Programmability of measurements and channels

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Programmability of channels

Deterministic programming

\[ \mathcal{P}_{V,\sigma}(\rho) \equiv \text{Tr}_2[V(\rho \otimes \sigma)V^\dagger] \]
Programmability of channels

\[ \mathcal{P}_{V,\sigma}(\rho) \doteq \text{Tr}_2[V(\rho \otimes \sigma)V^\dagger] \]

\[ \mathcal{P}_V \doteq \mathcal{P}_{V,\mathcal{A}} \]

No go theorem (Nielsen-Chuang)

It is impossible to program all unitary channels with a single \( V \) and a finite-dimensional ancilla.
Programmability of unitaries

**Nielsen-Chuang theorem**

Suppose distinct (up to a global phase) unitary operators $U_1, \ldots U_N$ are implemented by some programmable quantum gate array. Then the program register is at least $N$ dimensional. Moreover, the corresponding programs $|\psi_1\rangle, \ldots, |\psi_N\rangle$ are mutually orthogonal.

**Proof:** for arbitrary $|\psi\rangle \in H$

\[
V(|\psi\rangle \otimes |\psi_i\rangle) = U_i |\psi\rangle \otimes |\psi_i'\rangle
\]
\[
V(|\psi\rangle \otimes |\psi_j\rangle) = U_j |\psi\rangle \otimes |\psi_j'\rangle
\]

\[
\langle \psi_i | \psi_j \rangle = \langle \psi_i' | \psi_j' \rangle \langle \psi | U_i^\dagger U_j | \psi \rangle
\]

\[
\langle \psi_i' | \psi_j' \rangle \neq 0 \Rightarrow \frac{\langle \psi_i | \psi_j \rangle}{\langle \psi_i' | \psi_j' \rangle} = \langle \psi | U_i^\dagger U_j | \psi \rangle
\Rightarrow U_i^\dagger U_j = cI
\]

\[
\langle \psi_i' | \psi_j' \rangle = 0 \Rightarrow \langle \psi_i | \psi_j \rangle = 0 \blacksquare
\]
Programmability of unitaries

The joint unitary that programs perfectly the unitary operators $U_1, \ldots, U_N$ is the controlled-$U$ operator (modulo local unitaries)

$$V = \sum_j U_j \otimes |\psi_j\rangle\langle\psi_j|$$
Problem: The most efficient Unitary

For given $d = \text{dim}(A)$ find the unitary operators $V$ that are the most efficient in programming channels, namely which minimize the largest distance of each channel $C \in \mathcal{C}$ from the programmable set $\mathcal{P}_V, A$:

$$\varepsilon(V) = \max_{C \in \mathcal{C}} \min_{P \in \mathcal{P}_V, A} \delta(C, P)$$
The most efficient Unitary

one would like to use \( \delta(C, P) = \|C - P\|_{CB} \)

instead we use

\[
\delta(C, P) = \sqrt{1 - F(C, P)}
\]

where \( F(C, P) \) is the Raginsky fidelity

\[
F(U, P) = \frac{1}{d^2} \sum_i |\text{Tr}[C_i^\dagger U]|^2
\]

\[
F(V) = \min_{U \in \mathcal{U}(H)} F(U, V), \quad F(U, V) = \max_{\sigma \in \mathcal{A}} F(U, P_{V, \sigma})
\]
The most efficient Unitary

**GNS representation**

- Bipartite states $|\Psi\rangle \in H \otimes K \iff$ operators $\Psi \in HS(K, H)$
  
  $$|\Psi\rangle = \sum_{nm} \Psi_{nm} |n\rangle \otimes |m\rangle.$$  
- Matrix notation (for fixed reference basis in the Hilbert spaces)

  $$A \otimes B |C\rangle = |AC B^\top\rangle,$$

  $$\langle A|B\rangle \equiv \text{Tr}[A^\dagger B].$$
The most efficient Unitary

\textit{GNS representation}

cyclic vector \(|I\rangle \in H \otimes K\)

\[ \Psi \in \text{HS}(K, H), \quad |\Psi\rangle = (\Psi \otimes I)|I\rangle \]

transposition

\[ |\Psi\rangle = (\Psi \otimes I)|I\rangle = (I \otimes \Psi^\dagger)|I\rangle \]

complex conjugation

\[ X^* = (X^\dagger)^\dagger \]

\[ (|v\rangle\langle v| \otimes I)|I\rangle = |v\rangle|v^*\rangle. \]
The most efficient Unitary

\[ V = \sum_k e^{i\theta_k} |\Psi_k\rangle \langle \Psi_k|, \]

