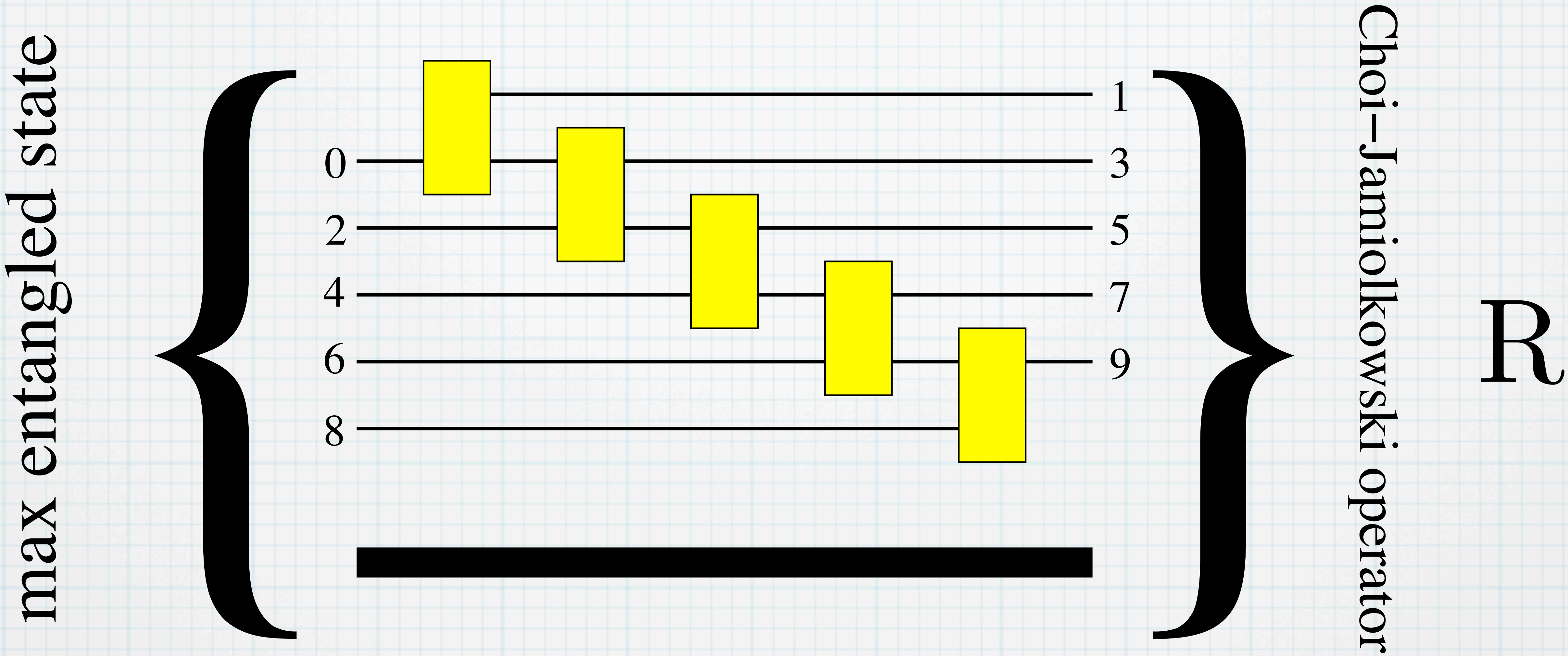

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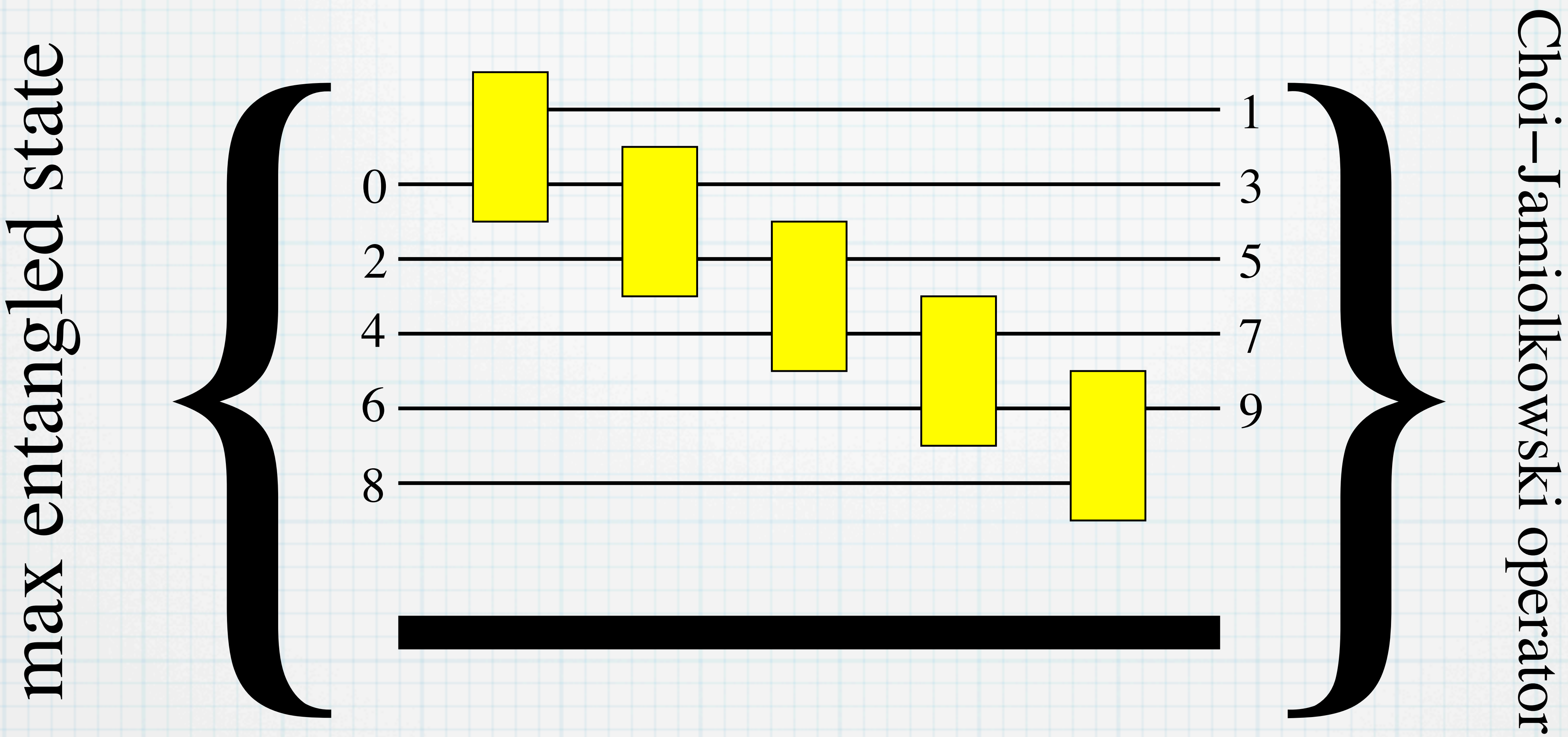


R

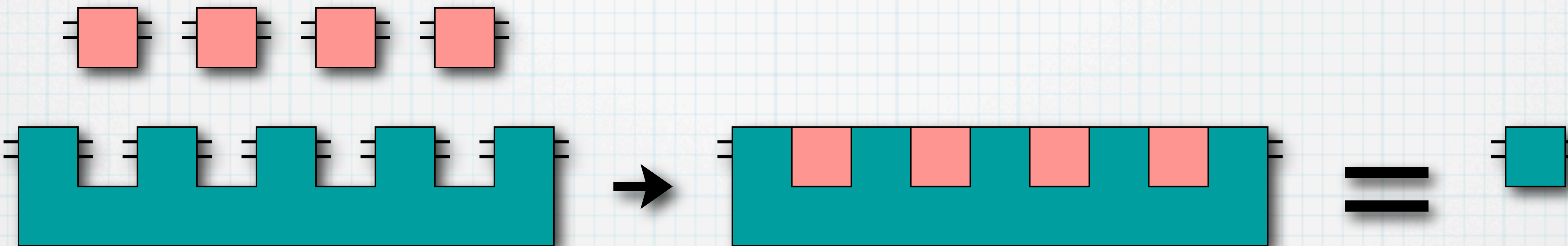
Causality constraints: (N+1 inputs/outputs)

$$\mathrm{Tr}_{2n-1}[R^{(n)}] = I_{2n-2} \otimes R^{(n-1)}, \quad n = 1, \ldots, N$$

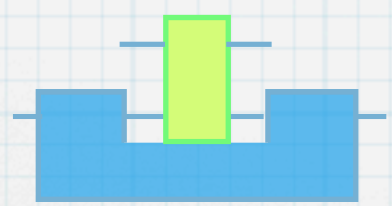
$$R^{(0)} = 1, \quad R^{(n)} = R$$

Supermaps

A quantum comb performs a transformation that is a generalization of the quantum operation: the so called "supermap"



A supermap sends a series of N channels to one channel, also when applied locally, e.g.



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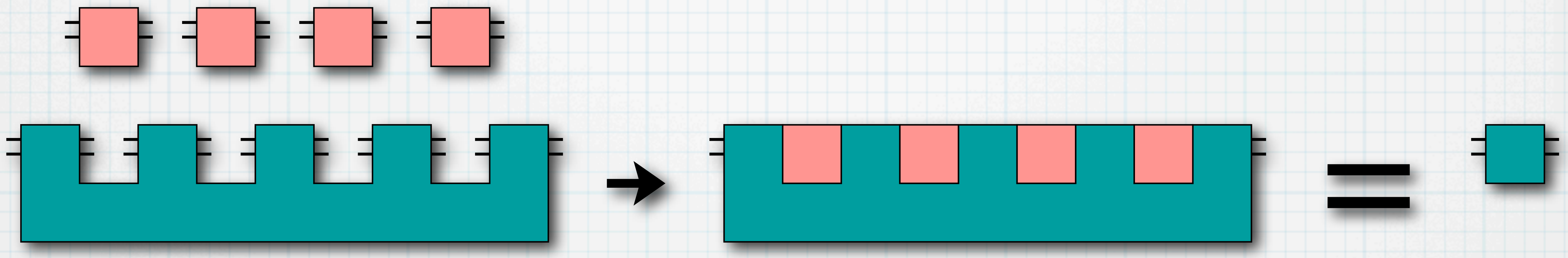
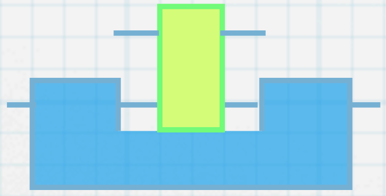


A supermap sends a series of N channels to one channel, also when applied locally, e.g. 

Mathematically it is represented by a CP N-linear map which sends N Choi operators to one Choi operator, and with his own Choi operator satisfying the causality constraints.

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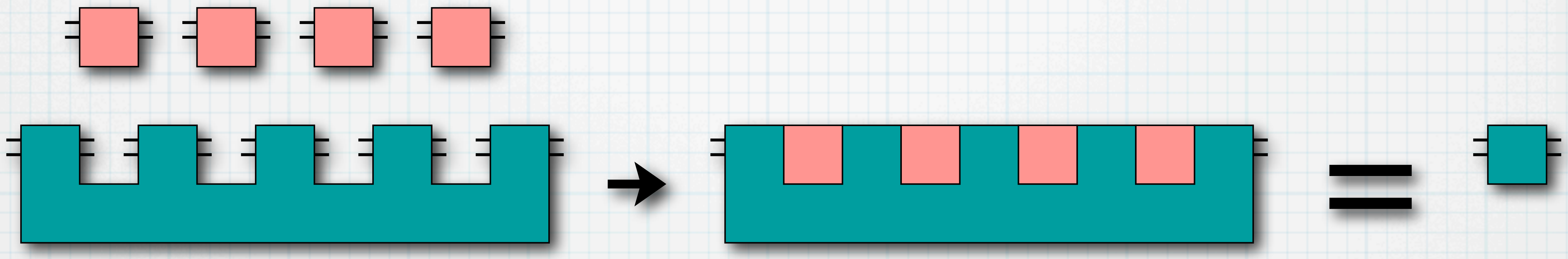
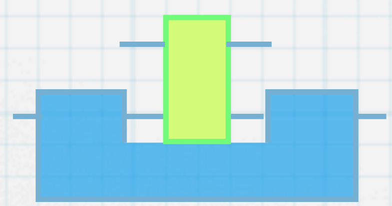


A supermap sends a series of N channels to one channel, also when applied locally, e.g. 

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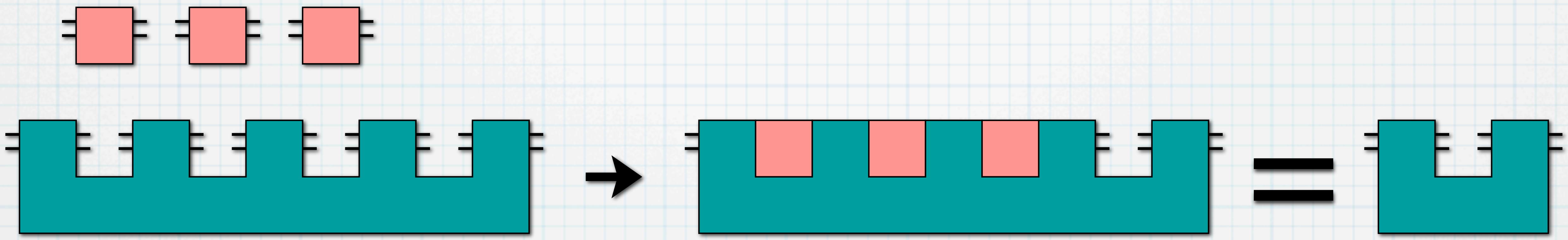
(we can likewise consider probabilistic supermaps).

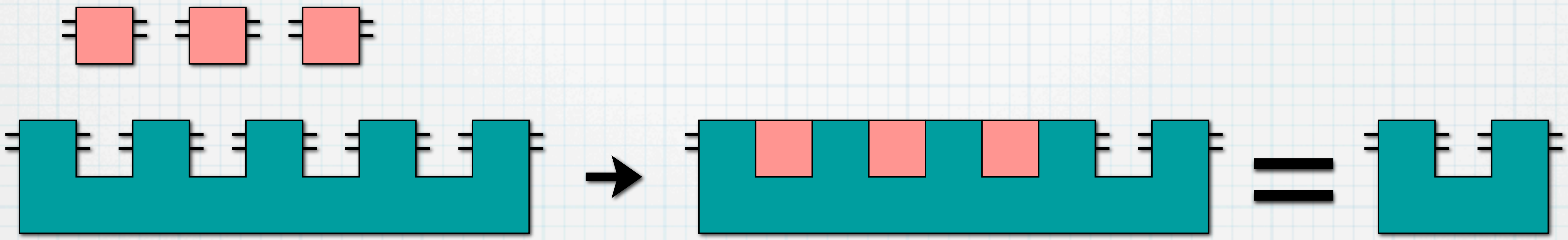
Supermaps

More generally, a quantum comb maps a series of channels into a comb

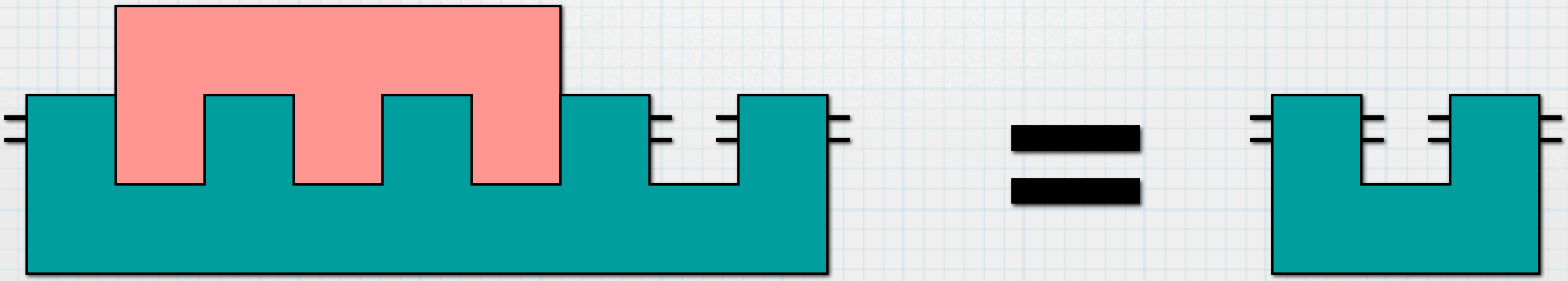


Supermaps

More generally, a quantum comb maps a series of channels into a comb



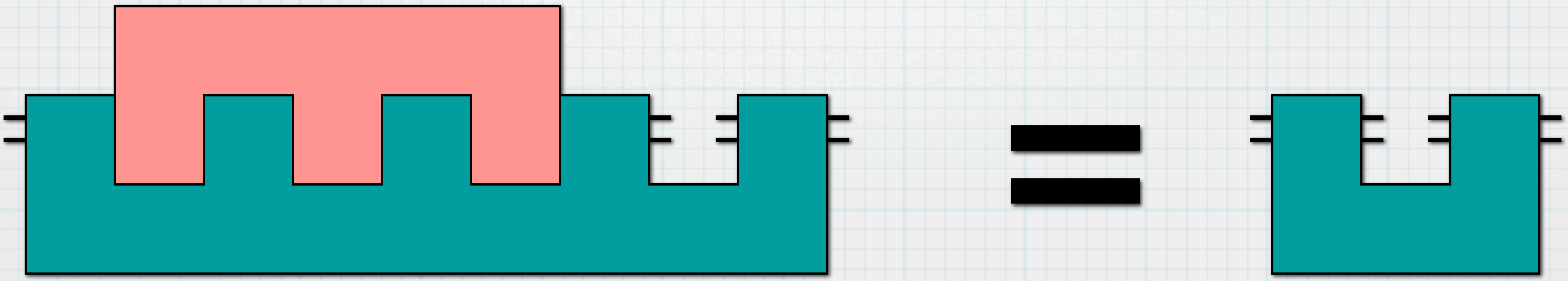
or, even more generally, a comb to a comb



PRL 101 060401 (2008)

Supermaps

The notion of supermap is the last level of generalization, i.e. “super-supermaps” (mapping supermaps to supermaps) are still supermaps = quantum combs.



