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Probabilistic Theories as Models for Exploring Operational Axiomatization of Quantum Mechanics

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Notation and conventions

Conventions: We use *uppercase Sanserif* letters—*e.g.* A, B, C, H, K, Span, Ker, etc.—to denote sets and/or linear spaces; with very few exceptions, as for the creation and annihilation operators a, $a^\dagger$, b, $b^\dagger$ of field modes, we will use *uppercase math* letters—*e.g.* A, B, C, P, Q, X, Y—to denote both matrices and operators; we keep *lowercase math* letters—*e.g.* a, b, c, f, v, w—for vectors. We use mathscript letters such as $\mathscr{A}$, $\mathscr{B}$, etc. to denote maps on operator spaces/algebras, and, finally, we use mathcal letters such as $\mathcal{A}$, $\mathcal{B}$, etc. to denote operator algebras. In the following we will use the following standard notation for Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

Introduction

A person approaching physical science is naturally led to wonder what is physics itself. Within this issue the question: “what is Quantum Mechanics?”, has a decisive role. From a rather abstract, but intuitive at the same time, point of view, physics is the description of the reality perceived by the ***human being***. The first aspect which is worth to speak about is the common attempt to make fundamental physics regardless of the human. In my opinion it’s deleterious and misleading; clearly each description of reality worked out by a human being cannot escape from its point of view even if the human would seem to be totally foreign to our description. It is not necessary a concrete role of the man in a theory to declare it man dependent (on the other hand it is the case in Quantum Mechanic), if a human think about a theory it automatically depends on its peculiar vision of the phenomenons. According to this seems to be convenient to acknowledge that physics, being a description of reality and not the reality itself, may be molded around us without it invalidates the generality of our results.
Which is the role of Quantum Mechanics in this context. Once again as abstract thought it can be regarded as the ***language*** in which physics is written. A language is a set of symbolic objects and rules to manage them in order to represent and describe something. We get a basic example of it from the different tongues on the earth, each of them is constituted by an alphabet and a set of grammar rules to compose words, phrases, etc.... It this sense a tongue is a network between people allowing to associate to everything a symbolic representative. If two language describe the same objects they are said to be equivalent, anyway they can be completely different. Coming back to tongues, English and Chinese are equivalent languages but they are really different. It is well known that Quantum Mechanics, regarded as physics language, has a striking predictive power in almost all its employment fields. Despite that there is no guarantee that it is the only admissible language. Therefore a very ambitious and outstanding result would be to select Quantum Mechanics as the only acceptable language of physics. In order to purse this goal we have to make rigorous the adjective “acceptable”.
This problem is included in the general context of Quantum Mechanics axiomatization. A very recent and fruitful approach to such problem consists in positioning Quantum Mechanics within the landscape of general ***probabilistic theories***. These last ones seem to be the most natural way to achieve an ***operational axiomatization*** of Quantum Mechanics. The basic structure of a probabilistic theory is an operational framework arising from a a very natural consideration:“the human being accedes to realty through experiments”. Each experiment has a result which is the information gained by the experimenter; thus a operational framework is a set of rules which allows the experi-

menter to make predictions on future events on the basis of suitable tests. The probability play a crucial role in this definition because the well know concept of "state" can be concretized as a set of probabilities for all possible outcomes of any test. Whence by operational axiomatization we mean the possibility of deriving Quantum Mechanics as the mathematical representation of a fair operational framework. The idea is to define a general class of probabilistic theories assuming only Postulates that need to be satisfied by any fair operational framework and then adding new operational Postulates until the only Quantum Mechanics, or at least an equivalence class of probabilistic theories equivalent to Quantum mechanics, satisfy all of them. A very general class of probabilistic theories is defined in the article ***"Probabilistic theories: what is special about Quantum Mechanics?"*** (see [D'A08]) were the minimal Postulates are:

> NSF: *no signaling from the future*, implying that it is possible to make predictions based on present tests;
>
> PFAITH: *existence of preparationally faithful states*, implying the possibility of preparing any states and calibrating any test.