**Krauss form**.

\[ P_{V, \sigma}(\rho) = \sum_{nm} C_{nm} \rho C_{nm}^\dagger, \quad C_{nm} = \sum_k e^{i\theta_k} \Psi_k |v_n^*\rangle \langle v_m| \Psi_k^\dagger \sqrt{\lambda_m} \]

|\nu_n\rangle denotes the eigenvector of \( \sigma \) corresponding to the eigenvalue \( \lambda_n \)

\[ \sum_{nm} |\text{Tr}[C_{nm}^\dagger U]|^2 = \sum_{kh} e^{i(\theta_k - \theta_h)} \text{Tr}[\Psi_k^\dagger U^\dagger \Psi_k \sigma^\dagger \Psi_h^\dagger U \Psi_h] \]

\[ = \text{Tr}[\sigma^\dagger S(U, V)^\dagger S(U, V)] \]

\[ S(U, V) = \sum_k e^{-i\theta_k} \Psi_k^\dagger U \Psi_k. \]

Max over \( \sigma \) \[ F(U, V) = \frac{1}{d^2} \|S(U, V)\|^2. \]
The most efficient Unitary

\[ F(U, V) = \frac{1}{d^2} \| S(U, V) \|^2. \]

\[ S(U, V) = \text{Tr}_1[(U^\dagger \otimes I)V^*] \]

Changing \( V \) by local unitary operators transforms \( S(U, V) \) in the following fashion

\[ S(U, (W_1 \otimes W_2)V(W_3 \otimes W_4)) = W_2^*S(W_1^\dagger UW_3^\dagger, V)W_4^* \]
The most efficient Unitary $d=2$

\[ V = (W_1 \otimes W_2) \exp[i(\alpha_1 \sigma_1 \otimes \sigma_1^T + \alpha_2 \sigma_2 \otimes \sigma_2^T + \alpha_3 \sigma_3 \otimes \sigma_3^T)](W_3 \otimes W_4) \]

Figure 1. Quantum circuit scheme for the general joint unitary operator $V$:
Here we use the notation $G_\phi = \exp(i\phi \sigma_G)$ with $G = X, Y, Z$.

study only joint unitary operators of the form

\[ V = \exp[(i(\alpha_1 \sigma_1 \otimes \sigma_1^T + \alpha_2 \sigma_2 \otimes \sigma_2^T + \alpha_3 \sigma_3 \otimes \sigma_3^T)] \]
The most efficient Unitary $d=2$

\[ V = \exp \left[ (i(\alpha_1 \sigma_1 \otimes \sigma_1^T + \alpha_2 \sigma_2 \otimes \sigma_2^T + \alpha_3 \sigma_3 \otimes \sigma_3^T)) \right] \]

**eigenvectors:**

\[ |\Psi_j\rangle = \frac{1}{\sqrt{2}} |\sigma_j\rangle \quad (Bell \ basis) \]

\[ S(U,V) = \frac{1}{2} \sum_{j=0}^{3} e^{-i\theta_j} \sigma_j U \sigma_j \quad \theta_0 = \alpha_1 + \alpha_2 + \alpha_3, \quad \theta_i = 2\alpha_i - \theta_0 \]

**Bloch representation:**

\[ U = n_0 I + i n \cdot \sigma \]

\[ n_k \in \mathbb{R} \text{ and } n_0^2 + |n|^2 = 1 \]

\[ S(U, V) = \tilde{n}_0 I + \tilde{n} \cdot \sigma, \]

\[ \tilde{n}_j = t_j n_j, \quad 0 \leq j \leq 3, \quad t_0 = \frac{1}{2} \sum_{j=0}^{3} e^{-i\theta_j}, \]

\[ t_j = e^{-i\theta_0} + e^{-i\theta_j} - t_0, \quad 1 \leq j \leq 3, \quad t_j = |t_j| e^{i\phi_j}, \quad 0 \leq j \leq 3, \]
The most efficient Unitary $d=2$

$$V = \exp[(i(\alpha_1 \sigma_1 \otimes \sigma_1^T + \alpha_2 \sigma_2 \otimes \sigma_2^T + \alpha_3 \sigma_3 \otimes \sigma_3^T))]$$