Link product

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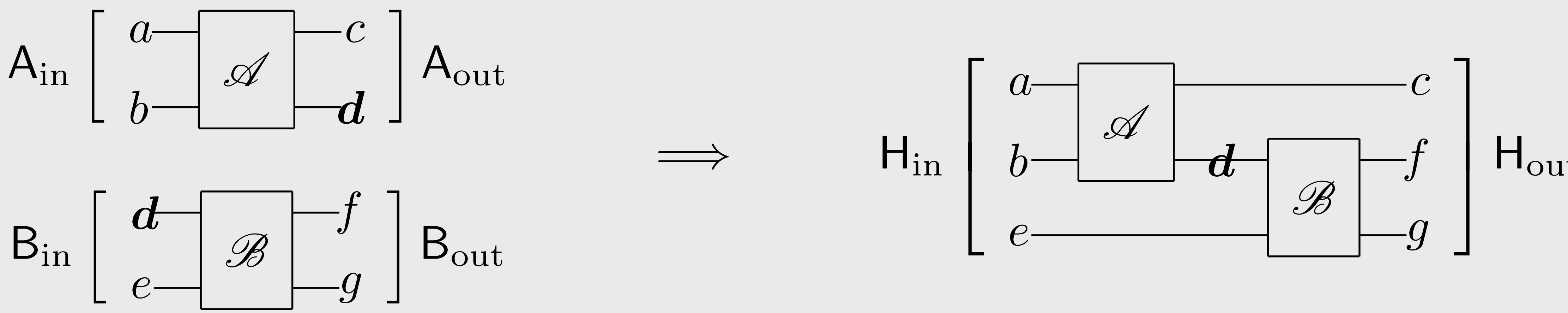
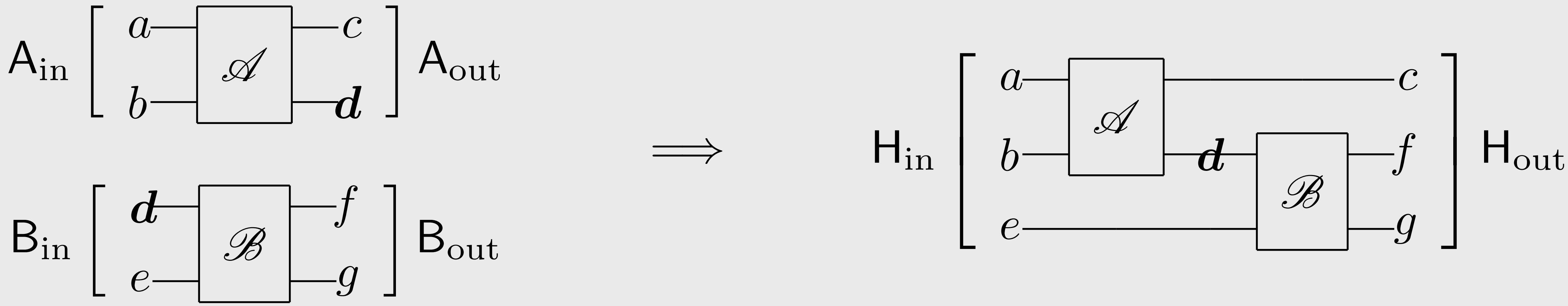


Link product

PRL 101 060401 (2008)



Choi-operator calculus

$$A \in \mathsf{B}(\mathsf{A}_{\mathrm{out}} \otimes \mathsf{A}_{\mathrm{in}}) = \mathsf{B}(\mathsf{H}_a \otimes \mathsf{H}_b \otimes \mathsf{H}_c \otimes \mathsf{H}_{\boldsymbol{d}}), \qquad \mathsf{J} \equiv \mathsf{H}_{\boldsymbol{d}}$$

$$B \in \mathsf{B}(\mathsf{B}_{\mathrm{out}} \otimes \mathsf{B}_{\mathrm{in}}) = \mathsf{B}(\mathsf{H}_{\boldsymbol{d}} \otimes \mathsf{H}_e \otimes \mathsf{H}_f \otimes \mathsf{H}_g)$$

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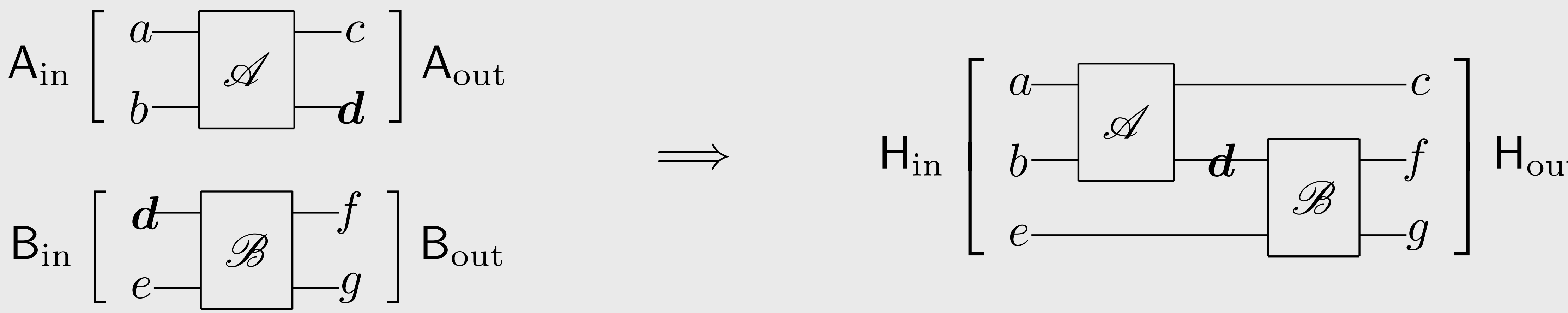
Link product



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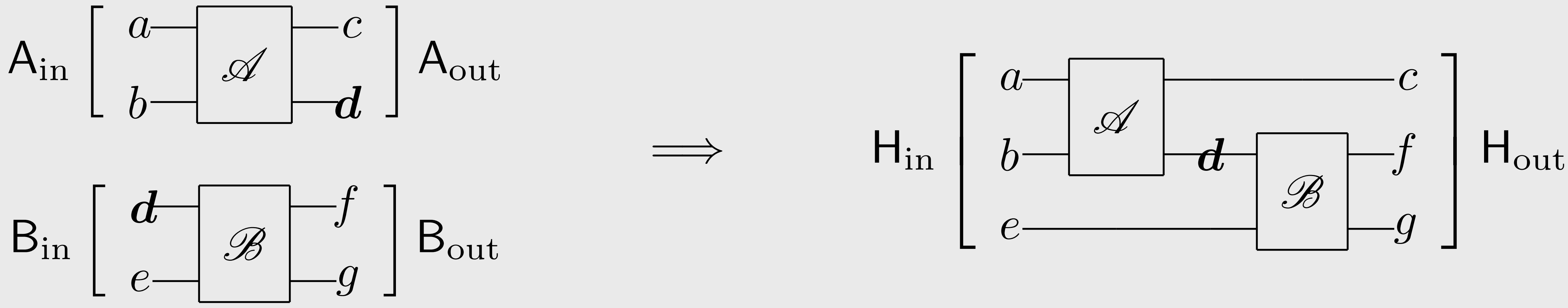
$$AB := (A \otimes I_{e,f,g})(I_{a,b,c} \otimes B)$$

quantum information theory group

PRL 101 060401 (2008)

Link product



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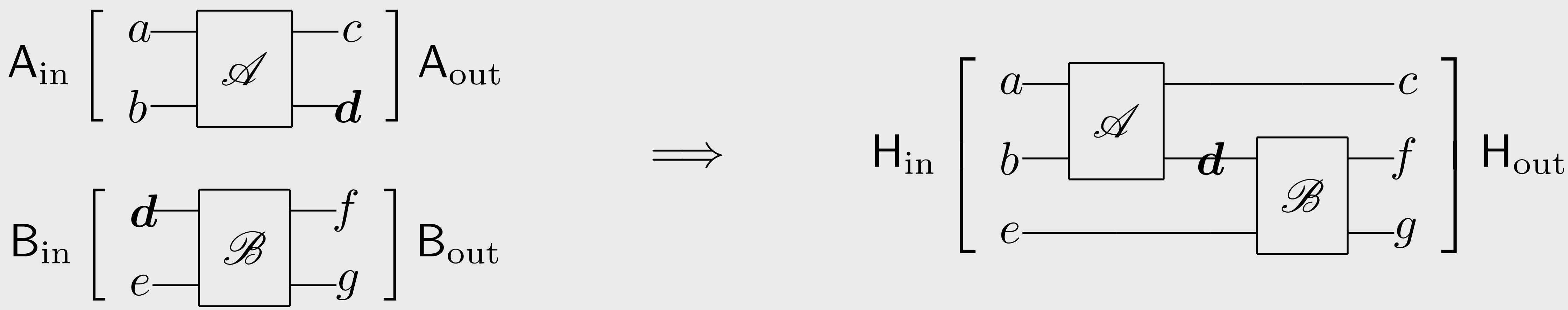
$$\boxed{A * B = \mathrm{Tr}_{\mathsf{J}}[A^{\theta_{\mathsf{J}}} B] \in \mathsf{B}(\mathsf{H}_{\mathrm{out}} \otimes \mathsf{H}_{\mathrm{in}})}$$

PRL 101 060401 (2008)

Link product



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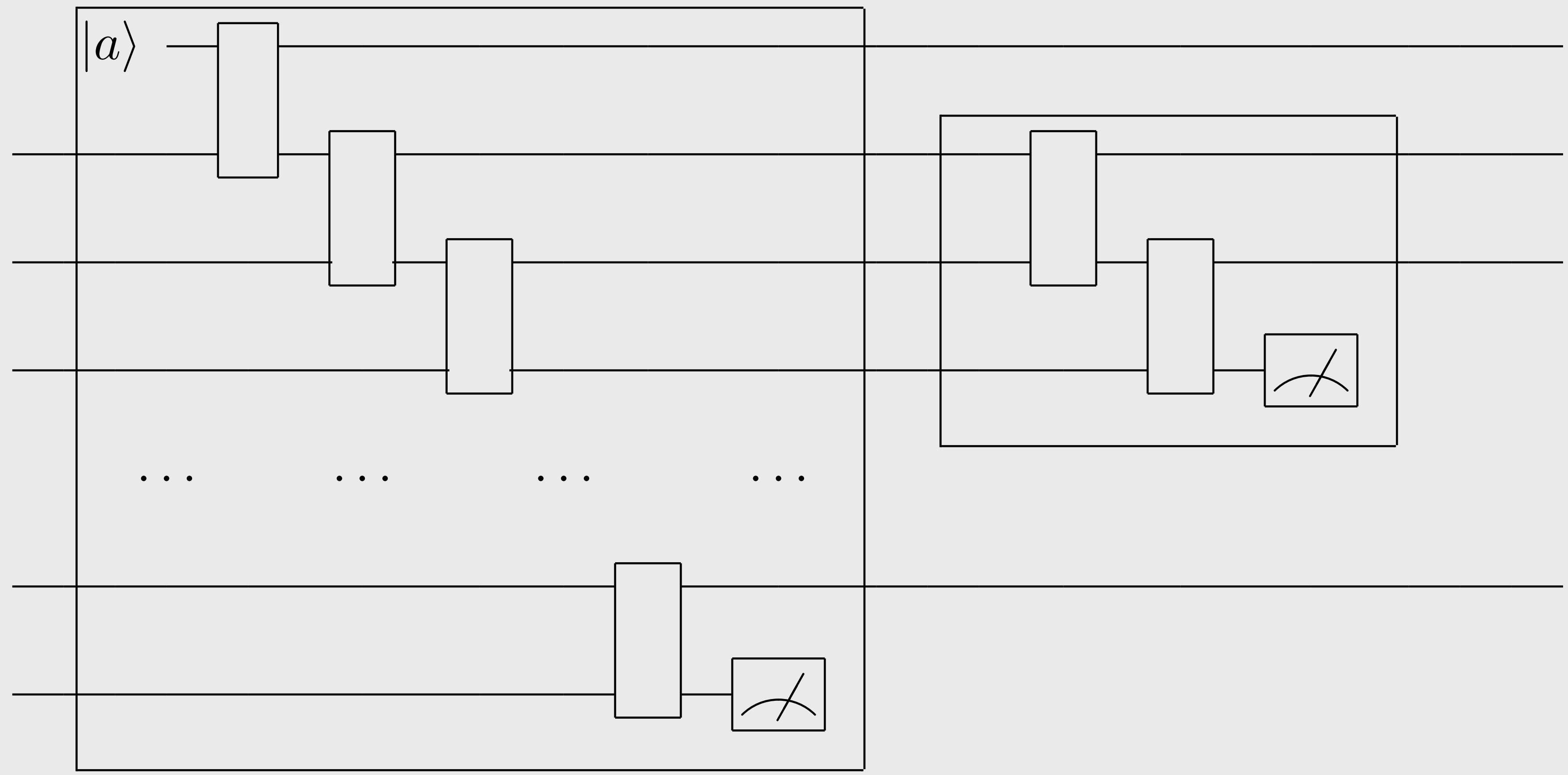
The link-product is commutative!

QUit quantum information theory group

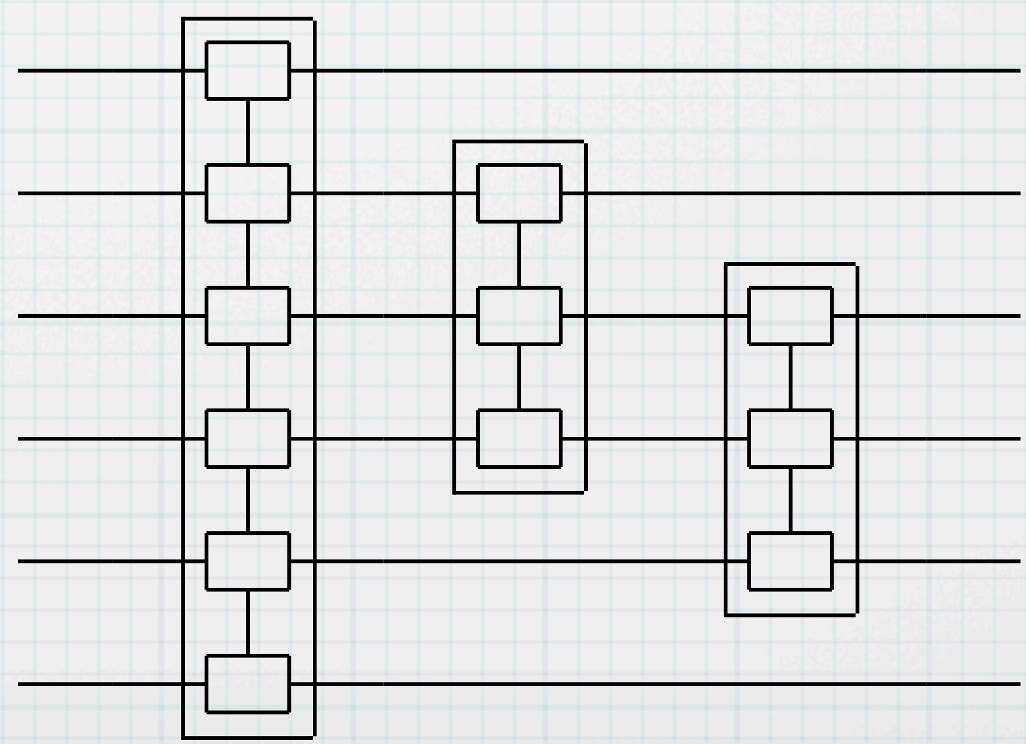
PRL 101 060401 (2008)

Link product

$R_1 * R_2 \qquad \Longleftrightarrow$



$$R_1 * R_2 * R_3 \qquad \Longleftrightarrow$$



PRL 101 060401 (2008)

Link product

Special cases:

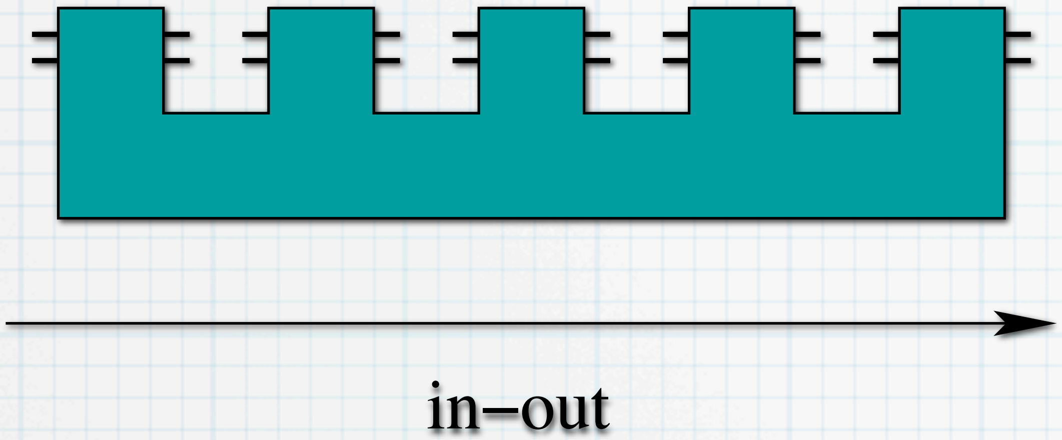
$$\mathscr{M}(\rho) = R_{\mathscr{M}} * \rho$$ quantum operation

$$\mathrm{Tr}[P^t \rho] = P * \rho$$ POVM

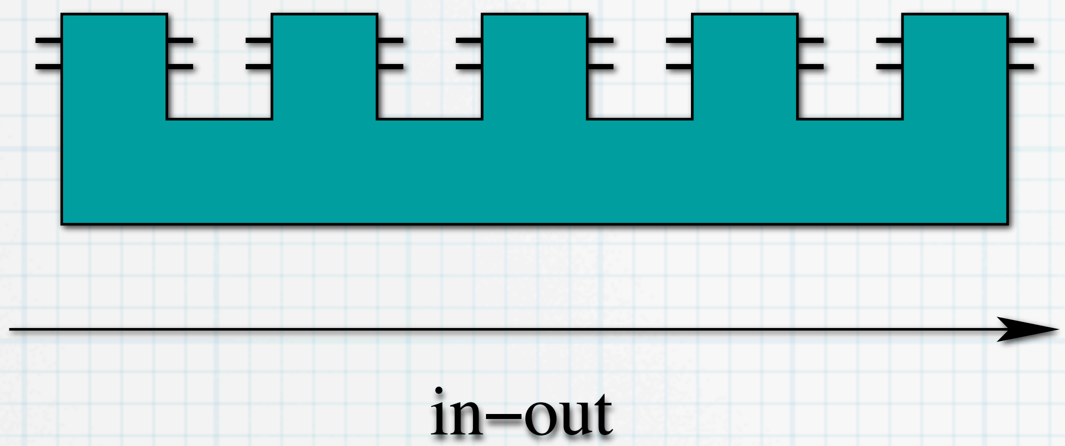
$$\rho \bigotimes \sigma = \rho * \sigma$$ tensor product

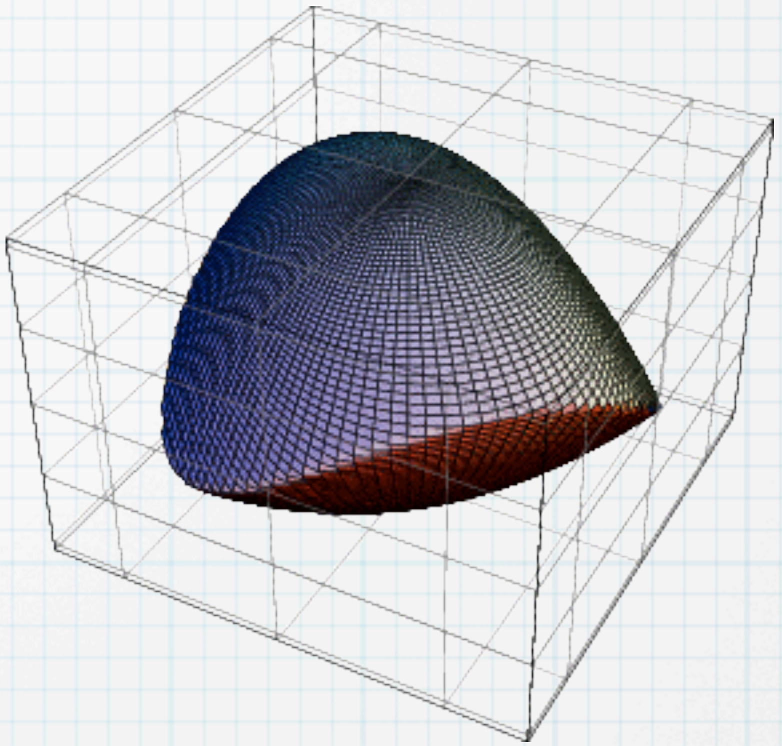
$$\mathrm{Tr}_{\mathsf{H}}[R] = R * I_{\mathsf{H}}$$ partial trace

Circuits Architecture Optimization



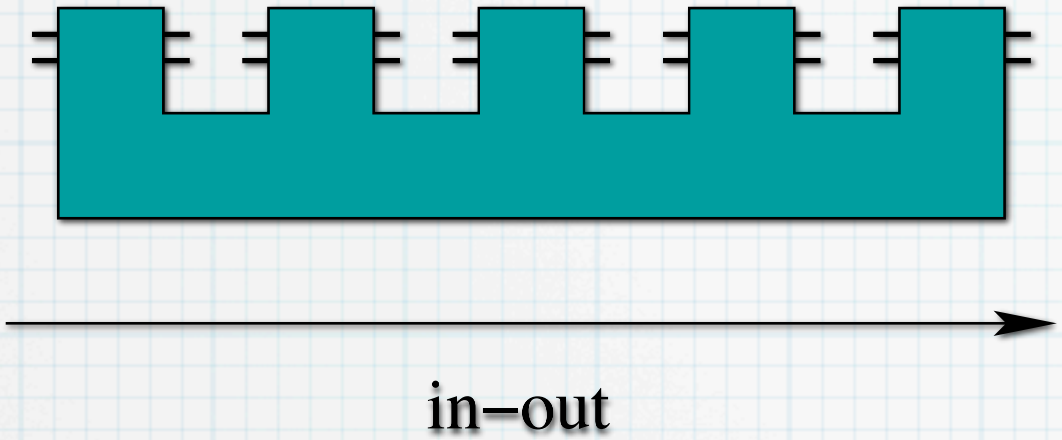
Circuits Architecture Optimization

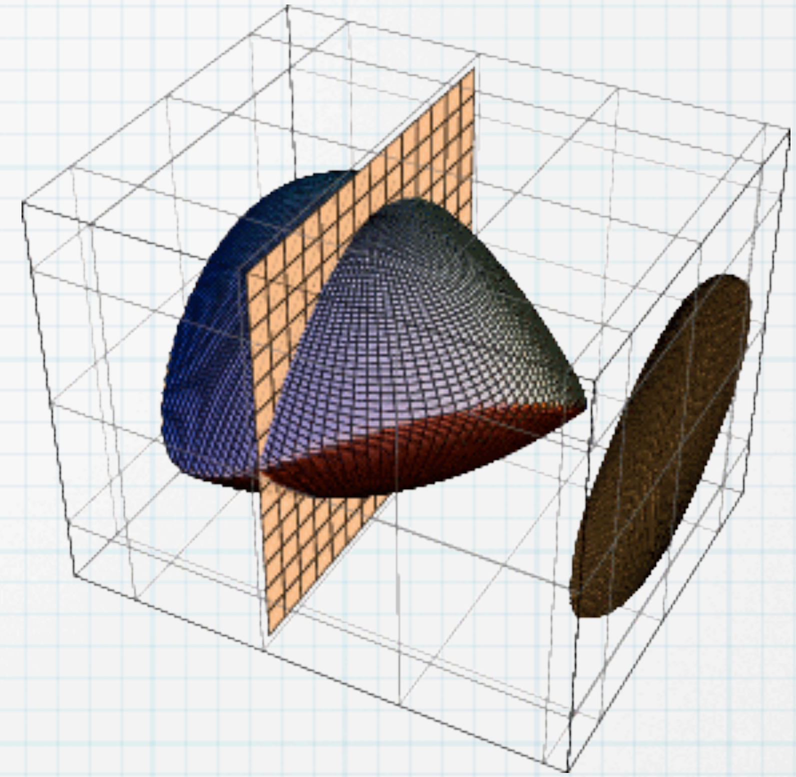




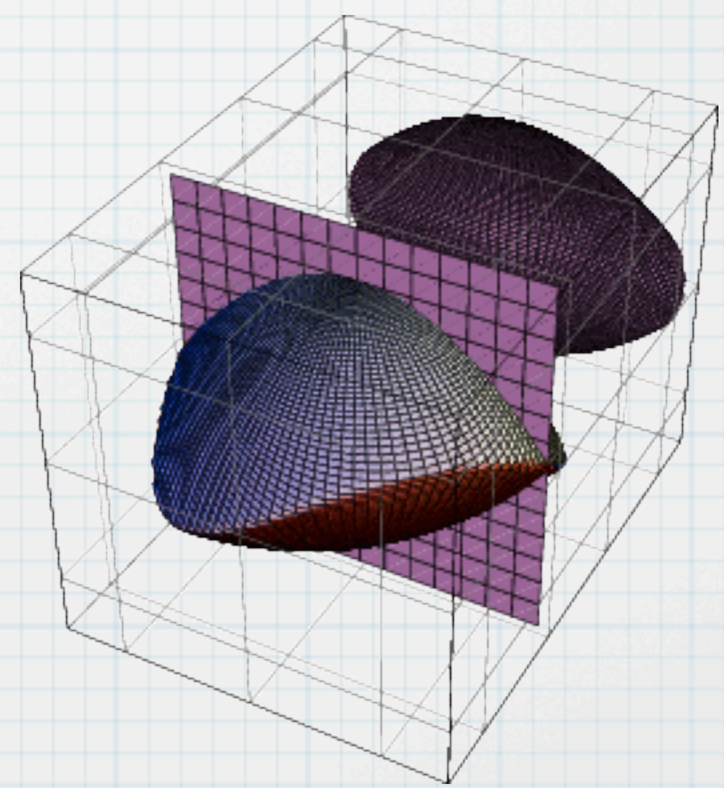
- The Choi operators of a fixed input-output comb structure make a convex set