From these Postulates derive a lot of features for the class of probabilistic theories in exam as we will see in Cap. 2 of the thesis. However that framework includes also theories with non-local correlations stronger than the quantum ones, *e.g.* the Popescu-Rohrlich boxes analysed in Cap. 3, which violate the quantum Cirel'son bound [Cir94] although they are still compatible with the no-signaling principle; also this aspect will be further discussed in the thesis. In [D'A08] other Postulates are introduced and their consequence are still being investigated in new publication and, in part, in this thesis too. However it's perplexing to derive Quantum Mechanics from these principles only. In the particular context of the probabilistic theories, from the main operational notion of tests, it's possible to define the cones of transformations, states and effects of a theory (in a nutshell effects are equivalence classes of transformations), and usual mathematical structures are achievable for the linear spaces in which they lie. What is really special about Quantum Mechanics is that not only transformations but also effects make a C^**-algebra*.[1] Unfortunately composition of effects, unlike their sum, has no operational meaning and to select the quantum-classical hybrid among the whole set of possible probabilistic theories we need a ***mathematical Postulate***. In [D'A08] the goal is reached by the Postulates

> AE: *atomicity of evolution*;
>
> CJ: *Choi-Jamiolkowski isomorphism*[Cho75, Jam72].

In the following we will see that atomicity of evolution is a very natural postulate while the Choi-Jamiolkowski isomorphism, although looks like reasonable in an operational context, is a mathematical postulate; optimistically it will be find out to be a consequence of some others operational principles.

The thesis is organised in three Parts whose content is briefly summarized in the following.

[1] More precisely as specified in [D'A08] this is true for all hybrid quantum-classical theories,*i.e.* corresponding to Quantum Mechanics plus super-selection rules.

Part I. States, effects and transformations of a probabilistic theory are elements of convex sets where the convex structure is inherited from the probabilistic character of the frame. Therefore in Cap. 1 we briefly define the main concepts concerning convex sets and special emphasis will be placed upon transformations preserving convexity. After introducing the Block representation for affine transformations of a convex set we will deal the problem of classifying its contractions.
The second chapter is fully devoted to the probabilistic theories general formulation given in [D'A08]. The starting point will be the notion of test enabling an operational definition of ***system*** with his set of transformations, states and effects. As already suggested will be introduced a rather general class of theories assuming the only Postulates NSF and PFAITH. Integral part of a probabilistic theory are the ***multipartite systems*** and the correlations between systems. In this context arises the non locality property which is a common features of a lot of probabilistic theories and not a peculiarity of the quantum one. Thus the composition of systems will be defined and from a very general notion of dynamical independence will be easy to show that all the theories in exam satisfy the Einstein locality principle, namely it's impossible to find detectable effects on a system from whatever is done on another non-interacting system. This is a reasonable condition in a fair operational framework and another reasonable requirement is the main consequence of the PFAITH Postulate, namely the ***local observability principle***. According to this last one it's possible to achieve an informationally complete test using only local tests allowing for example the preparation of non local states acting locally on a system. Subsequently some additional postulates will be introduced,*e.g.* the Postulates FAITH and PURIFY. The first one is something similar to the dual version of PFAITH while PURIFY concern the possibility of achieve a purification of all states as in Quantum Mechanic. The consequence of this postulates are still being investigated and their effectiveness in the operational axiomatization of Quantum Mechanics will be tested in Part II of this thesis too.

Part II. The probabilistic theories world is still unexplored and then there is a scanty intuition about it. To be more precise the intuition could yet be misleading because it comes from Quantum Mechanics which should be our goal and not our investigation instrument. In fact it's easy to mistakenly assume quantum-like features as general properties of probabilistic theories. Another remarkable difficulty comes from the absence of probabilistic models, different from Quantum Mechanics, through which test any new postulate. It will be very useful after introducing a new postulate try to construct a theory satisfying all the postulates assumed till then. According to these necessities in Part II some concrete probabilistic models will be constructed. Naturally all of them will satisfy the the basic NSF and PFAITH postulates. The first probabilistic model is a generalisation of the well known ***Popescu-Rohrlich boxes model***, see [RP95], which achieve the greatest violation of the CHSH inequality compatible with the no-signaling principle. The easy immersion of a preexisting model in our general probabilistic framework might be interpreted as a test of the framework itself.
In Cap. 4 we will consider the ***two-clocks probabilistic model***. This name come from the the local system denoted *clock* because of its convex set of states which is a disc. Clearly a lot of theories would have a disc as local set of states so we will focus our attention on a particular situation. We will take as completely positive maps all the maps