$$\|S(U, V)\|^2 = u \cdot t + \sqrt{u \cdot Tu}.$$

where $u = (n_0^2, n_1^2, n_2^2, n_3^2)$, $t = (|t_0|^2, |t_1|^2, |t_2|^2, |t_3|^2)$, and $T_{ij} = |t_i|^2|t_j|^2 \sin^2(\phi_i - \phi_j)$. One has the bounds

$$u \cdot t + \sqrt{u \cdot Tu} \geq u \cdot t \geq \min_j |t_j|^2$$

$$F(V) = \frac{1}{d^2} \min_j |t_j|^2.$$

take $\theta_0 = 0, \theta_1 = \pi/2, \theta_2 = \pi, \theta_3 = \pi/2$, corresponding to the eigenvalues $i, 1, -i, 1$ for $V$. Another solution is $\theta_0 = 0, \theta_1 = -\pi/2, \theta_2 = \pi, \theta_3 = -\pi/2$. 
The most efficient Unitary $d=2$

$$V = \exp[(i(\alpha_1 \sigma_1 \otimes \sigma_1^T + \alpha_2 \sigma_2 \otimes \sigma_2^T + \alpha_3 \sigma_3 \otimes \sigma_3^T))]$$

$$F \doteq \max_{V \in U(H^{\otimes 2})} F(V) = \frac{1}{d^2} = \frac{1}{4}$$

the corresponding optimal $V$ has the form

$$V = \exp \left[ \pm i \frac{\pi}{4} (\sigma_x \otimes \sigma_x \pm \sigma_z \otimes \sigma_z) \right]$$

Figure 2. Quantum circuit scheme for the optimal unitary operator $V$
The most efficient Unitary d=2 controlled-\(U\)

\[
V = \sum_{k=1}^{2} V_k \otimes |\psi_k\rangle\langle\psi_k|, \quad \langle\psi_1|\psi_2\rangle = 0, \quad V_1, V_2 \text{ unitary on } H \simeq \mathbb{C}^2
\]

\[
F(U, V) = \frac{1}{4} |\text{Tr}[V_h^\dagger U]|^2 \quad h = \arg \max_k |\text{Tr}[V_k^\dagger U]|
\]

\[
F(V) = \min_U F(U, V) = 0.
\]
Programmability of POVMs

\[ P_{Z,\sigma} \doteq \text{Tr}_2[(I \otimes \sigma)Z] \]

\[ \mathcal{P}_Z \doteq P_{Z,\mathcal{A}} \]

\[ Z \doteq \{Z_1, Z_2, \ldots, Z_N\} \]

\[ \mathbf{P} \doteq \{P_1, P_2, \ldots, P_N\} \]
Programmability of POVMs

Deterministic channel programmability

Deterministic POVMs programmability
**No go theorem**

It is impossible to program all observables with a single joint observable $Z$ and a finite-dimensional ancilla.

$$P_{Z,\sigma} = \text{Tr}_2[(I \otimes \sigma)Z]$$

$$P_Z = P_{Z,A}$$
No go theorem

Suppose $M$ distinct observables $X_1, X_2, \ldots, X_M$ are implemented by some programmable quantum gate array. Then the program register is at least $M$ dimensional. Moreover, the corresponding programs $|\psi_1\rangle, \ldots, |\psi_M\rangle$ are mutually orthogonal.

**Proof:**

$X_l = \text{Tr}_2[(I \otimes |\psi_l\rangle\langle\psi_l|)Z]$

$|x_l^{(j)}\rangle\langle x_l^{(j)}| = \text{Tr}_2[(I \otimes |\psi_l\rangle\langle\psi_l|)Z^{(j)}]$

$\langle x_l^{(n)}| |\psi_l\rangle Z^{(j)} |x_l^{(m)}\rangle |\psi_l\rangle = \delta_{jn}\delta_{jm}$

$Z^{(j)} |x_l^{(j)}\rangle |\psi_l\rangle = |x_l^{(j)}\rangle |\psi_l\rangle$

$Z^{(i)} Z^{(j)} = \delta_{ij} Z^{(j)}$

$Z^{(j)} |x_l^{(i)}\rangle |\psi_l\rangle = \delta_{ij} |x_l^{(i)}\rangle |\psi_l\rangle$

$\langle \psi_l|\psi_k\rangle \langle x_l^{(j)}|x_k^{(i)}\rangle = 0, \ i \neq j$

$X_l \neq X_k \ \Rightarrow \ \langle \psi_l|\psi_k\rangle = \delta_{lk}$
Programmability of observables