Circuits Architecture Optimization





- The Choi operators of a fixed input-output comb structure make a convex set
- Causality constraints correspond to a hyperplane section of the convex
- Group-covariance gives another linear constraint:



$$[R, V_g] = 0 \implies R = \bigoplus_j R_j \otimes \mathbb{1}_{m_j}$$

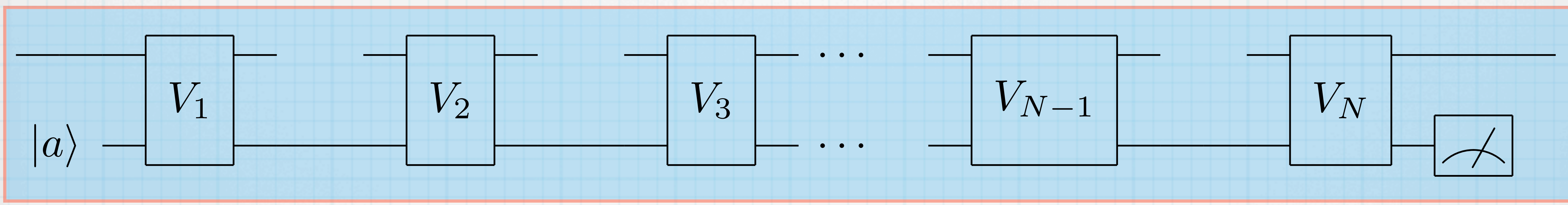
QUit quantum information theory group

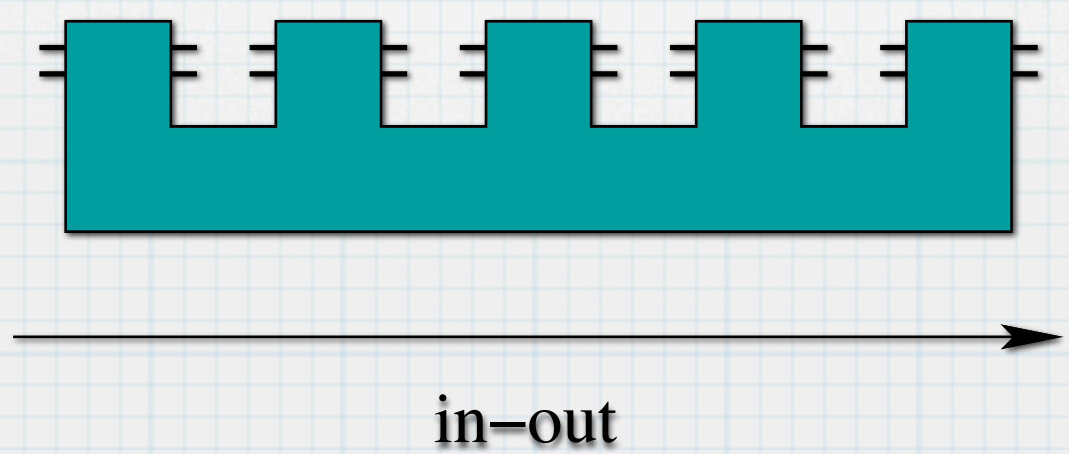
The mathematical formulation is reduced to a convex problem!

Realization theorem

Chiribella, D’Ariano, Perinotti, PRL **101** 060401 (2008) EL **83** 30004 (2008)

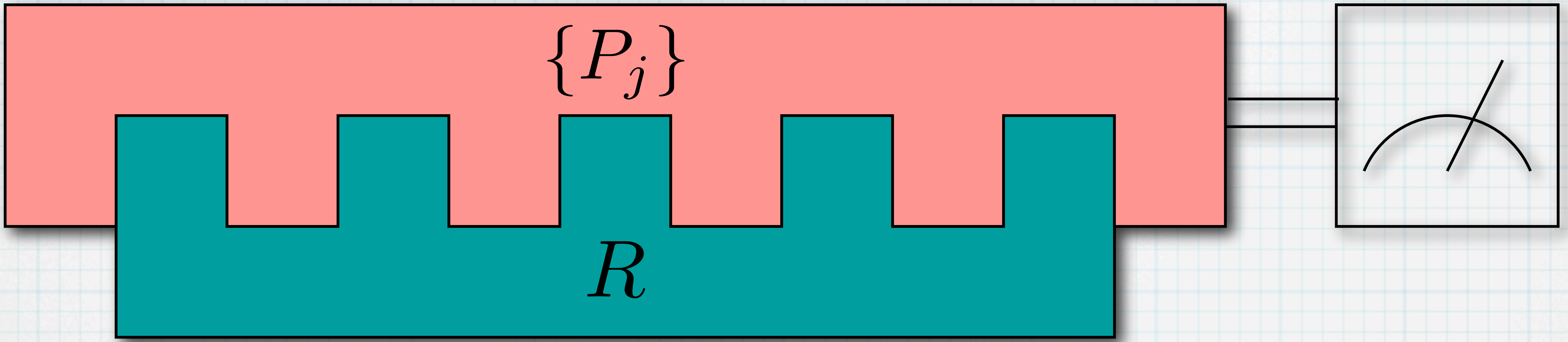
Theorem: Every Choi operator on given input-output spaces and satisfying given causality conditions is realized by the comb of isometries





For realization of isometries see: Buscemi, D'Ariano, and Sacchi, PRA 68 042113 (2003)

Quantum board testers



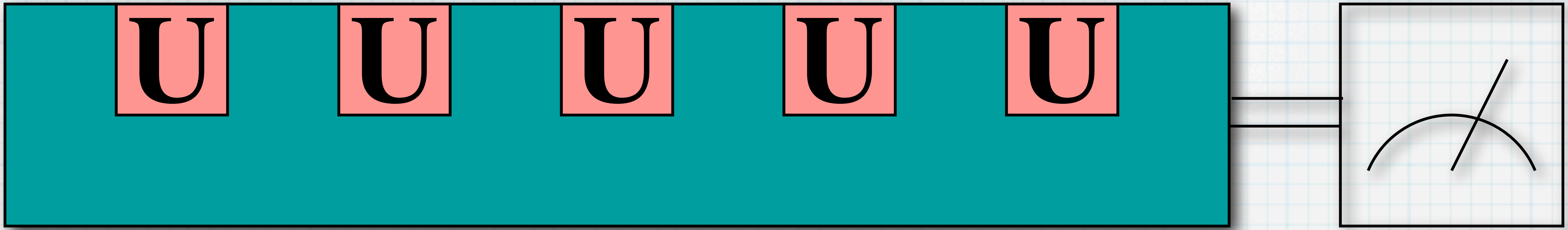
Tester
Born rule:

$$\mathrm{Tr}[P_j R] = p_j, \qquad \sum_j P_j = \Xi$$

causality constraints:

$$\mathrm{Tr}_{2n+1}[\Xi^{(n)}] = I_{2n} \otimes \Xi^{(n-1)}, \; n = 0, 1, \ldots, N$$
$$\Xi^{(N)} \equiv \Xi, \; \mathrm{Tr}_1[\Xi^{(0)}] = 1$$

Estimating tester



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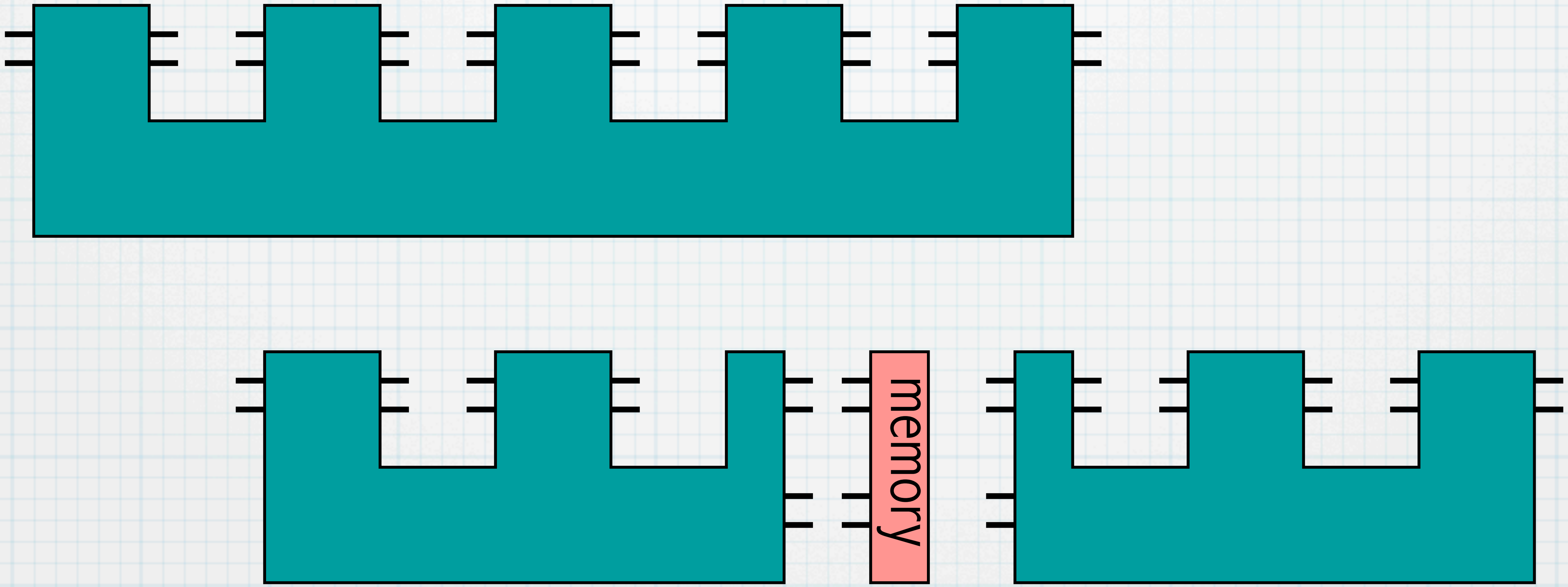
Using quantum memory

delay the use of subcircuits by breaking the comb into subcombs + quantum memory



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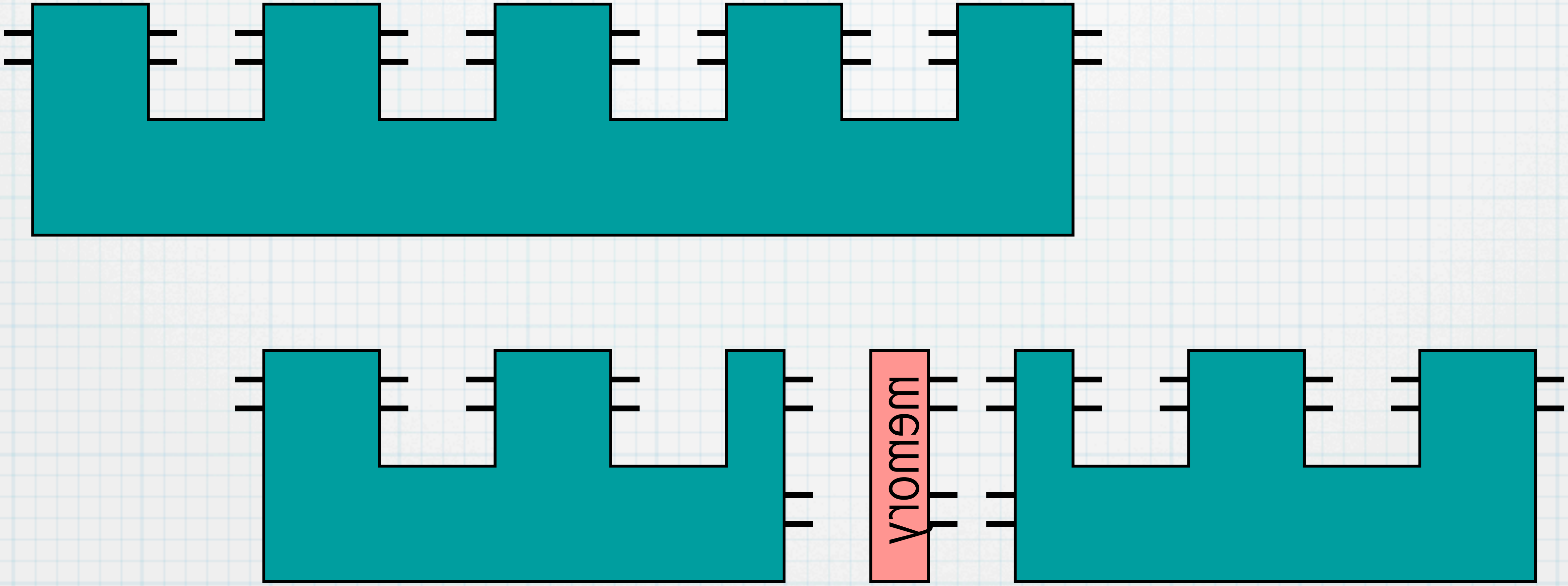


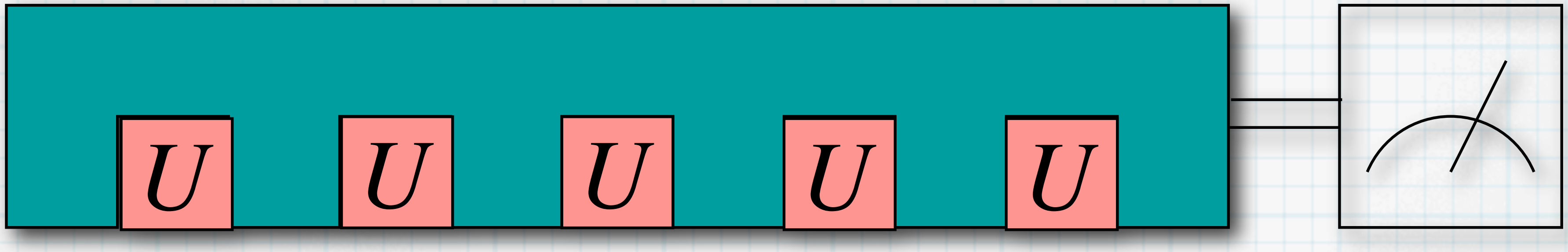
Applications

Application 1: discrimination and estimation of unitaries (optimal oracle-calling quantum algorithms)

Discrimination of unitaries

Chiribella, D'Ariano, Perinotti, PRL 101 180501 (2008)

Optimal discrimination between two possible unitary operators U_1 U_2



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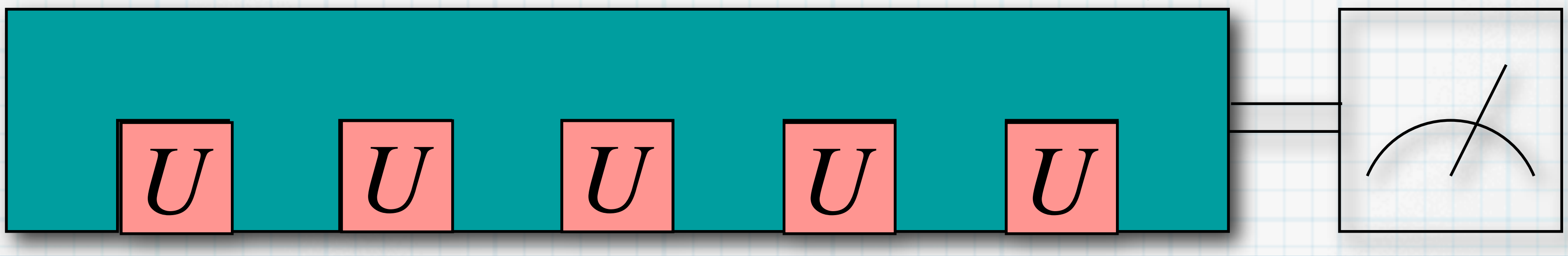
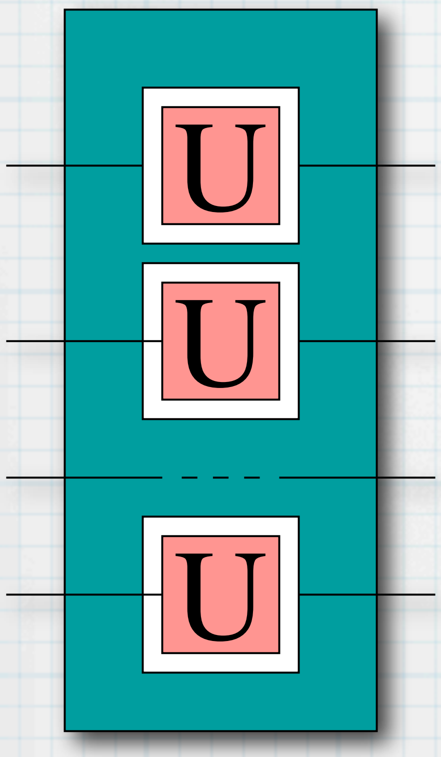
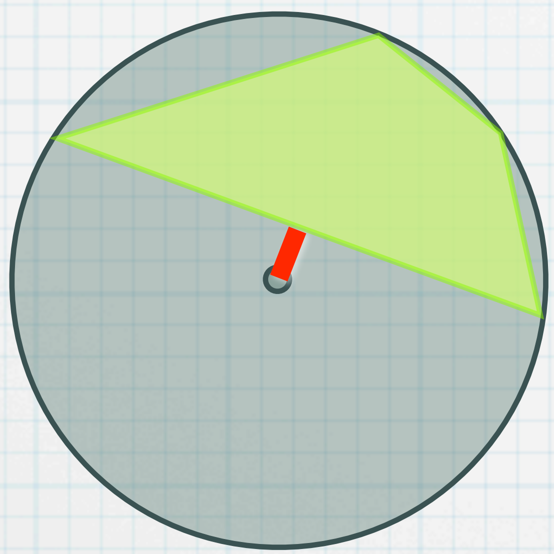




$$V = U_1^\dagger U_2$$



G. M. D'Ariano, P. Lo Presti, M. G. A. Paris, PRL 87, 270404 (2001); A. Acín, PRL 87, 177901(2001).

angular spread $\Delta(U)$

$$\Delta(U \otimes V) = \Delta(U) + \Delta(V)$$

$$N = \left\lceil \frac{\pi}{\Delta\phi} \right\rceil$$

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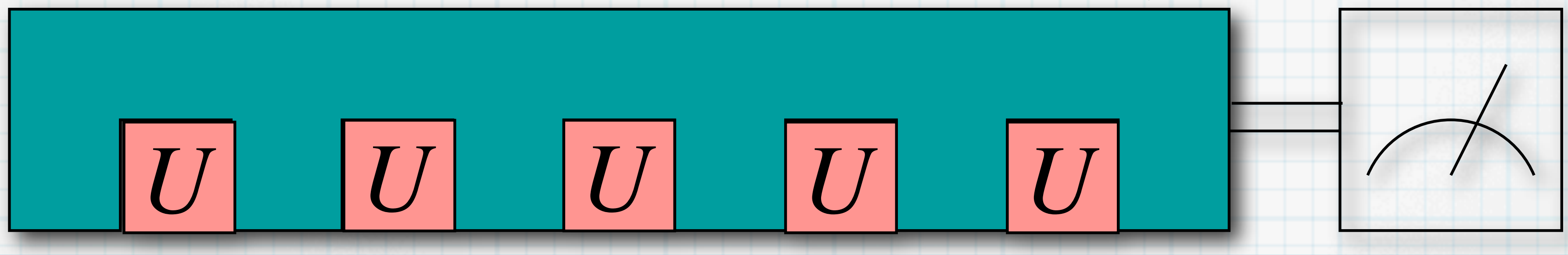