preserving the local cone of states. This model will satisfying for example the Postulate PURIFY introduced in Part. I (which is not a property of the Popescu-Rohrlich model), on the other hand the model lack in some fundamental quantum features and this exclude the possibility of achieving Quantum Mechanics just adding PURIFY to the main postulates. A stronger and promising version of Postulate PURIFY, not satisfied by our model, is being investigated in [CDP09]. The two-clocks probabilistic theories admit an underline ***hidden quantum model*** which substantially correspond to the ***equatorial spins***. Despite of its quantum-like form, the model will violate the local observability principle allowing the existence of ***ghosts***, namely different transformation indistinguishable by local tests. Subsequently the will be defined the ***spin-factors***, a sort of n-dimensional generalisation of the 2-dimensional clock. In the last chapter the classical mechanics will be immersed in the probabilistic theories framework regarding it as the special case having a simplex as convex set of states and describing the ***trit*** system, namely a not trivial generalisation of the well known ***bit***.

In **Part III**

Introduzione

Part I

Mathematical and theoretical instruments

affine set
affine combination
affine dimension

Chapter 1

Convex sets

In this chapter we briefly report the main definitions about convex sets which will be used in the thesis. Finally we will investigate the problem of classify the contractions of a convex set. To reach this goal and to achieve an easy geometrical interpretation of the transformation's action, the block representation will be introduced.

1.1 Main definitions

The principal font for this section is [BV04, BV04].

1.1.1 Affine sets

A set $\mathsf{A} \in \mathbb{R}^n$ is an **affine set** if the line through to any two distinct points in A lies in A, namely

$$\forall x_1, x_2 \in \mathsf{A} \text{ and } \forall t \in \mathbb{R} \qquad t x_1 + (1-t) x_2 \in \mathsf{A}. \tag{1.1}$$

Then a set is affine if it contains all the linear combinations of any two points inside it and using induction it can be shown that also any affine combinations of its points is included in A. We refer to a point of the form $t_1 x_1 + \ldots + t_k x_k$, with $t_1 + \ldots + t_k = 1$, as **affine combination** of the points $x_1, \ldots, x_k$. An important feature of affine sets is that they have "no origin". Given an affine set A it can always be expressed as a subspace[1]plus an offset as follows

$$\mathsf{A} = \mathsf{S} + x_0 = \{s + x_0 | \, s \in \mathsf{S}\}. \tag{1.2}$$

Obviously the subspace S doesn't depend on the particular x_0 which can be arbitrarily chosen in A. We define the **affine dimension** of an affine set A as the dimension of the subspace $\mathsf{S} = \mathsf{A} - x_0$ with x_0 any element of A[2].

[1] A subspace of an affine set is a subset closed under multiplication by scalars an sum.

[2]Notice that in general the affine dimension of a set doesn't coincide with other definitions of dimension. Think about the circle in $\mathbb{R}^2$ whose affine dimension is two and not one as usual.

affine hull
convex set
convex combination
convex hull
cone
convex cone
conic combination

Affine hull. The set of all affine combinations of points in some set $\mathsf{A} \in \mathbb{R}^n$ is called the **affine hull** of A:

$$\mathsf{Aff}(\mathsf{A}) = \{t_1 x_1 + \ldots + t_k x_k \,|\, x_1, \ldots, x_k \in \mathsf{A},\ t_1 + \ldots + t_k = 1\}. \tag{1.3}$$

Notice that $\mathsf{Aff}(\mathsf{A})$ is the smallest affine set containing A

1.1.2 Convex sets

A set C is a **convex set** if the line segment between any two poins in C lies in C, namely

$$\forall x_1, x_2 \in \mathsf{C} \ \text{ and } \ \forall 0 \leqslant t \leqslant 1 \qquad t x_1 + (1-t) x_2 \in \mathsf{C}. \tag{1.4}$$

Then every point in a convex set can be seen by every other point along a straight path lying in the set. Naturally

$$\textit{affine set} \ \Rightarrow \ \textit{convex set}. \tag{1.5}$$

A point of the form $t_1 x_1 + \ldots + t_k x_k$, with $t_1 + \ldots + t_k = 1$ and $t_i \geqslant 0$, $i = 1, \ldots, k$, is called **convex combination** of the points $x_1, \ldots, x_k$. A set is convex if and only if it contains all the convex combinations of its points.