The joint observable that programs perfectly the observables $X_1, X_2, \ldots, X_M$ is the controlled-O operator

$$Z = \sum_l X_l \otimes |\psi_l\rangle\langle\psi_l|$$

which can be implemented with a fixed local observable and a controlled-U

$$Z = V^\dagger (X \otimes I)V$$

$$V = \sum_l U_l \otimes |\psi_l\rangle\langle\psi_l| \quad X_l = U_l^\dagger XU_l$$
Problem: The most efficient observable

For given \( d = \dim(\mathcal{A}) \) and \( N = |\mathbf{Z}| = |\mathbf{P}| \), find the observables \( \mathbf{Z} \) that are the most efficient in programming POVM’s, namely which minimize the largest distance of each POVM from the programmable set:

\[
\varepsilon(\mathbf{Z}) = \max_{Q \in \mathcal{P}_N} \min_{P \in \mathcal{P}_Z, \mathcal{A}} \delta(P, Q)
\]
Programmability of observables

programmability with **accuracy** $\varepsilon^{-1}$:

$$\varepsilon(Z) = \max_{P \in \mathcal{P}_Z} \min_{Q \in \mathcal{P}_N} \delta(P, Q)$$

$$\delta(P, Q) = \max_{\rho} \sum_i |\text{Tr}[\rho(P_i - Q_i)]|$$

Using a joint observable $Z$ of the form

$$Z_i = U^\dagger (|\psi_i\rangle\langle\psi_i| \otimes I_A) U, \quad U = \sum_{k=1}^{\dim(A)} W_k \otimes |\phi_k\rangle\langle\phi_k|$$

with $\{\psi_i\}$ and $\{\phi_k\}$ orthonormal sets and $W_k$ unitary, we can program observables with accuracy $\varepsilon^{-1}$ using an ancilla with **polynomial** growth

$$\dim(A) \leq \kappa(N) \left(\frac{1}{\varepsilon}\right)^{N(N-1)}$$
For qubits: *linear growth*!

Program for the observable \( \mathbf{P} = \{ U_g^{(1/2)} | \pm \frac{1}{2} \rangle \langle \pm \frac{1}{2} | U_g^{(1/2)\dagger} \} \)

\[
\sigma = U_g^{(j)} |jj\rangle \langle jj| U_g^{(j)\dagger}
\]

in dimension \( \text{dim}(\mathcal{A}) = 2j + 1 \), with joint observable

\[
\mathbf{Z} = \{ \Pi^{(j \pm \frac{1}{2})} \}
\]

gives the programmability accuracy

\[
\varepsilon(\mathbf{Z}) = \frac{2}{2j + 1} \quad \text{dim}(\mathcal{A}) = 2\varepsilon^{-1}
\]
Exact Programmability of POVMs

Covariant measurements are exactly programmable

G-covariant POVM densities (Holevo theorem)

\[ P_g \, dg = U_g \xi U_g^\dagger \, dg, \quad g \in G \]

programmable as

\[ P_g = \text{Tr}_2[(I \otimes \sigma)F_g], \quad \xi = V\sigma^TV^\dagger \]

with covariant Bell POVM density

\[ F_g = (U_g \otimes I)|V\rangle\langle V|(U_g^\dagger \otimes I) \]
Exact Programmability of POVMs


Unitary operator $U$ connecting the Bell observable with local observables

$$U(|m\rangle \otimes |n\rangle) = \frac{1}{\sqrt{d}} |U_{m,n}\rangle$$

of the controlled-$U$ form

$$U = \sum_{n} |n\rangle\langle n| \otimes W^{n}$$

e. g. for projective $d$-dimensional UIR of the Abelian group $G = Z_{d} \times Z_{d}$

$$U_{m,n} = Z^{m} W^{n}, \quad Z = \sum_{j} \omega^{j} |j\rangle\langle j|, \quad W = \sum_{k} |k\rangle\langle k \oplus 1|, \quad \omega = e^{\frac{2\pi i}{d}}.$$
Conclusions

Programmable channels:
- Nielsen-Chuang theorem revisited
- Exact programming for finite set of unitaries: controlled-U
- Optimal programming in 2x2 dimensions: two controlled-NOT

Programmable POVMs:
- No go theorem
- Exact programming for finite set of observables: controlled-O
  - controlled-O: polynomial complexity programming
  - for qubits: linear complexity programming