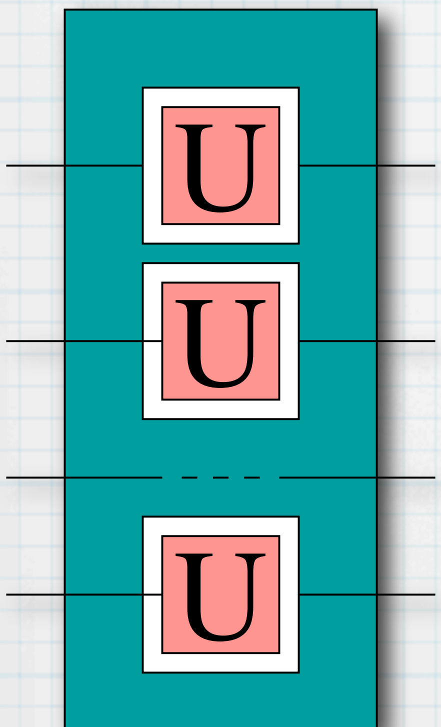

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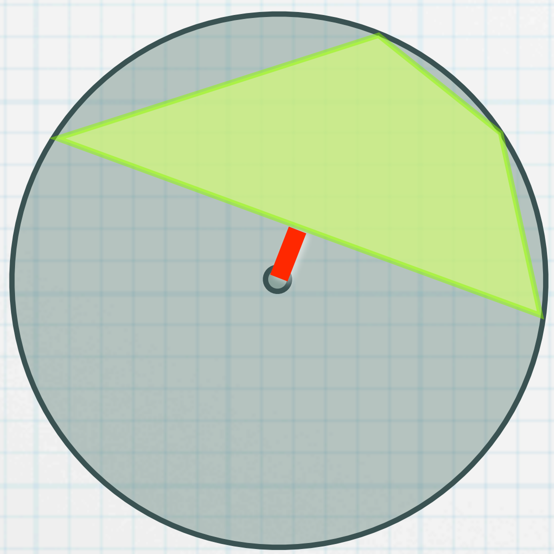

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spread lemma: $\Delta(AB) \leq \Delta(A) + \Delta(B)$ A M Childs, J Preskill, and J Renes, JMO 47, 155-176 (2000).

$$\Delta[W(U \otimes I)W^\dagger(U \otimes I)] \leq \Delta(U^{\otimes 2})$$

The spread of the tester is not larger than that of $U^{\otimes N}$ and U^N

Discrimination of unitaries

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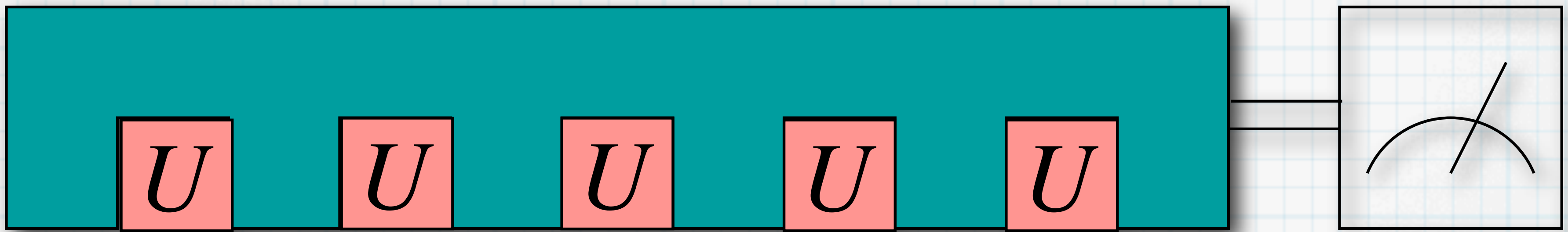

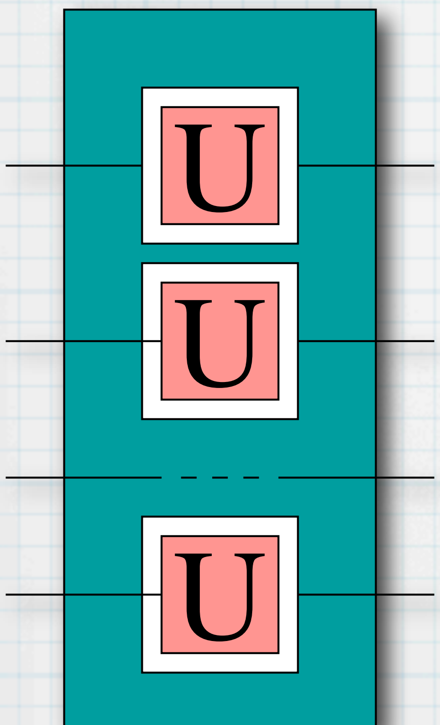

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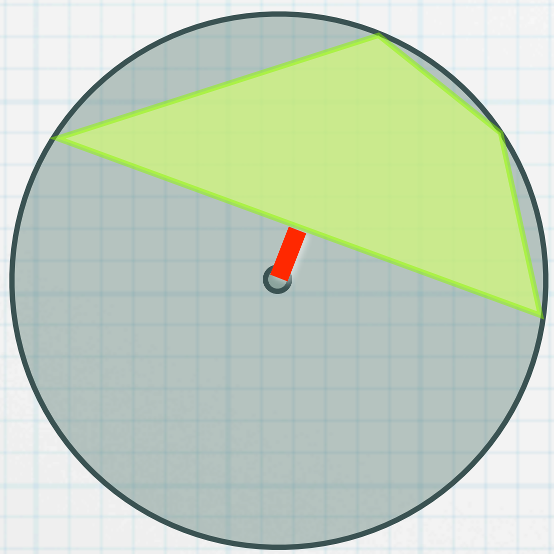

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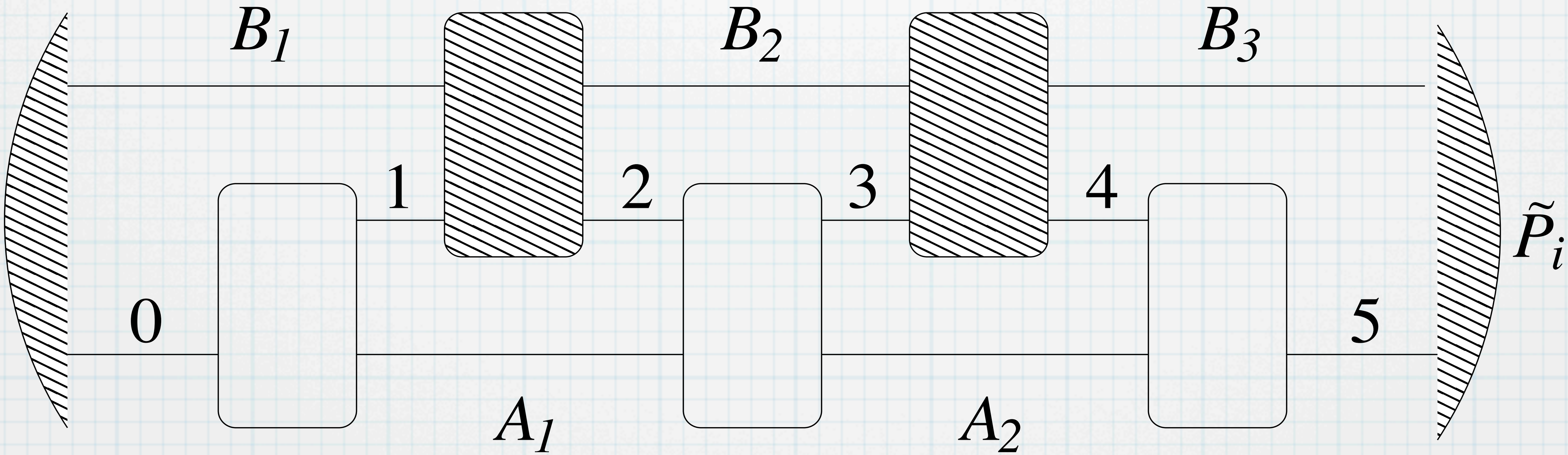
The spread of the tester is not larger than that of $U^{\otimes N}$and U^N

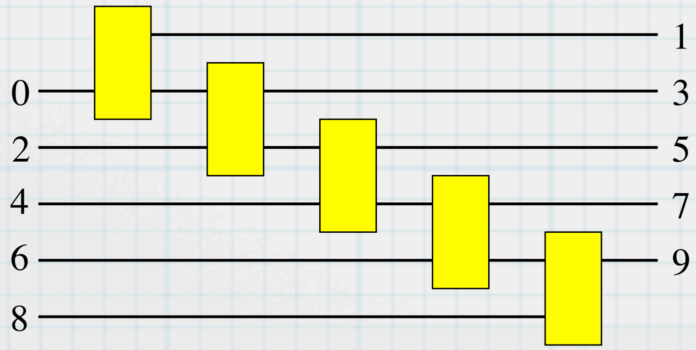
The parallel diposition is already optimal

Discrimination of memory channels

PRL **101** 180501 (2008)

There are memory channels that can be discriminated perfectly with a single use by a quantum tester, and not conventionally



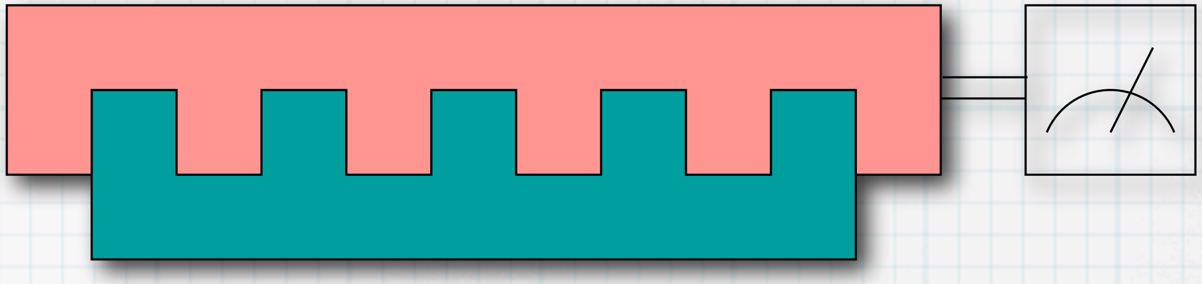


Discrimination of memory channels

Chiribella, D'Ariano, Perinotti, (unpublished)

Tester Born rule:

$$\mathrm{Tr}[P_j R] = p_j, \qquad \sum_j P_j = \Xi$$



Perfect discrimination for:

$$R_1(I \otimes \Xi)R_2 = 0$$

Discrimination of unitaries

- What happens for more than two unitaries?
- What happens for non-unitary channels?

Covariant estimation of unitaries

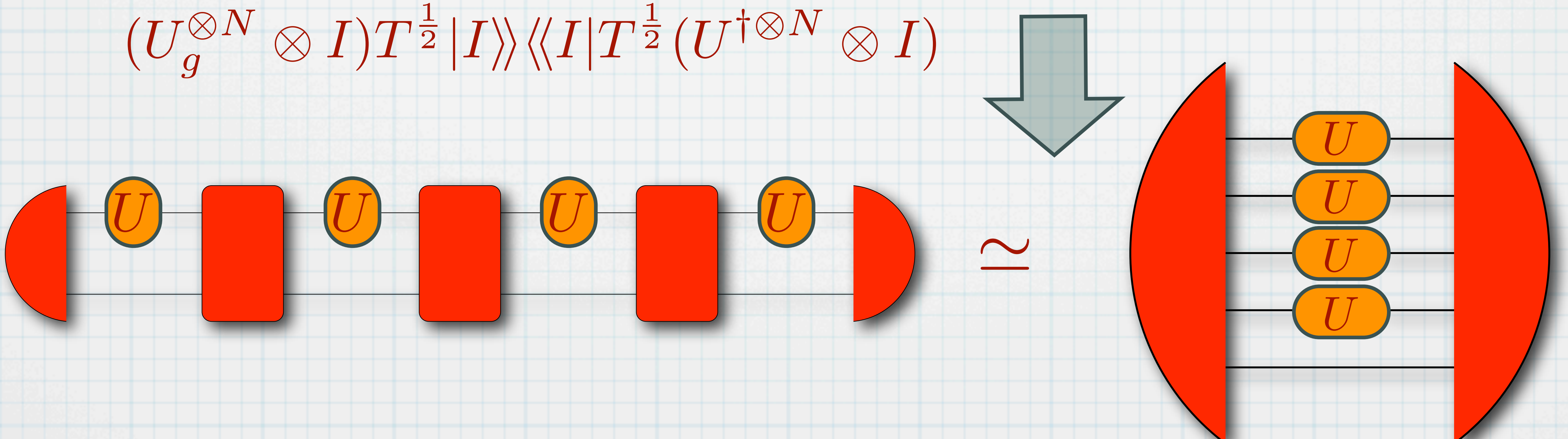
- Covariant unitary estimation problem (group of unitaries, Haar-distributed)
- Problem: find the optimal tester for estimating the group element

One can prove that the optimal tester is covariant:

$$T = \int_G \mathrm{d}\, g T_g \qquad T_h = (U_h^{\otimes N} \otimes I)\Theta(U_h^{\dagger\otimes N} \otimes I) \quad \Rightarrow \quad [T, U_h^{\otimes N} \otimes I] = 0$$

Then: $T^{\frac{1}{2}}(|U_g\rangle\!\rangle\langle\!\langle U_g|)^{\otimes N} T^{\frac{1}{2}} = T^{\frac{1}{2}}(U_g^{\otimes N} \otimes I)|I\rangle\!\rangle\langle\!\langle I|(U^{\dagger\otimes N} \otimes I)T^{\frac{1}{2}} =$

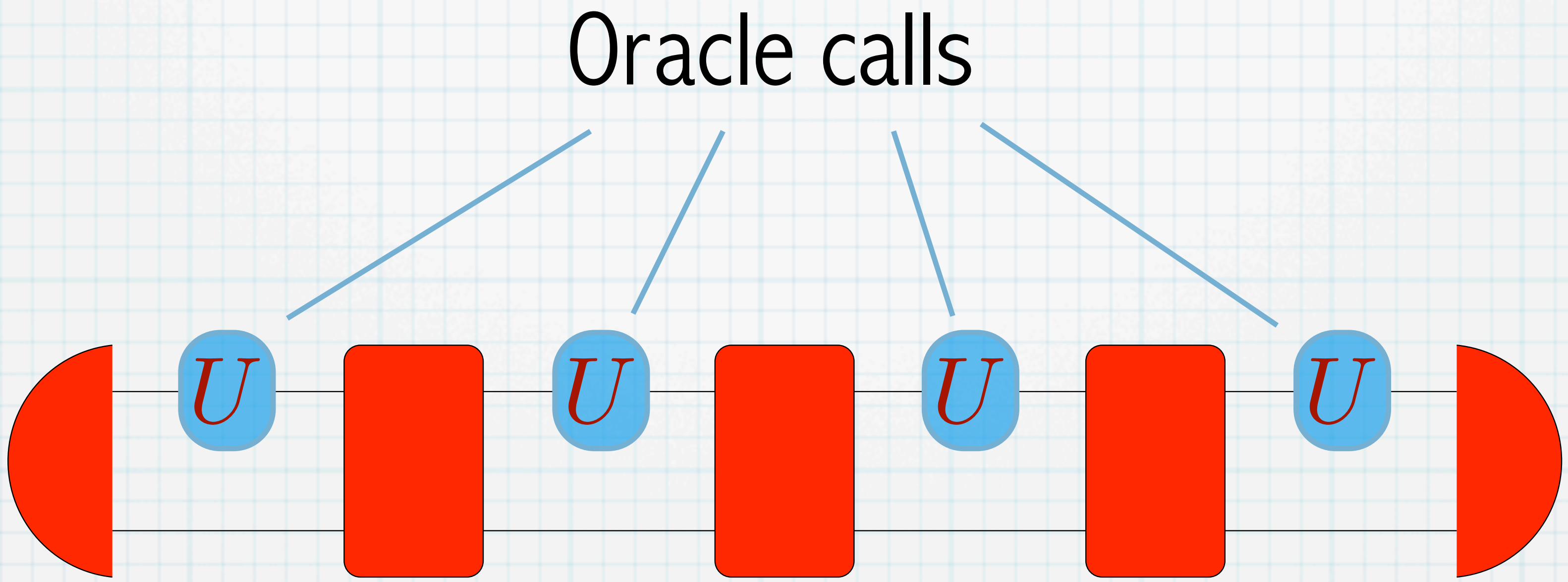
$$(U_g^{\otimes N} \otimes I)T^{\frac{1}{2}}|I\rangle\!\rangle\langle\!\langle I|T^{\frac{1}{2}}(U^{\dagger\otimes N} \otimes I)$$



Any covariant tester is equivalent to a parallel scheme

Discrimination of unitaries

Oracle-calling quantum algorithm as optimal discrimination of unitaries



Discrimination of unitaries

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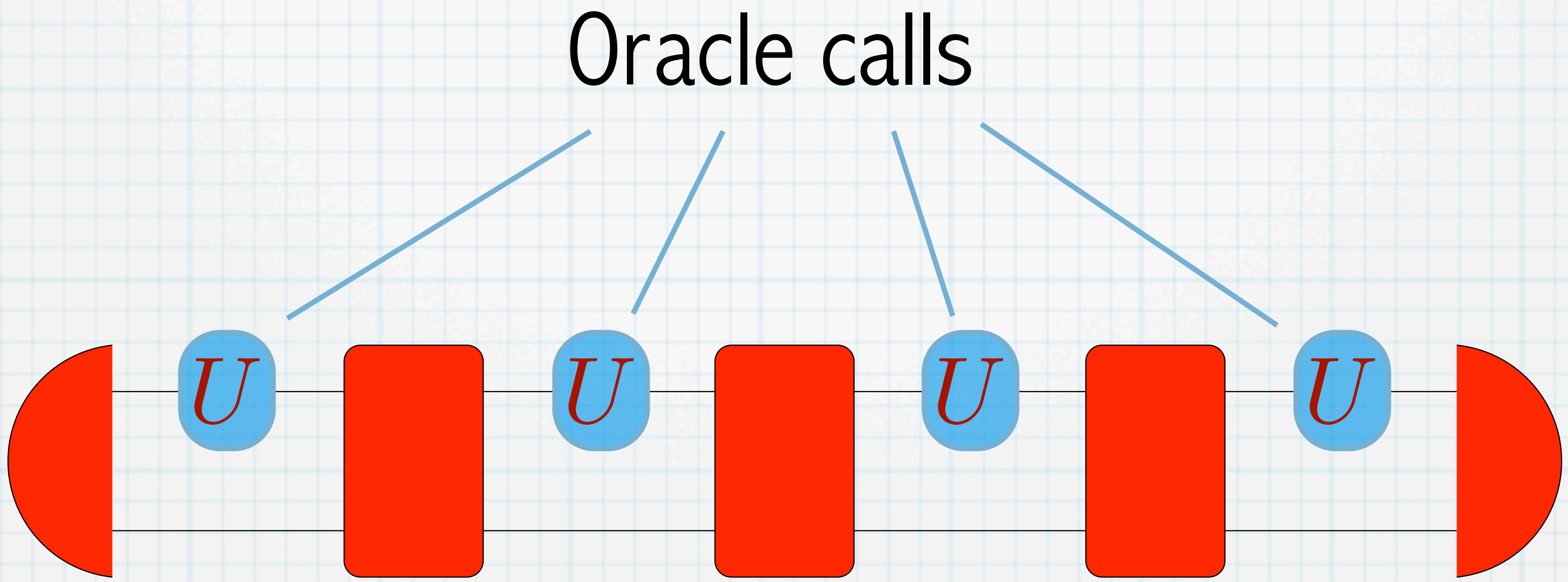

Hidden-subgroup algorithms (Deutsh-Jozsa, Simon, etc): parallel calls are optimal