Convex hull. Given a set C its **convex hull** is the set of all convex combinations of points in C:

$$\mathsf{Co}(\mathsf{C}) = \{t_1 x_1 + \ldots + t_k x_k \,|\, x_i \in \mathsf{C},\ t_i \geqslant 0,\ i = 1, \ldots, k,\ t_1 + \ldots + t_k = 1\}. \tag{1.6}$$

The convex hull of a set C is the smallest convex set containing C. The idea of convex combination can be generalized to include infinite sums, integrals, and, in the most general form, probability distributions. In the last case let $\mathsf{C} \in \mathbb{R}^n$ be a convex set and $x \in \mathsf{C}$ a random vector with probability one then we get $\mathbf{E}x \in \mathsf{C}$. If a probability density $p(x)$ exist we can write

$$\int_{\mathsf{C}} p(x)\, dx = 1 \qquad \mathbf{E}x = \int_{\mathsf{C}} p(x) x \, dx \in \mathsf{C}. \tag{1.7}$$

Observation 1.1 *The idea of convex combination is tightly connected to a probabilistic framework because it can be regarded as a* weighted average *of points. Intuitively a theory of probabilistic nature would be structured on convex sets.*

1.1.3 Cones

A set C is called a **cone**, or *non negative homogeneous*, if

$$t x \in \mathsf{C} \qquad \forall x \in \mathsf{C},\ \forall t \geqslant 0. \tag{1.8}$$

In particular a set C is a **convex cone** if it is convex and a cone, namely

$$\forall x_1, x_2 \in \mathsf{C} \ \text{ and } \ t_1, t_2 \geqslant 0 \qquad t_1 x_1 + t_2 x_2 \in \mathsf{C}. \tag{1.9}$$

A point of the form $t_1 x_1 + \ldots t_k x_k$, with $t_1, \ldots, t_k \geqslant 0$, is called **conic combination** of the points $x_1, \ldots, x_k$. A set C is a convex cone if and only if it contains all conic combinations of its elements.

conic hull
hyperplane
halfspaces
ball
ellipsoid

Conic hull. The **conic hull** of a set C is the set of all conic combinations of elements in C, namely

$$\mathsf{Co}_+(\mathsf{C}) = \{t_1 x_1 + \ldots + t_k x_k \mid x_i \in \mathsf{C},\ t_i \geqslant 0,\ i = 1, \ldots, k\}. \tag{1.10}$$

As usual the conic hull of a set is the smallest convex cone containing such set.

1.1.4 Some significant convex sets

Hyperplanes. A **hyperplane** is a set of the form

$$\{x \mid a^\tau x = b\} \tag{1.11}$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$. This is an affine set and we can express it in the form

$$\{x \mid a^\tau (x - x_0) = 0\}, \tag{1.12}$$

where x_0 is any point in the hyperplane (*i.e.* , any point such that $a^\tau x_0 = b$).

Halfspaces. A hyperplane cuts $\mathbb{R}^n$ into two halfspaces. A (closed) **halfspaces** is a set given by

$$\{x \mid a^\tau x \leqslant b\}, \tag{1.13}$$

wherw $a \neq 0$. These sets are convex but clearly they are not affine spaces.

Euclidean balls. A **ball** in $\mathbb{R}^n$ is a set having the form

$$\mathsf{B}(x_c, r) = \{x \mid \|x - x_c\|_2 \leqslant r\} = \{x \mid (x - x_c)^\tau (x - x_c) \leqslant r^2\}, \tag{1.14}$$

where $r \geqslant 0$, and $\|\cdot\|_2$ is the usual Euclidean norm. x_c is the center of the ball while r is its radius. It's easy to prove that an Euclidean ball is a convex set.

Ellipsoids. The Euclidean balls are a particular kind of ellipsoids. An **ellipsoid** in $\mathbb{R}^n$ is a set which has the form

$$\mathsf{E} = \{x \mid (x - x_c)^\tau P^{-1} (x - x_c) \leqslant 1\}, \tag{1.15}$$

where $P = P^\tau$ is a symmetric positive definite matrix. As usual x_c is the center of the ellipsoid. The matrix P determines how far the ellipsoid extends in every directions from x_c; the lengths of the semi-axes of the set E are given by the square roots of the eigenvalues of the matrix P. Ellipsoid are naturally convex sets. The intersection of a 2-dimensional hyperplane with the 3-dimensional norm cone (see the following example) give a conic of the norm cone. The conics are classified according to the discriminant of their representative matrix. There are three kinds of conics whose equation is exactly as in Eq. (1.15) with the sign $\leqslant$ replaced by the equal. These three cases obviously correspond to circular, elliptical and degenerate conics. In the last case the matrix P is singular, while in the circular case we get $P = r^2 I$ where r is the radius of the circle and I is the identical matrix.