Discrimination of unitaries

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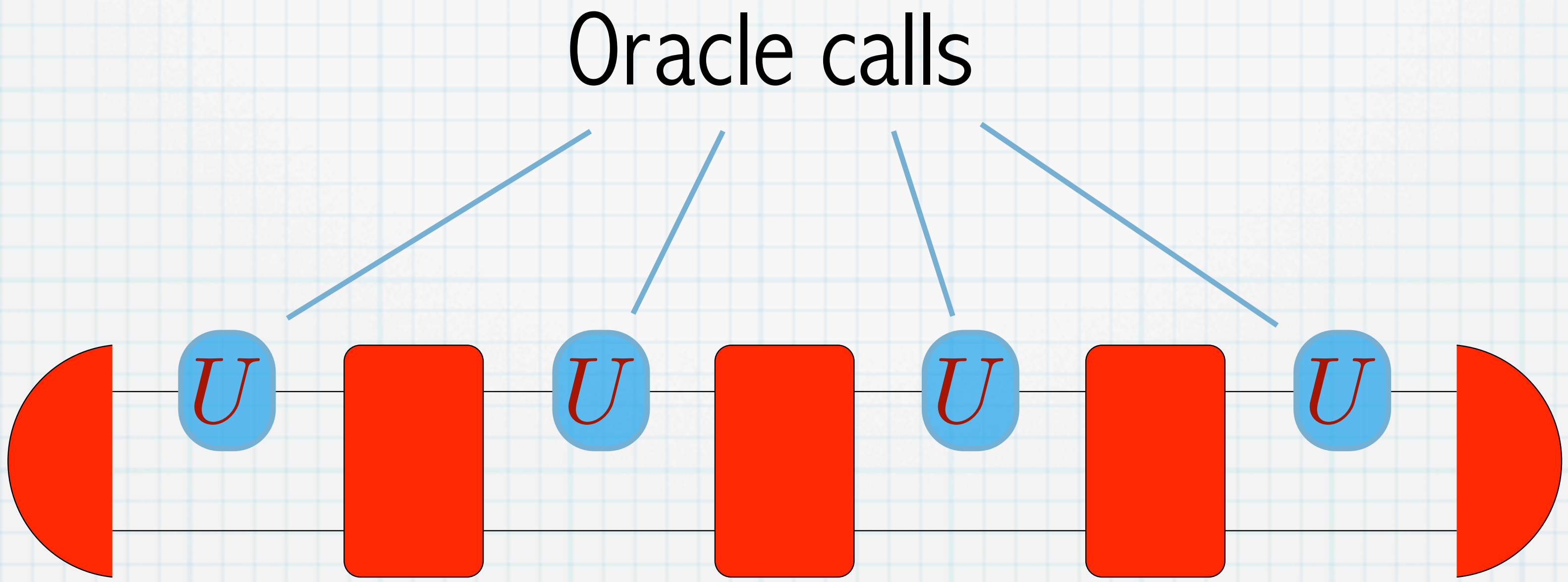

Hidden-subgroup algorithms (Deutsh-Jozsa, Simon, etc): parallel calls are optimal

Algorithms with no hidden subgroup (e.g. Grover) need a comb [C. Zalka, PRA 60, 2746 (1999)]

Discrimination of unitaries

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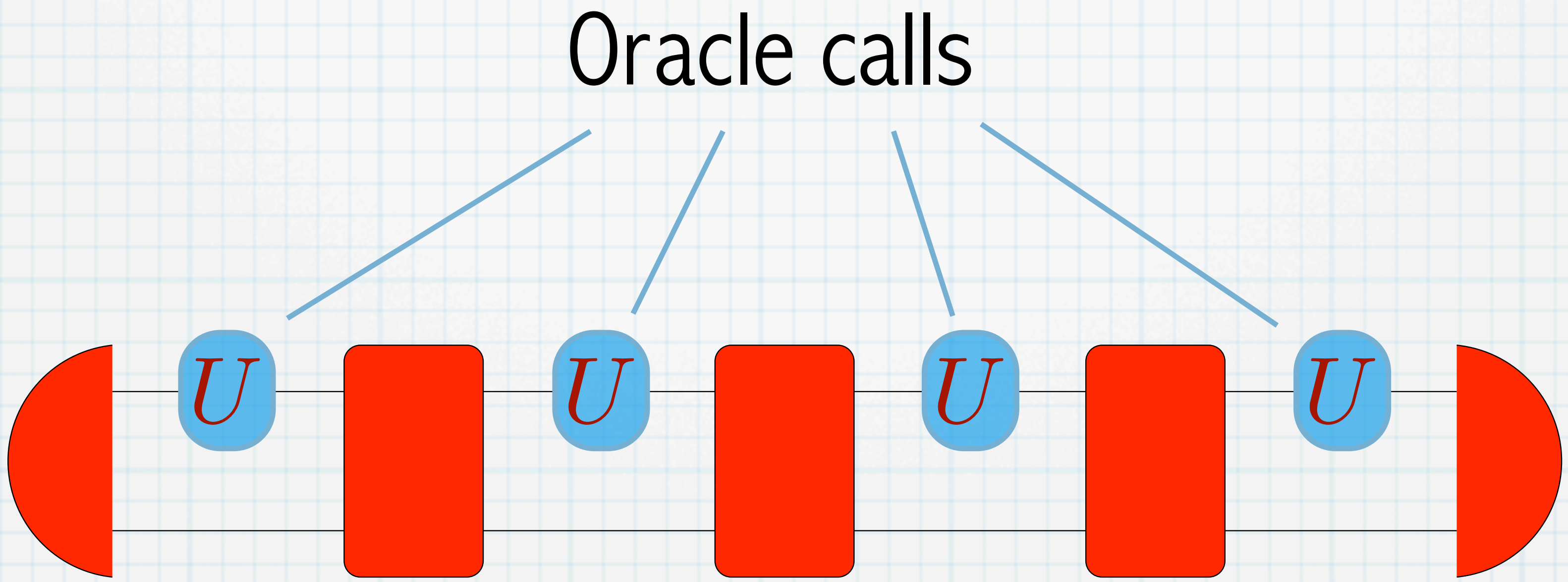

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Q-combs: systematic method to determine optimal oracle-calling algorithms

Discrimination of unitaries

When do we need a tester (not just a parallel discrimination):

Discrimination of unitaries

When do we need a tester (not just a parallel discrimination):

- For discrimination of non-unitary channels
- For discrimination within non-covariant sets
- For discrimination of memory channels

Operational network distance

PRL **101** 180501 (2008)

Existence of optimal non parallel optimal discrimination schemes

Operational network distance

PRL **101** 180501 (2008)

Existence of optimal non parallel optimal discrimination schemes

The proper distance for memory channels must be defined in terms of optimal discriminating testers

$$D(\mathscr{C}^{(N)}, \mathscr{D}^{(N)}) := \max_{\Xi^{(N)}} \left\| \left(I \otimes \Xi^{(N)\frac{1}{2}} \right) \Delta \left(I \otimes \Xi^{(N)\frac{1}{2}} \right) \right\|_1$$

$$\Delta := C - D$$

Operational network distance

PRL **101** 180501 (2008)

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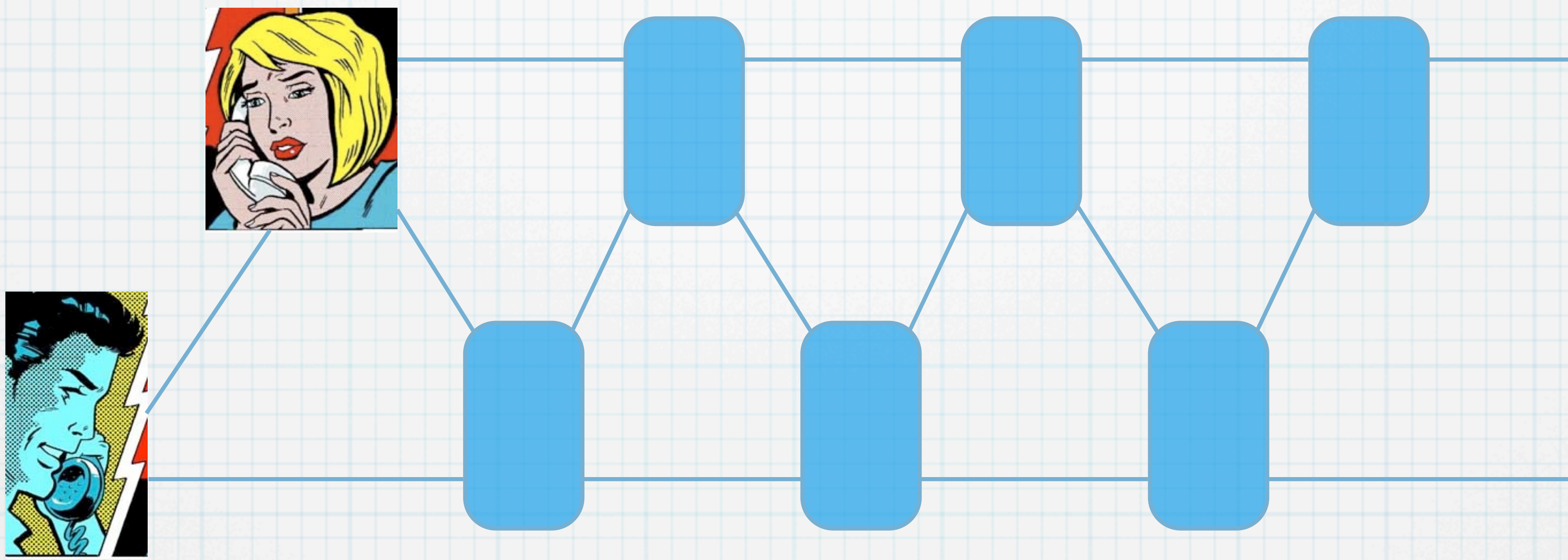
$$\Delta := C - D$$

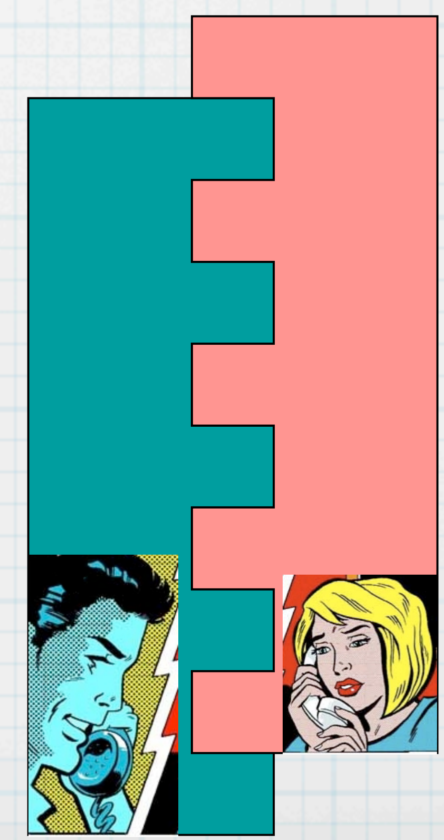
CB-norm distance only accounts for parallel discrimination schemes

QUit quantum information theory group

Application 2: quantum protocols (cryptography, game-theory)

Quantum protocols

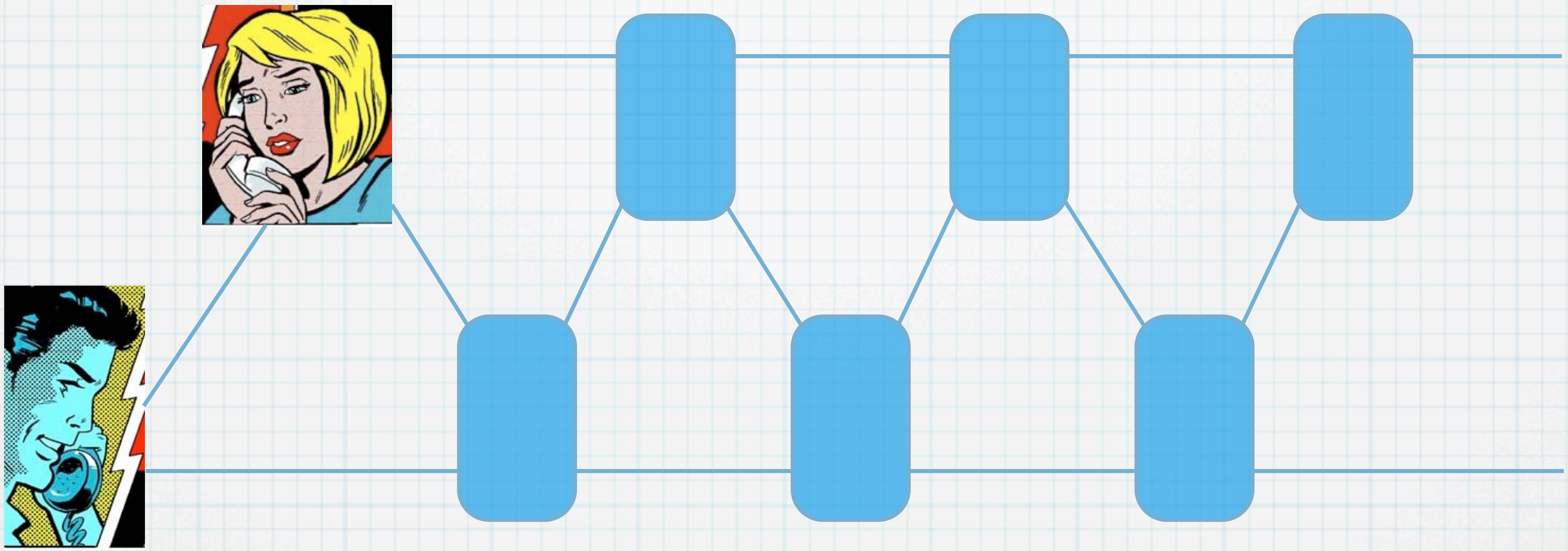




Quantum combs describe the most general strategies in multi-party protocols and games

G. Gutoski and J. Watrous, Proc. STOC, 565-574, (2007)

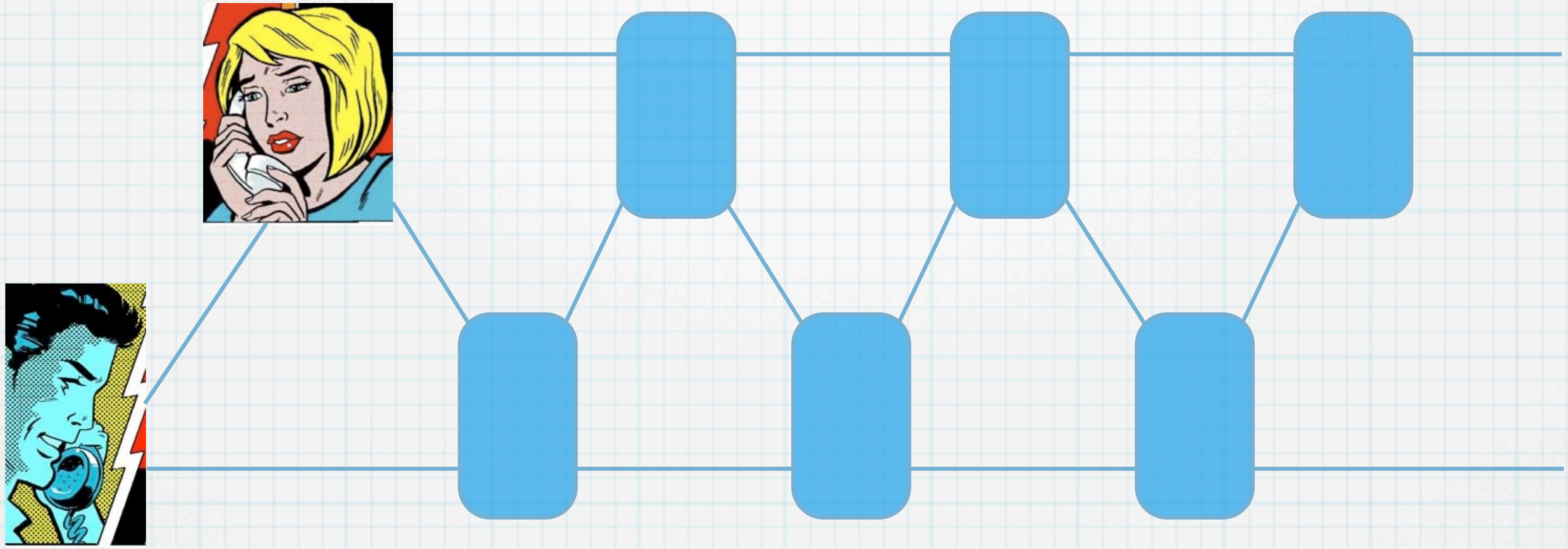
Quantum bit commitment



Quantum combs is the most suitable mathematical formulation of Alice and Bob’s strategies in a quantum bit commitment protocol

Sketch of impossibility proof

Chiribella, D'Ariano, Perinotti, Schlingemann, and Werner (in preparation)



Thm: a QBC concealing protocol cannot be binding

Proof: continuity of the comb-Stinespring versus the operational distance between strategies

Application 3: Quantum-algorithm learning

Quantum-algorithm learning

Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow

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- Alice owns quantum circuit that performs a very valuable algorithm U that she wants to keep undisclosed.

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Quantum-algorithm learning

Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow

- Alice owns quantum circuit that performs a very valuable algorithm U that she wants to keep undisclosed.
- Bob needs to run Alice's algorithm on an input state that will be available tomorrow, but he can borrow the circuit from Alice only today for just a limited number of uses N, and with the circuit sealed.

Quantum algorithm learning

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- After that the only thing that remains available to Bob for tomorrow is the output state, which Bob can store on a quantum memory.

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- The only thing that Bob can do today, with the circuit available, is to use it on a input state known to him.
- After that the only thing that remains available to Bob for tomorrow is the output state, which Bob can store on a quantum memory.
- Therefore, Bob needs a quantum device that is capable of retrieving from the output state, namely recovering U and then running it on a new unknown state.

Quantum algorithm learning

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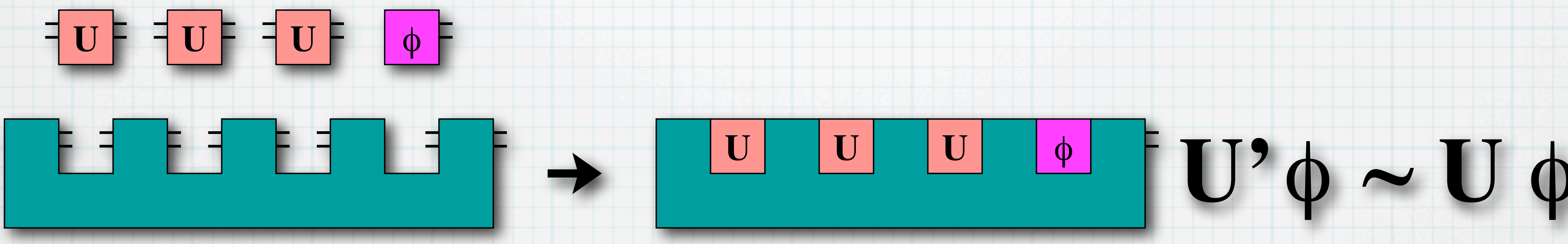
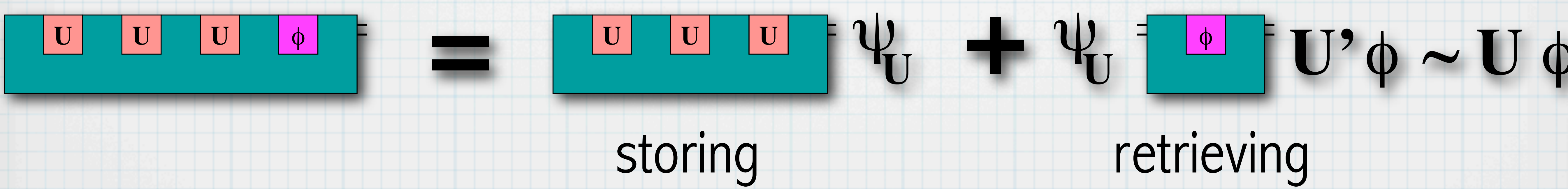
- Exact storing of quantum states is possible (quantum memory is a technological problem)
- Perfect storing of undisclosed unitaries over a quantum state is impossible even in-principle (Nielsen-Chuang no-programming theorem)

Quantum algorithm learning

Problem: run an unknown unitary that is available today on a quantum state that will be available tomorrow





Quantum algorithm learning

Problem: a black box computes an unknown function y = f(x)

We can evaluate f on a finite set of points $x_1, \ldots, x_N$

getting outcomes $y_1, \ldots, y_N$

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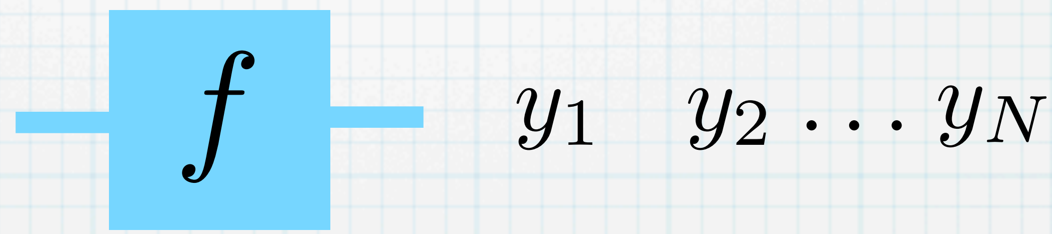

Subsequently, we are asked to compute f on a new point x ,

without using the black box $f(x) = \ ?$

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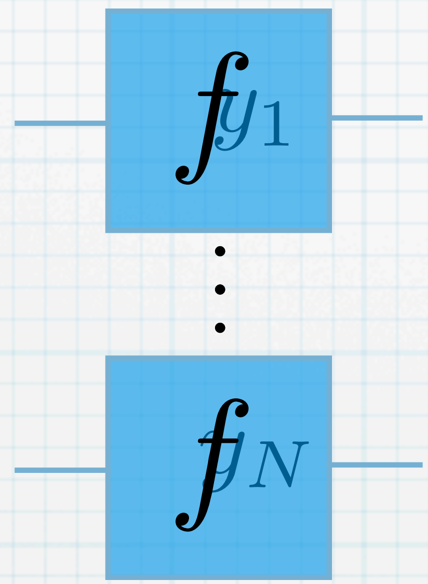
In classical computer science, statistical learning provides a method to solve this problem

Quantum algorithm learning

classical networks for learning:

Comparing x with f(x) for N times is not the only possibility:
this just corresponds to the parallel configuration



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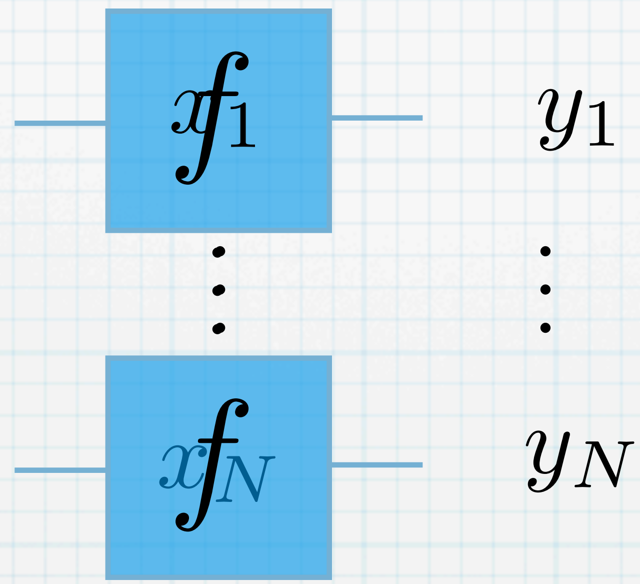

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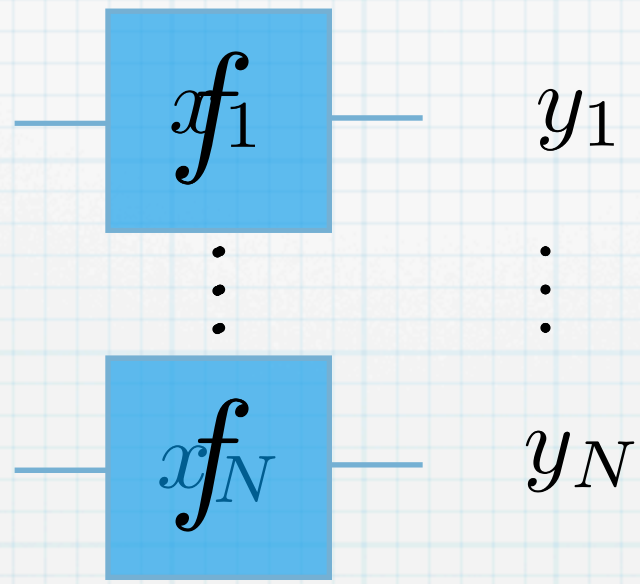
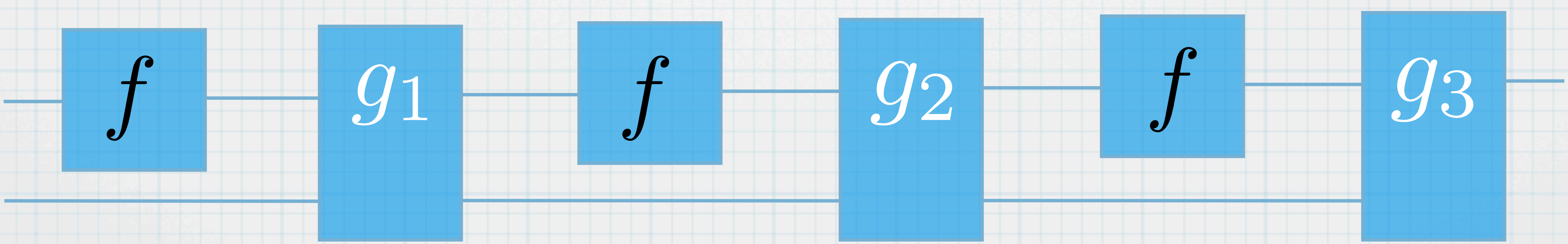


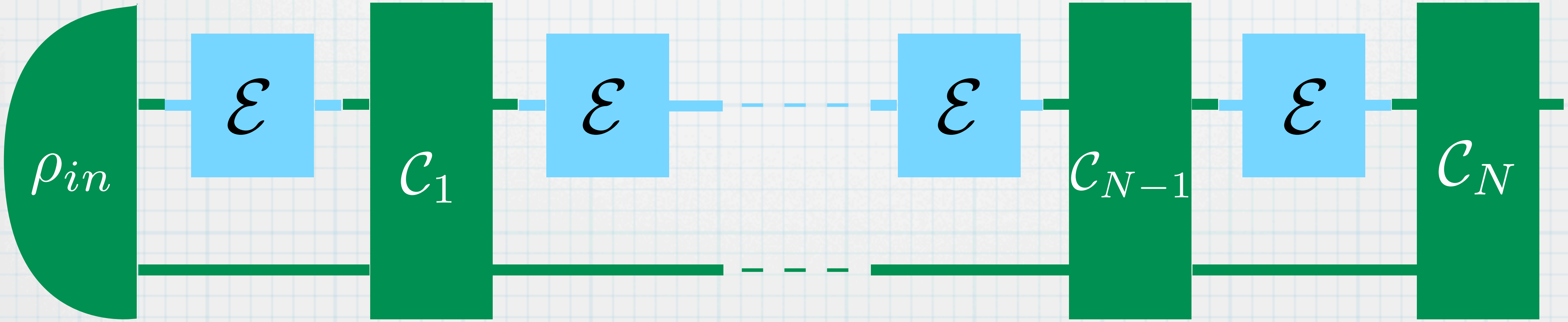
To learn better, one could use a sequential network:



where $g_1, g_2, \ldots, g_N$ are known functions

Quantum algorithm learning

- Unknown function f $\longrightarrow$ unknown quantum channel $\mathcal{E}$
- Classical program $\longrightarrow$ quantum network
- Input X $\longrightarrow$ quantum state ρ_{in}
- Output Y $\longrightarrow$ quantum state ρ_{out}



Quantum algorithm learning

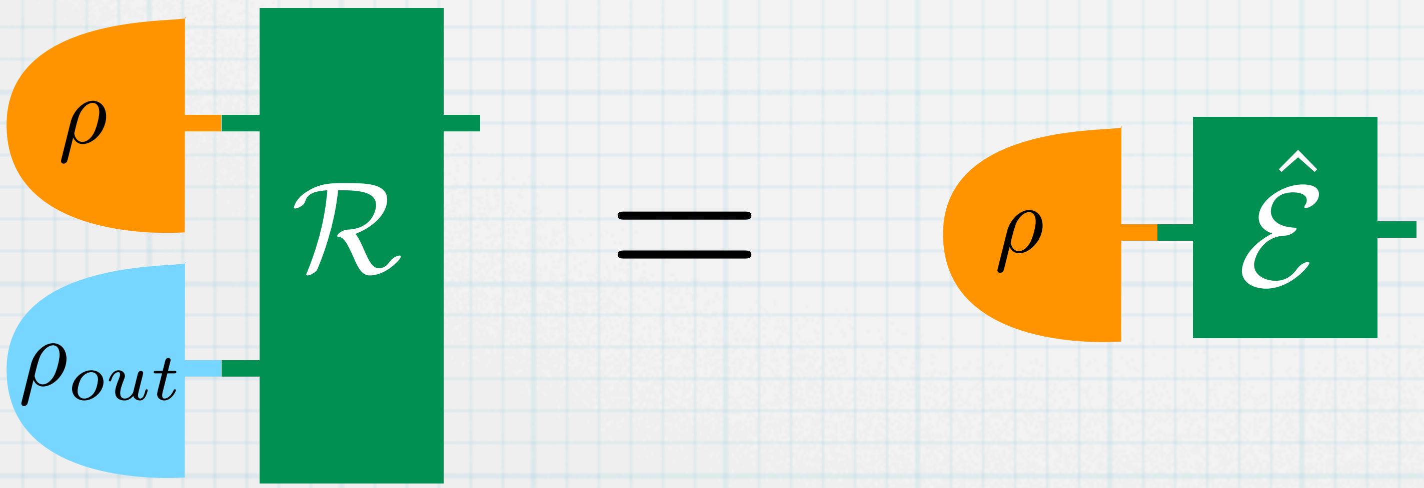
- Classical guess $\qquad\qquad\qquad\qquad$ Quantum “guess”

$$Y \to \hat{f} \qquad \longrightarrow \qquad \rho_{out} \to \hat{\mathcal{E}}$$

Physical implementation of the quantum guess:

retrieving channel $\mathcal{R}$

It retrieves the unknown transformation from the output state ρ_{out} and performs it on a new state ρ



Quantum algorithm learning

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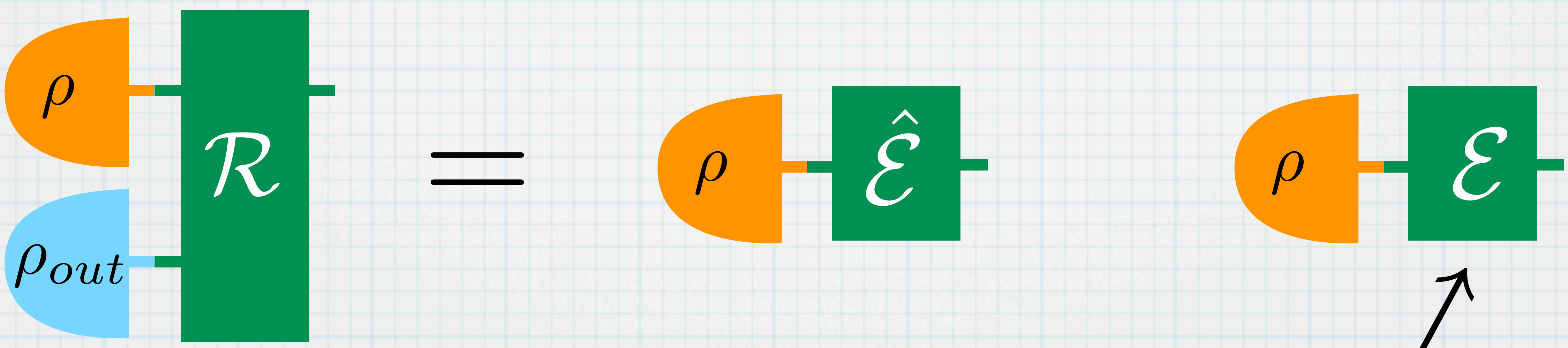


Target: implementing the unknown channel with maximum fidelity

Quantum algorithm learning

find the optimal quantum comb

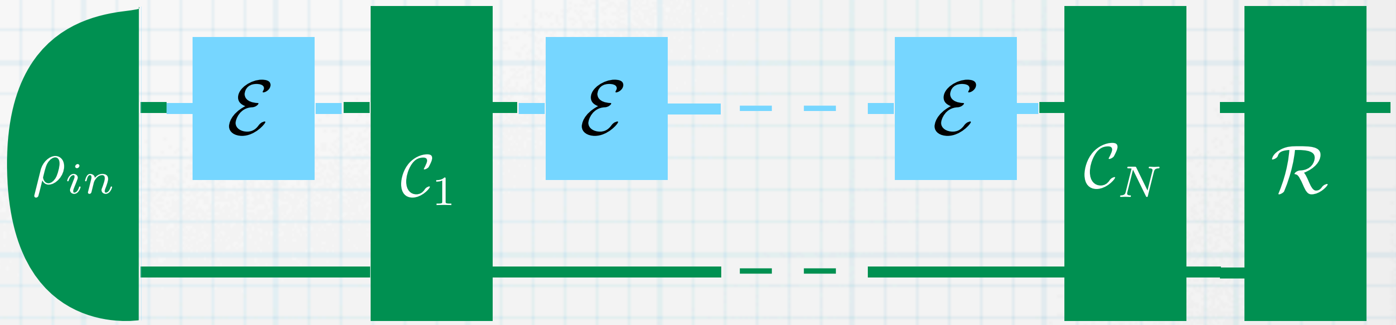
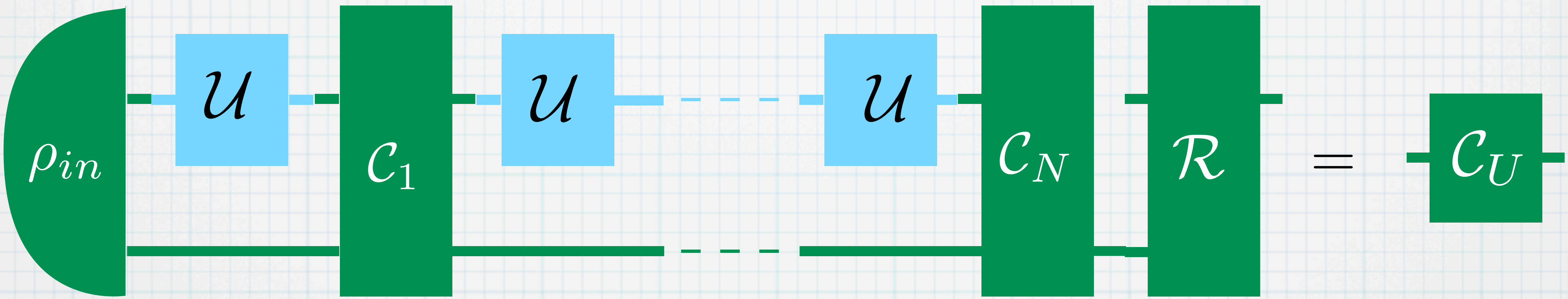


Figure of merit: input-output fidelity

$$F(\mathcal{E}, \hat{\mathcal{E}}) = \int \mathrm{d}\varphi \;\; F(\mathcal{E}(\varphi), \hat{\mathcal{E}}(\varphi)) \qquad F(\rho, \sigma) = \mathrm{Tr}\left[(\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}})^{\frac{1}{2}} \right]$$

Quantum algorithm learning

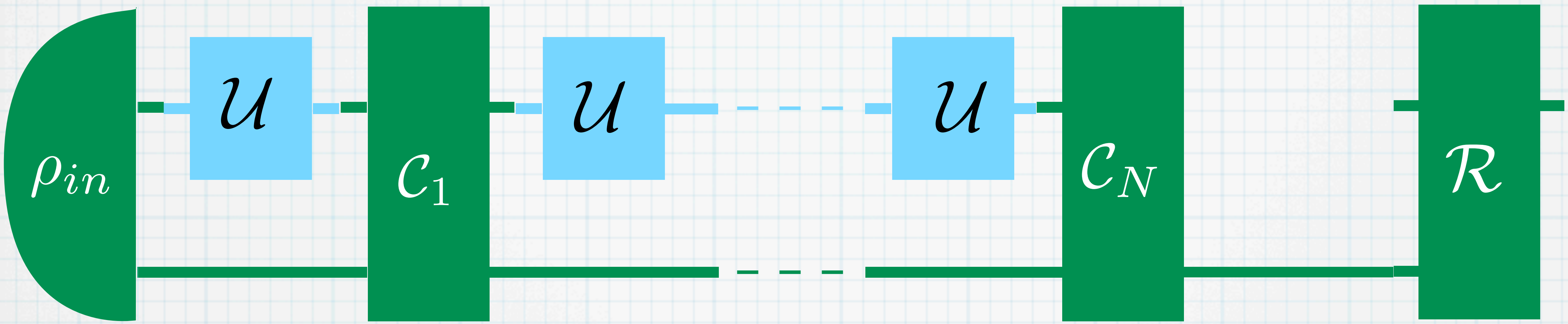
Consider the case where the set of channels is a group of unitary transformations.



Assuming a uniform prior for the unknown unitaries, we have the average fidelity

$$F = \int \mathrm{d}U \quad F(\mathcal{U}, \mathcal{C}_U)$$

Quantum algorithm learning



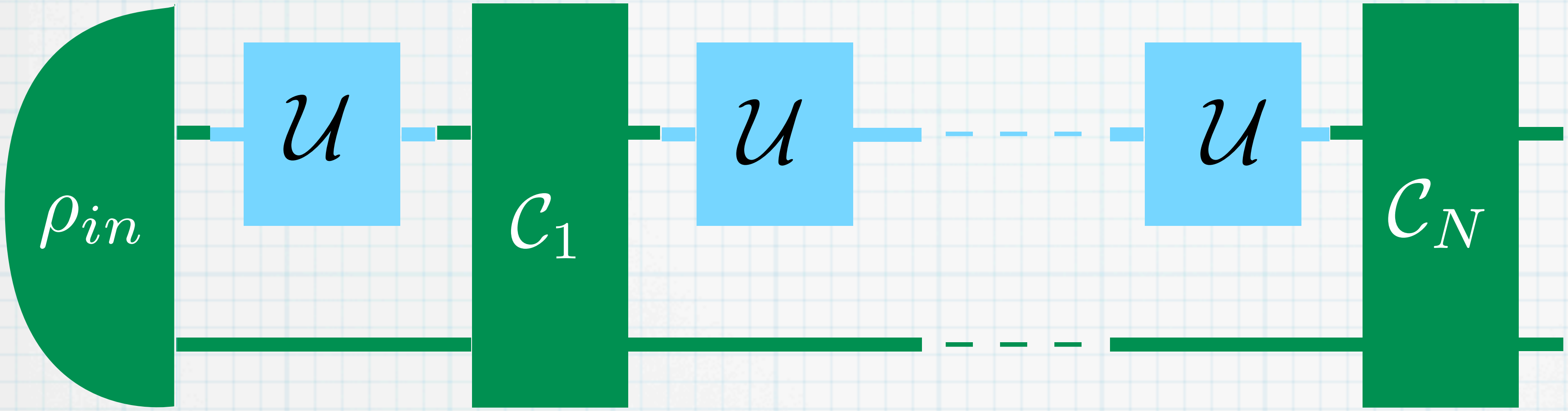
Comb of the learning network: $L = R * C_N * \cdots * C_2 * C_1 * \rho_{in}$

Fidelity: $$F = \frac{1}{d^2} \int \mathrm{d}U \ \ \langle\langle U | \langle\langle U^* |^{\otimes N} | \ L \ | U \rangle\rangle | U^* \rangle\rangle^{\otimes N}$$

We can always optimize over covariant combs:

$$[L, U \otimes V^* \otimes U^{* \otimes N} \otimes V^{\otimes N}] = 0 \qquad \forall U, V$$

Quantum algorithm learning



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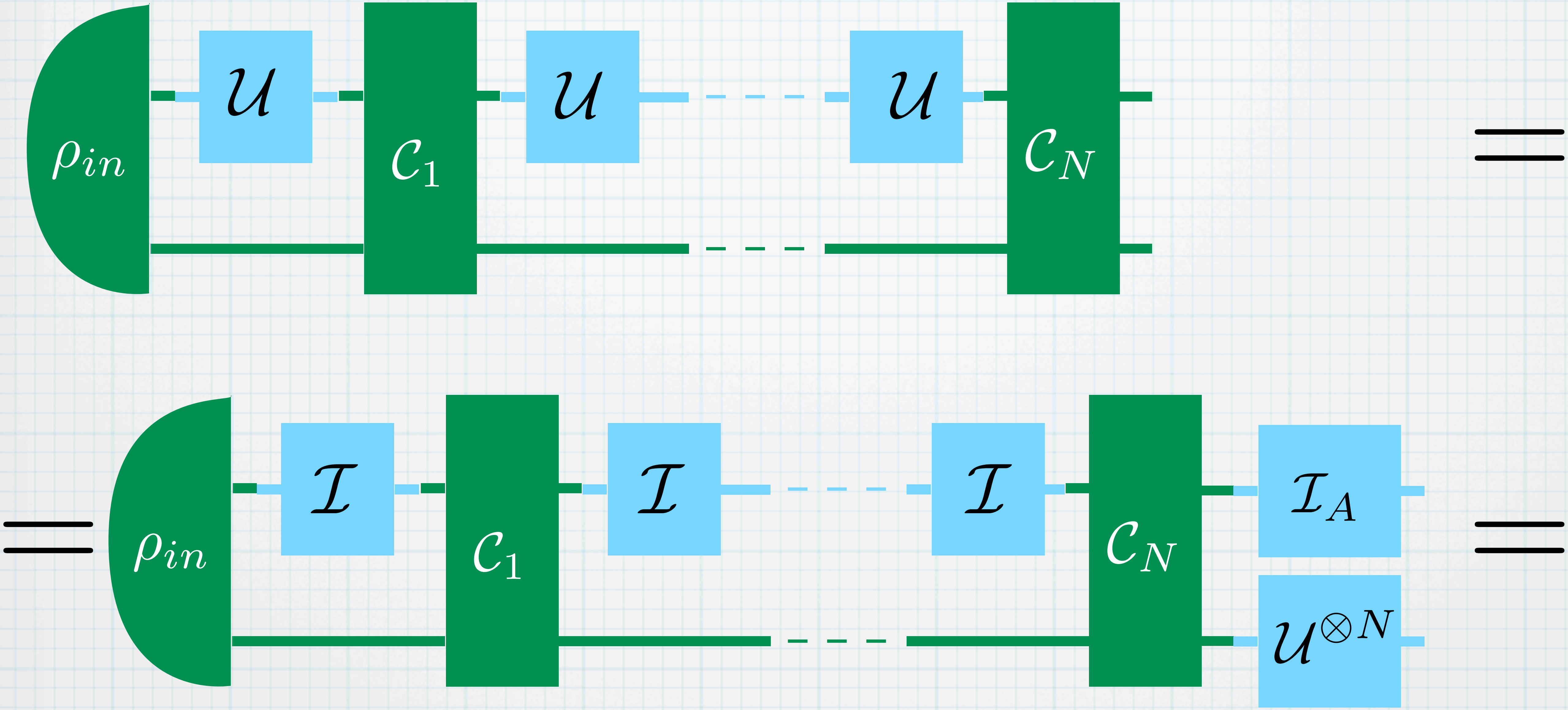
Quantum algorithm learning