norm cone
polyhedron
polytope
simplex
probability simplex
probability vectors

Norm cone. Let $\|\cdot\|$ be any norm in $\mathbb{R}^n$. The **norm cone** associated to the norm $\|\cdot\|$ is the set

$$\mathsf{C} = \{(x,t) \,|\, \|x\| \leqslant t\} \subseteq \mathbb{R}^{n+1}. \tag{1.16}$$

In a nutshell this is the cone in the common sense.

Polyhedra or polytopes. A **polyhedron** or **polytope**[3] is simply given as the solution set of a finite number of linear equalities and inequalities in the following way

$$\mathsf{P} = \{a_j{}^\tau x \leqslant b_j,\ j = 1,\ldots,p,\ c_j{}^\tau x = d_j,\ j = 1,\ldots,k\}. \tag{1.17}$$

Then a polyhedra is the intersection of a finite number of halfspaces and hyperplanes and it is easy to prove that they are convex set.

Simplexes. Simplexes are particular Polyhedra. Taking $k+1$ points $x_0,\ldots,x_k$ in $\mathbb{R}^k$ affinely independent[4] the **simplex** determined by them is

$$\mathsf{C} = \mathsf{Co}\{x_0,\ldots,x_k\} = \{t_0x_0 + \ldots + t_kx_k \,|\, t \succeq 0, \mathbf{1}^\tau t = 1\} \tag{1.18}$$

where the symbol $\succeq$ denotes componentwise inequality[5] and $\mathbf{1}$ is the wector with all entrues one. This simplex is also called k-dimensional simplex in $\mathbb{R}^n$ because it's affine dimension is k. In Chap.6 we will find a probabilistic model concerning a particular simplex denoted **probability simplex**. It is the $(n-1)-$*dimensional* simplex determined by the unit vectors $e_1,\ldots,e_n \in \mathbb{R}^n$ which satisfy

$$x \succeq 0, \qquad \mathbf{1}^\tau x = 1. \tag{1.19}$$

The vectors in this simplex corespond to probability distributions (and then are nominated **probability vectors**) on a set with n elements, with x_i interpreted as the probability of the ith element.

1.1.5 Affine maps

A map $\mathscr{A} : \mathbb{R}^n \to \mathbb{R}^m$ is an *affine* map if it is the sum of a linear map and a constant, *i.e.* it is of the form $\mathscr{A}(x) = Ax + b$, where $A : \mathbb{R}^n \to \mathbb{R}^m$ and $f \in \mathbb{R}^m$. An important property of affine maps is that they preserve convexity. If $\mathsf{C} \subset \mathbb{R}^n$ is a convex set and $\mathscr{A} : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map then the image of C under $\mathscr{A}$

$$\mathscr{A}(\mathsf{C}) = \{\mathscr{A}(x)|\ x \in \mathsf{C}\} \tag{1.20}$$

is still a convex set. Moreover if $A : \mathbb{R}^k \to \mathbb{R}^n$ is affine then the inverse image of C under $\mathscr{A}^{-1}$ is a convex set.

[3] Sometimes one between polytope and polyhedra is reserved for bounded sets but there isn't an universally adopted terminology.

[4] The points $x_0,\ldots,x_k$ are affinely independent if $x_1 - x_0,\ldots,x_k - x_0$ are linearly independent.

[5] Componentwise or vector inequality in $\mathbb{R}^n$:$w \succeq v$ means $w_i \geqslant v_i$ for $i = 1,\ldots,n$.

1.1.6 Perspective map

perspective map

The **perspective map** $\mathscr{P} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a map with domain $\mathsf{dom}\mathscr{P} = \mathbb{R}^n \times \mathbb{R}_{++}$[6] defined as

$$\mathscr{P}(z, t) = z/t. \tag{1.21}$$

Whence the perspective map acts on the vectors in $\mathbb{R}^n \times \mathbb{R}_{++}$ making their last component equal to one (it "normalizes" the vectors) and then it drops this last component achieving a vector in $\mathbb{R}^n$. A graphical interpretation of the perspective map is given by the action of a pin-hole camera in $\mathbb{R}^{n+1}$. A pin-hole camera in $\mathbb{R}^3$, whose representation is given in Fig.1.1, consists of an opaque horizontal plane $x - 3 = 0$, with a single pin-hole at the origin which allows light to pass. Then we get an image in the horizontal plane $x_3 = -1$ as follows: an object at the point x, above the camera, forms an image at the new point $-(x_1/x_3, x_2/x_3, 1)$. Then, dropping the last component of the image point (which is always -1), the image point of x appears at $-(x_1/x_3, x_2/x_3)$, namely at $-\mathscr{P}(x)$.

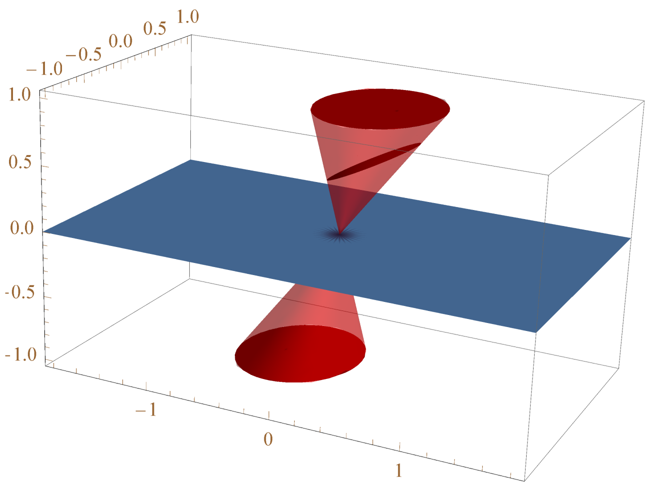


Figure 1.1: The pin-hole camera implementation of the perspective map. The gray opaque plane at $x = 0$ has a single hole in the origin. In Red, Inside the cone of light, we can see a general object (in this case a convex set) above the camera. The red figure in the plane at $x = -1$ is the image of the red object achieved by the "light" passing through the hole. the corespondent red object in the plane at $x = 1$ is the image of the object above the camera under the perspective map.

As for the affine maps also the perspective ones preserve convexity. In fact if a set $\mathsf{C} \in \mathsf{dom}\mathscr{P}$ is convex, then its image under the the perspective map $\mathscr{P}$

$$\mathscr{P}(\mathsf{C}) = \{\mathscr{P}(x) \,|\, x \in \mathsf{c}\} \tag{1.22}$$

[6] We denote by $\mathbb{R}_{++}$ the set of positive numbers

linear fractional map
projective map
dual cone

is still a convex set. The inverse image of a convex set under the perspective map is also convex.

1.1.7 Linear-fractional map

A **linear fractional map** or **projective map** is the composition of an affine map with the perspective map[7]. Consider an affine map $\mathscr{A} : \mathbb{R}^n \to \mathbb{R}^{m+1}$ and represent this map as follows

$$\mathscr{A}(x) = \begin{bmatrix} \bar{A} \\ a^\tau \end{bmatrix} x + \begin{bmatrix} k \\ q \end{bmatrix}, \tag{1.23}$$

where $\bar{A} \in \mathbb{R}^{m \times n}$, $k \in \mathbb{R}^m$, $a \in \mathbb{R}^n$ and $q \in \mathbb{R}$. Then the map $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$\mathscr{F} = \mathscr{P} \circ \mathscr{A}, \tag{1.24}$$

is called a linear-fractional map in fact it is expressed as

$$\mathscr{F}(x) = \frac{(\bar{A}x + k)}{(a^\tau x + q)}, \qquad \mathsf{dom}\mathscr{F} = \{x \,|\, a^\tau x + q > 0\}. \tag{1.25}$$

Notice that $\mathscr{F}$ is an affine function if $a = 0$ and $d > 0$, in such case $\mathsf{dom}\mathscr{F}$ is $\mathbb{R}^n$.

Projective interpretation A very useful interpretation of the map is the projective one. It always possible to represent a linear-fractional map as a matrix

$$A = \left(\begin{array}{c|c} \bar{A} & k \\ \hline a^\tau & q \end{array} \right), \tag{1.26}$$

which acts on points of form $(x, 1)$ returning $(\bar{A}x + k, a^\tau x + q)$. then by scaling or normalizing so that the last component is one, we yield the points $(\mathscr{F}x, 1)$. The matrix A is obviously the matrix representing the affine map in Eq. (1.23).