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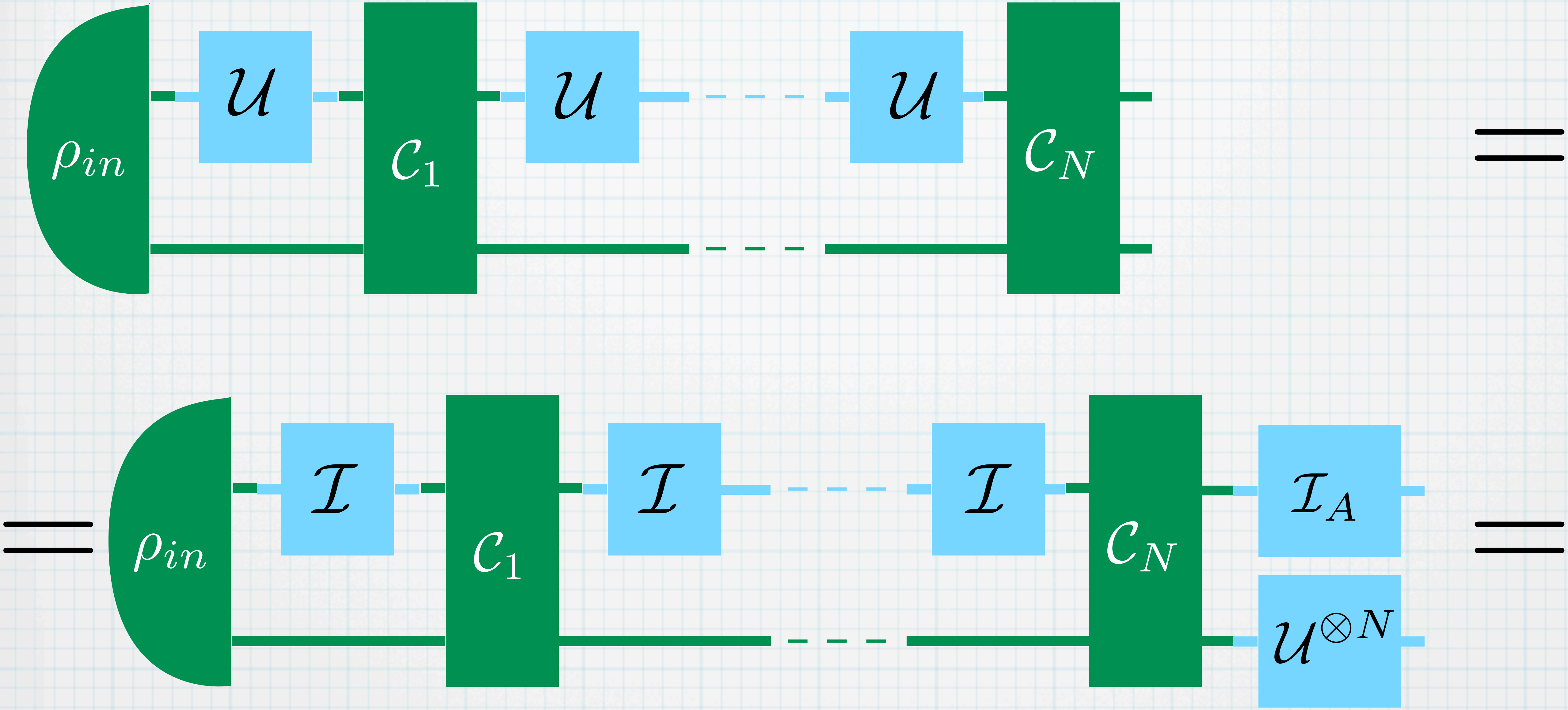
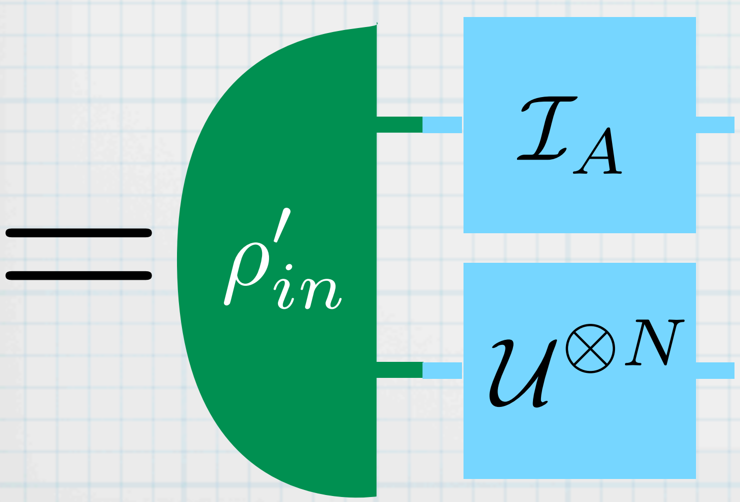




Any covariant network is equivalent to a parallel scheme with ancilla!

Learning can be parallelized, in the same way as estimation

Quantum algorithm learning

Decomposing the unitaries as $$U^{\otimes N} \otimes I_A = \bigoplus_J (U_J \otimes I_{m_J})$$

one can prove that the optimal input states have the form

$$|\psi\rangle = \bigoplus_J a_J \frac{|I_J\rangle\rangle}{\sqrt{d_J}} \qquad a_J \geq 0$$

where $|I_J\rangle\rangle \in \mathcal{H}_J^{\otimes 2}$ is a maximally entangled state

This is the same form of the optimal states for
estimation of the unknown unitary U with N copies

G Chiribella, G M D'Ariano, and M F Sacchi, Phys. Rev. A 72, 043448 (2005).

Quantum algorithm learning

Bisio, Chiribella, D'Ariano, Facchini, Perinotti (unpublished)

Theorem: for any group of unitaries, for an input state of the optimal form

$$|\psi\rangle = \bigoplus_J a_J \frac{|I_J\rangle\rangle}{\sqrt{d_J}} \qquad a_J \geq 0$$

the optimal retrieving channel to extract U from the states

$$(U^{\otimes N} \otimes I_A)|\psi\rangle = \bigoplus_J a_J \frac{|U_J\rangle\rangle}{\sqrt{d_J}} \qquad a_J \geq 0$$

is achieved by a "measure-and-prepare" scheme.

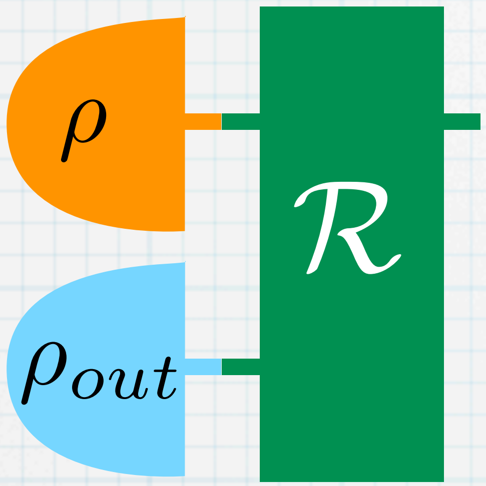
(estimation of the unknown unitary U:

for outcome $\hat{U}$, just perform the unitary $\hat{U}$)

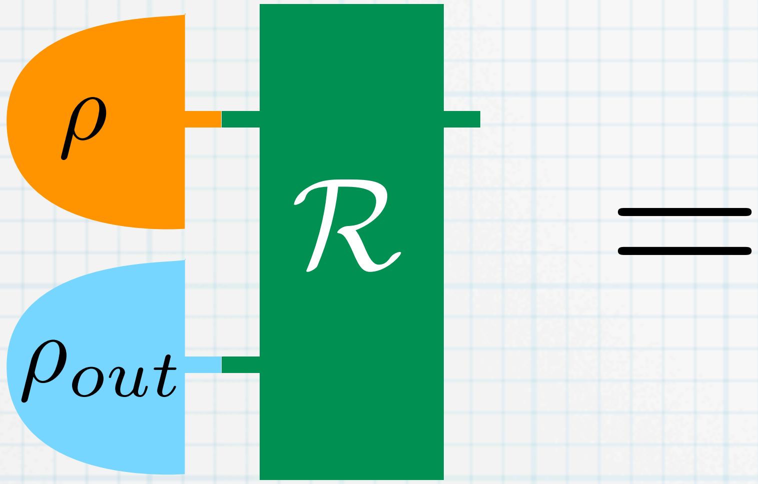
For the optimal POVM, see G Chiribella, G M D'Ariano, and M F Sacchi, Phys. Rev. A 72, 043448 (2005).

Quantum algorithm learning

Quantum algorithm learning



Quantum algorithm learning



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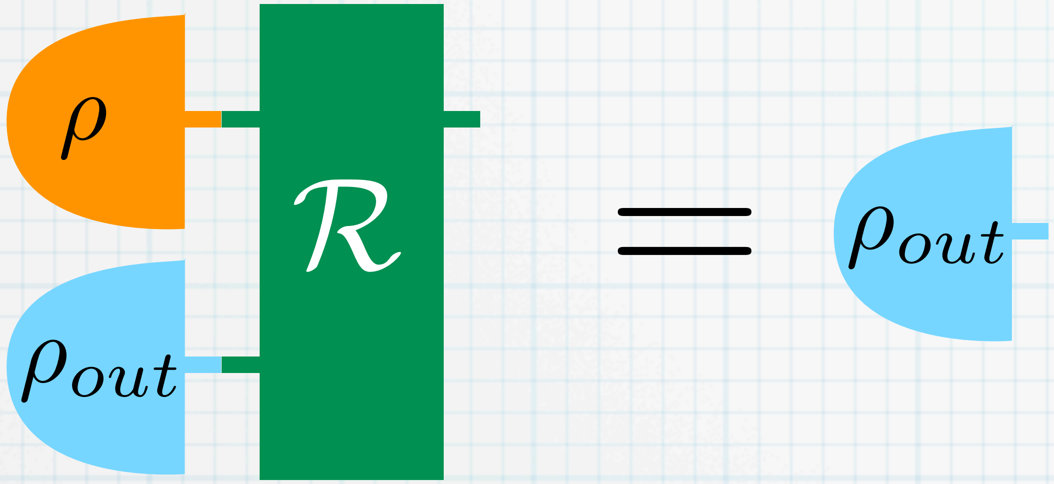

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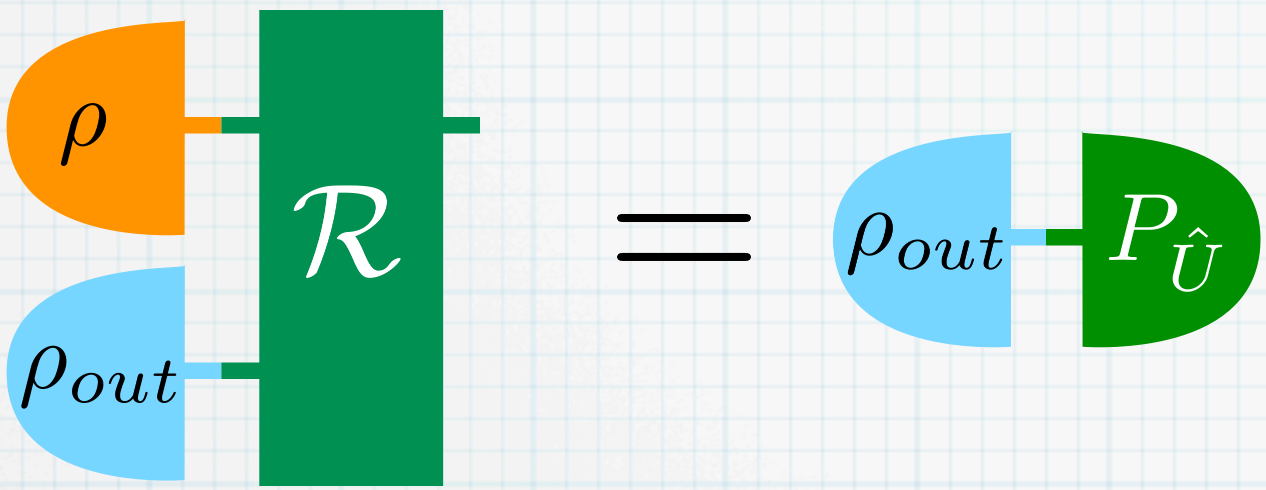

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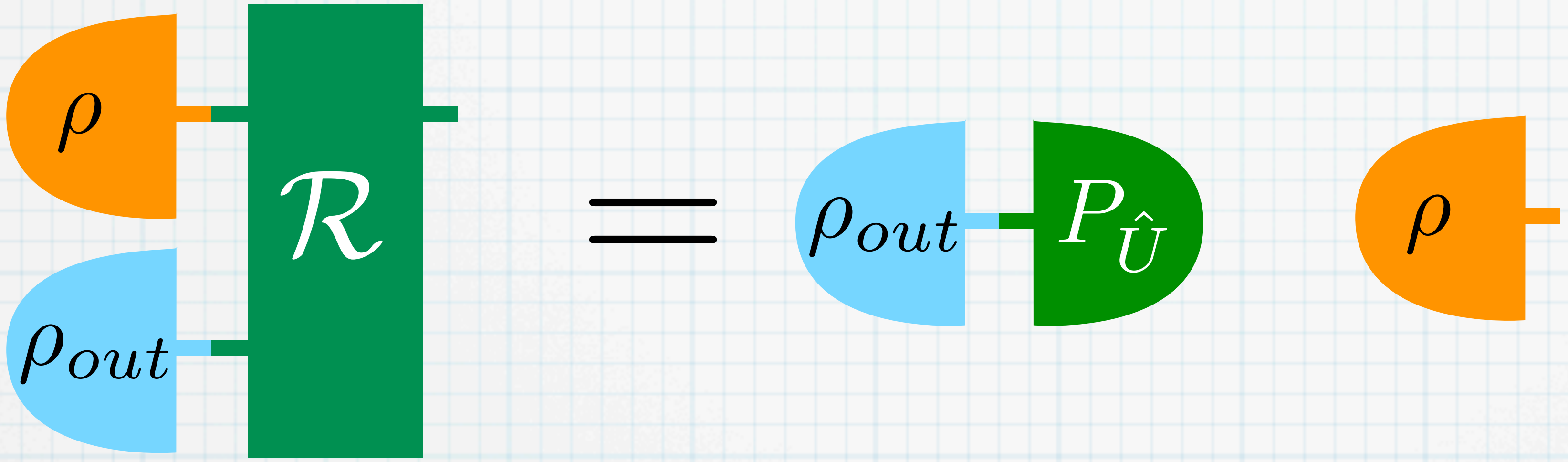

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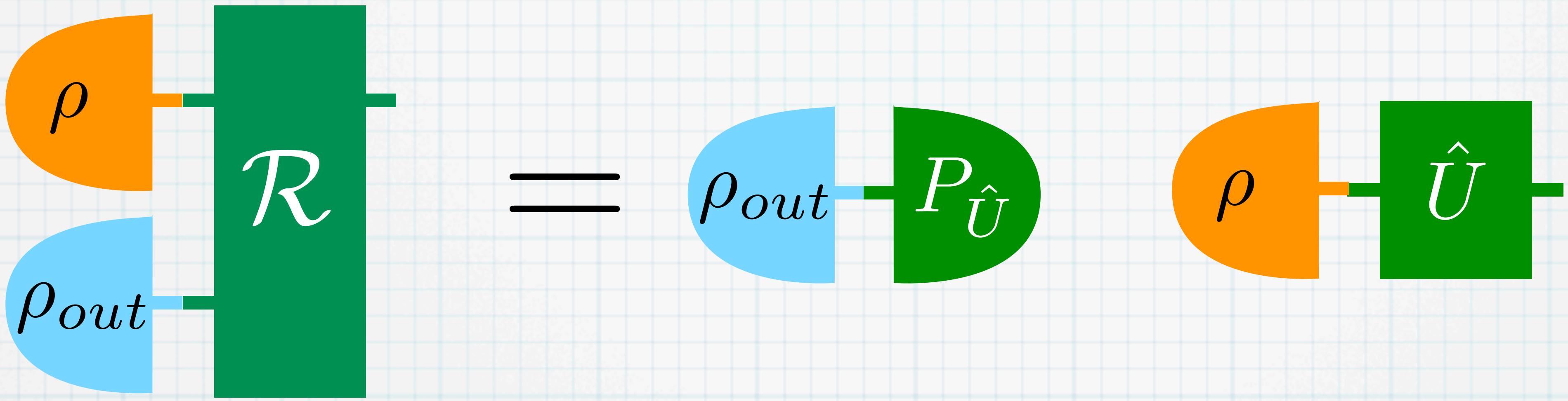

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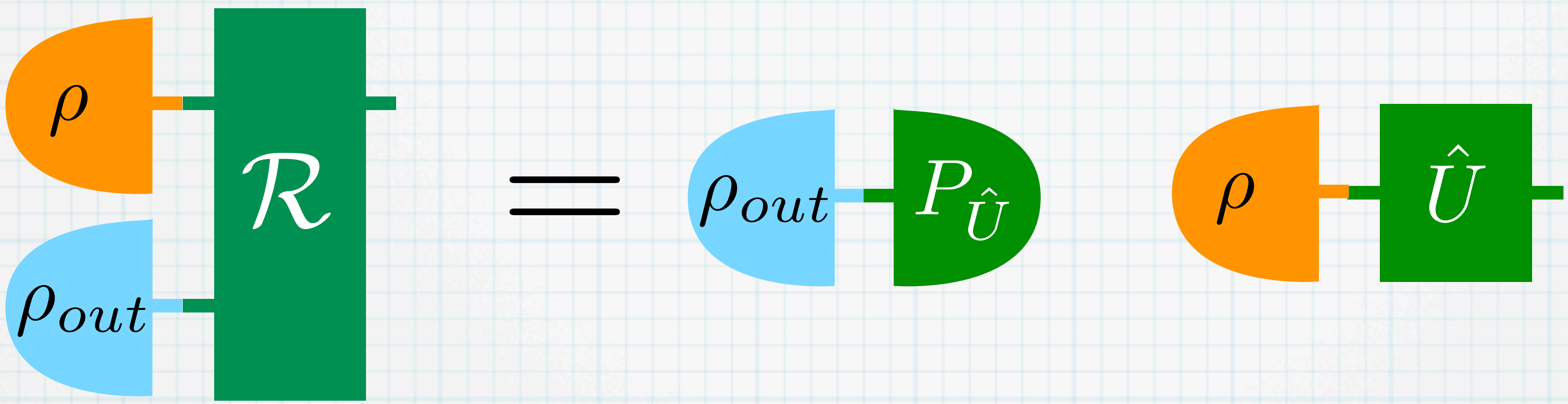

Optimal retrieving is “measure-and-prepare”: no need for quantum memory. We can measure immediately after applying U, and store the outcome $\hat{U}$ in a classical memory.

We can make as many copies as we want (a quantum memory is degraded at every access).

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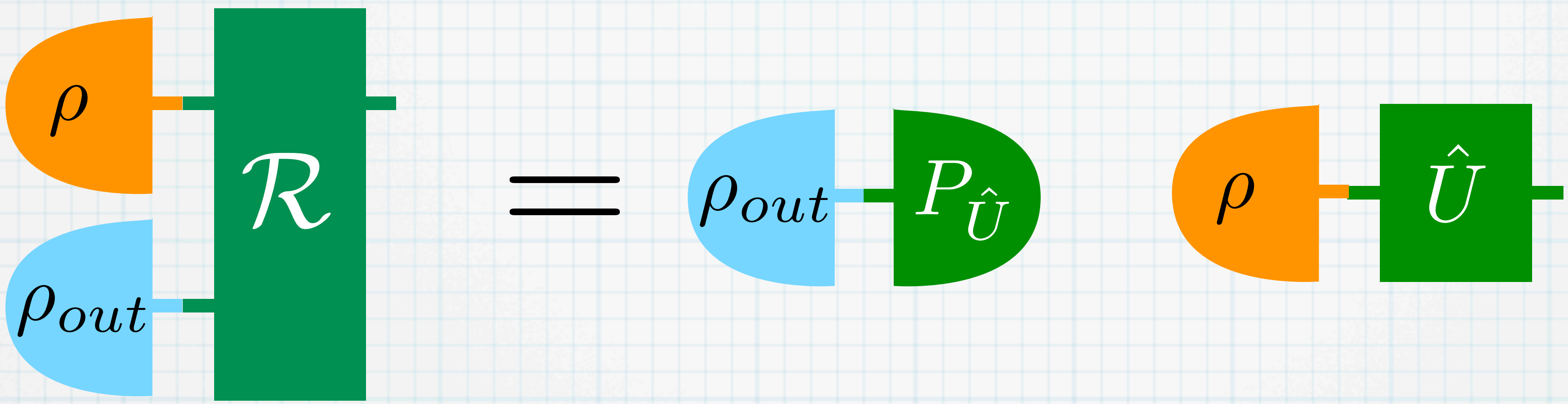
Quantum algorithm learning



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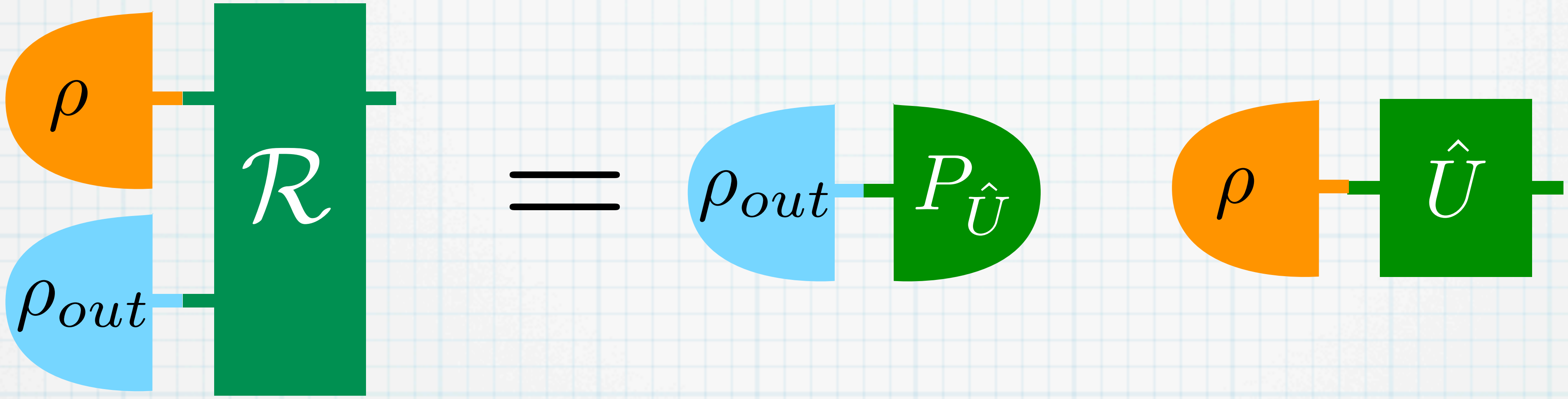
Parallel scheme + measure&reprepare still optimal for:

- N non-identical input unitaries (and/or non-identical target unitaries)
- perform the inverse of U: target $U^\dagger$ (error correction with correlated noise)

Quantum algorithm learning



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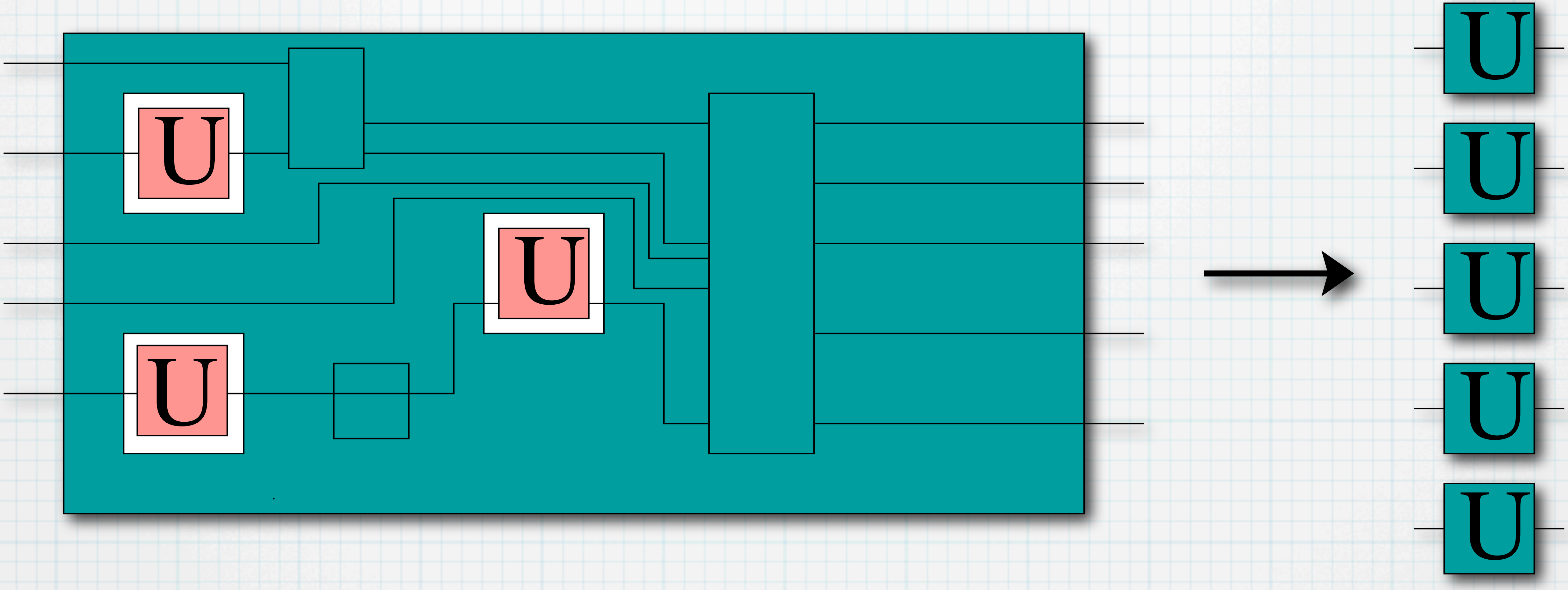
not necessarily optimal for

- learning general channels
- learning unitaries that do not form a group
- learning with restrictions on the available input states (entanglement)

Application 4: Optimal cloning of unitaries

Cloning of unitaries

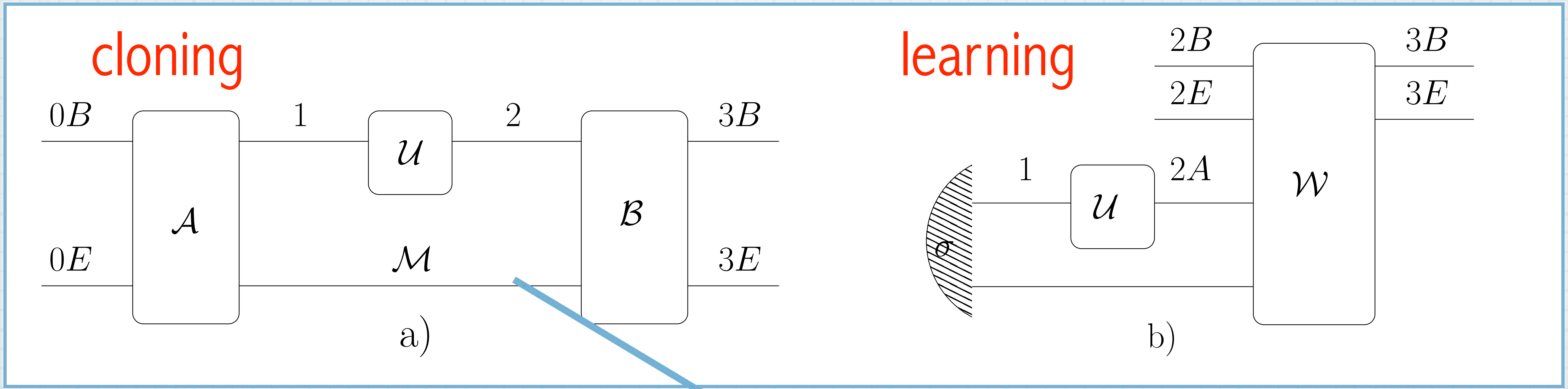
G. Chiribella, G. M. D'Ariano, P. Perinotti PRL **101** 180504 (2008)



$$F = \int \mathrm{d}U \, F(\mathscr{T}_U^{(N)}, \mathscr{T}_U^{\otimes N})$$ (channel fidelity)

Cloning of unitaries

Chiribella, D'Ariano, Perinotti, PRL **101** 180504 (2008)



(same for learning)

1-to-2 cloning $$F = \frac{d + \sqrt{d^2 - 1}}{d^3} > F_{est} = \frac{6}{d^4} (d \neq 2)$$

Much better than cloning states $(I \otimes U)|I\rangle\rangle$ → q-learning

Applying U to a state "degrades" U

Cloning unitaries is harder than cloning states (cryptography with information encoded on transformations is more secure)

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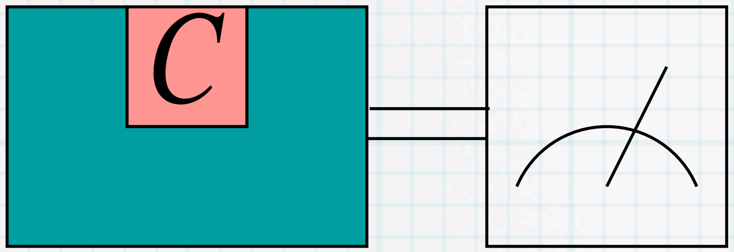
Application 5: Optimal quantum tomography

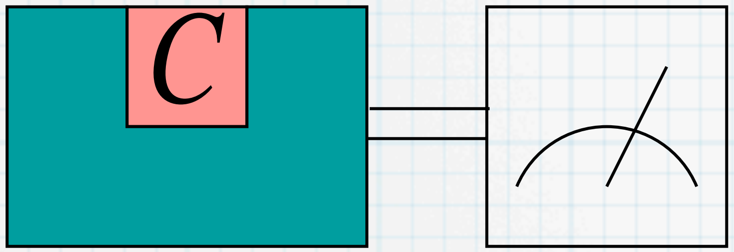
Optimal tomographers



(d^4 outcomes)

Informationally complete tester

Optimal tomographers



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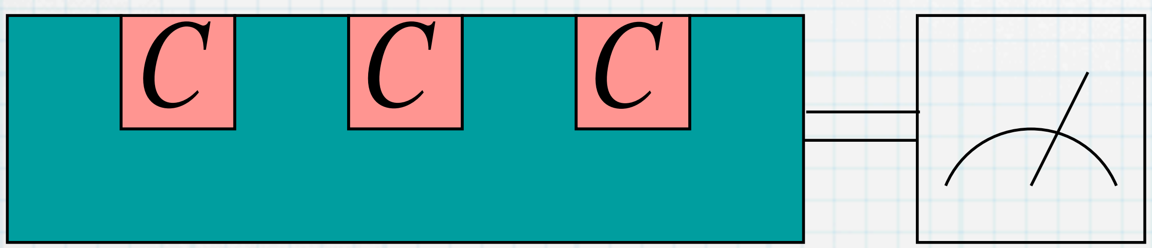
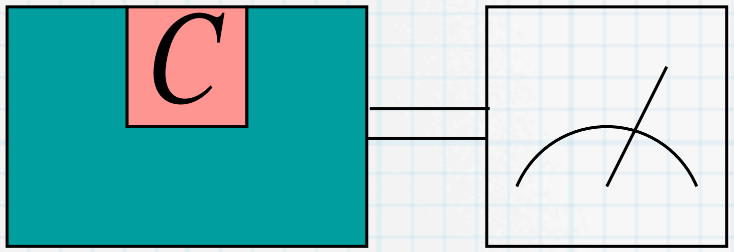


multiple uses

Optimal tomographers



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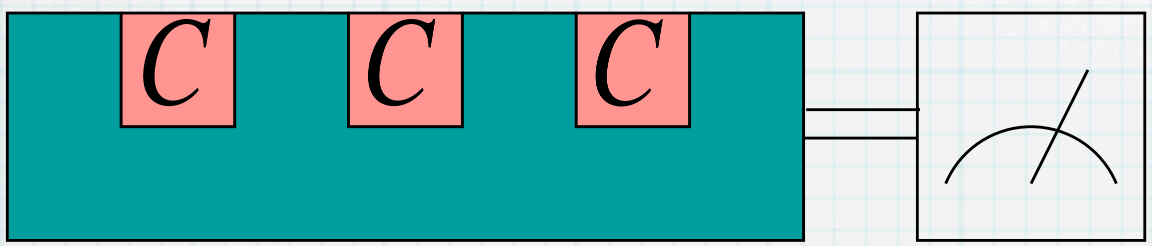

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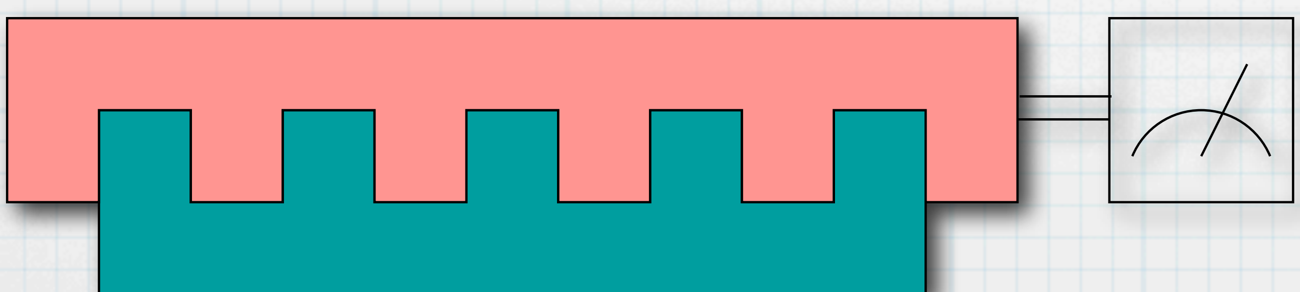
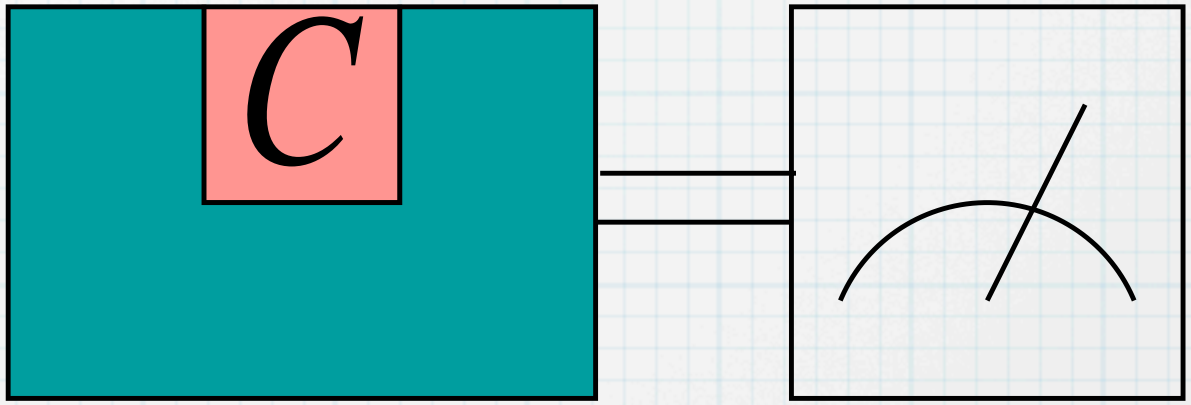


circuit board tomographer

Optimal tomography

Use different in and out dimensions to unify: states, channels, and POVMs

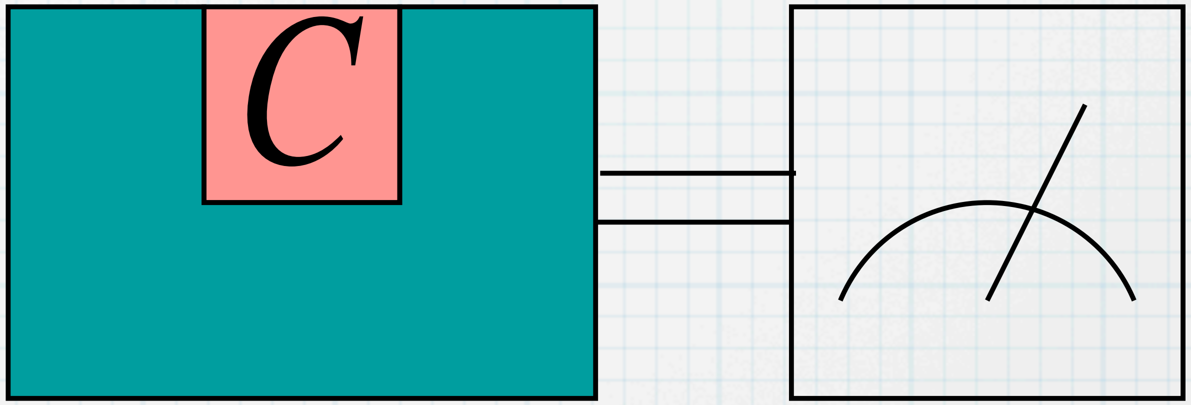


A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, P. Perinotti PRL 102 010404 (2009) arXiv: 0806.1172

Optimal tomography

- Prior distribution of channels corresponding to the depolarizing average channel
- Cost function = representation, (equally weighted orthonormal set of operators)
- Further selection:
  1) quantum operations,
  2) channels,
  3) unital channels

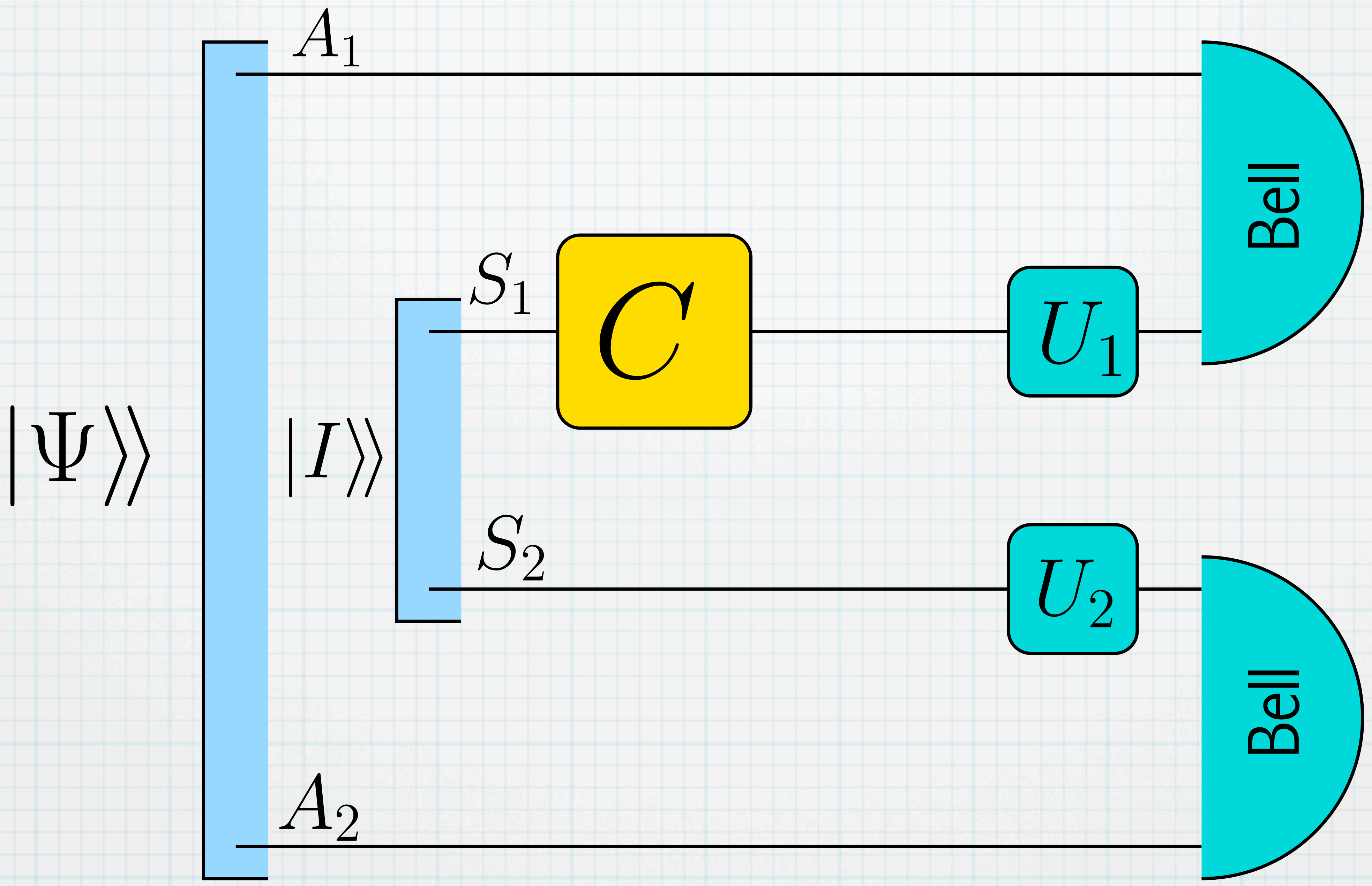
Use different in and out dimensions to unify: states, channels, and POVMs



A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, P. Perinotti PRL 102 010404 (2009)

arXiv: 0806.1172

Optimal tomography



$$\Psi = [d^{-1}(1-\beta)I + \beta\,|\psi\rangle\langle\psi|]^{\frac{1}{2}}$$

$\beta = \sqrt{(d+1)/(d^2+1)}$ quantum operations

$\beta = [(d-1)(2+\sqrt{2}(d^2-1))]^{-1/2}$ channels

$\beta = 0$ unital channels

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Conclusions

PRL **101** 060401 (2008)
PRL **101** 180501 (2008)
PRL **101** 180504 (2008)
PRL **102** 010404 (2009)
EL **83** 30004 (2008)

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- New Quantum Estimation Theory, with multiple copies, and optimization of the setup → optimization of quantum circuits architecture, engineering high-precision operations
  - Quantum circuit board = quantum comb = supermap
  - Comb algebra (link-product)
  - Convex optimization method
- Applications:
  - discrimination/estimation of unitaries and memory channels (optimal quantum oracle-calling algorithms)
  - quantum protocols
  - quantum-algorithm learning = storing undisclosable unitaries
  - cloning undisclosable unitaries
  - process tomography