1.1.8 Dual cones

Given a cone C, its **dual cone**, denoted C^*, is defined as the set

$$\mathsf{C}^* = \{y \,|\, x^\tau y \geqslant 0, \ \forall x \in \mathsf{C}\}. \tag{1.27}$$

[7]This map is very important for our future purposes, in fact in a probabilistic theory (see Sec.2.5) the conditioning after a transformation will be easily described by a map like this one.

generator set

Notice that C^*, which is a cone too, is always a convex cone even when the original cone is not. A suggestive geometrical view of a dual cone involves the idea of hyperplane; in fact y is an element of C^* if and only if $-y$ is the normal of a hyperplane that supports[8] C at the origin.

1.2 Contractions classification for a real n-dimensional convex set

In this section we give some general definitions and expose the problem of classify all the contraction for a n-dimensional convex set. The reason for optimizing the construction of such transformations by a general method will be clarified in Part (II) in which we will develop some probabilistic models (see Chap. (2) for the definition of probabilistic theory), including the quantum one, whose geometrical representation is of convex type. Naturally a probabilistic model for a system allows the evolution of the system itself and this feature is achieved by "transformations". Our goal is to classify all the transformations compatible with the underline geometrical structure. In fact we will show in Chap.2 that the admissible transformations for a probabilistic model are the contractions of its convex set of states[9].

1.2.1 Truncated dual cone

Given a n-dimensional convex set C, denote by $\mathsf{Aut(C)}$ the group of automorphism of C and by $\mathsf{G_C}$ the set of generators of C. The **generator set** $\mathsf{G_C}$ is defined to be the set whose orbit under the group action of $\mathsf{Aut(C)}$ is Extr(C). The group action is denoted as usual by

$$\text{Extr}(\mathsf{C}) = \mathsf{Aut(C)}\,\mathsf{G_C}. \tag{1.28}$$

$\mathsf{G_C}$ is said to be the generator of C under its group of automorphism because

$$\mathsf{C} = \mathsf{Co}(\mathsf{Aut(C)}\,\mathsf{G_C}) \tag{1.29}$$

Notice that the elements in $\mathsf{Aut(C)}\,\mathsf{G_C}$ are the vertexes of C[10]. We can always consider the $(n+1)$-dimensional convex set given by

$$\mathsf{C}_+ = \{\alpha\omega|\ \alpha \geqslant 0,\ \omega \in \mathsf{C}\}\,; \tag{1.30}$$

this set is the cone based on C or, which is the same, the conic hull of the elements in $\mathsf{Aut(C)}\,\mathsf{G_C}$

$$\mathsf{C}_+ = \mathsf{Co}_+(\mathsf{Aut(C)} \circ \mathsf{G}_{\mathsf{C}^*}) \tag{1.31}$$

[8] We haven't defined rigorously the concept of supporting hyperplane. The intuitive idea is that a hyperplane support a set C if such set is completely included in one of the two halfspaces generated by the hyperplane.

[9] In a probabilistic theory composition of systems is take into account and the set of physical transformations is in general strictly included into the set of contractions for the single convex set of states in order to preserve the structure of the composed system.

[10] The set $\mathsf{G_C}$ can be both a finite or an infinite set and, in the last case, it can be both a countable or uncountable set.

truncated dual

. Denoting by $\mathcal{L}(\mathsf{C}_+)$ the set of all the linear functional over C_+, we can define the dual cone of C_+ as[11]

$$\mathsf{C}_+^* = \{a \in \mathcal{L}(\mathsf{C}_+) | \, a(\omega) \geqslant 0 \; \forall \omega \in \mathsf{C}\}. \tag{1.32}$$

We can introduce a partial order relations on C_+^* as follows

$$a, b \in \mathsf{C}_+^* \qquad a \leqslant b \;\Leftrightarrow\; \omega(a) \leqslant \omega(b) \quad \forall \omega \in \mathsf{C}. \tag{1.33}$$

Via this partial order relation it's possible to define a new convex set which is the **truncated dual** cone C^* given by

$$\mathsf{C}^* = \{a \in \mathsf{C}_+^* | \, a \leqslant e\}. \tag{1.34}$$

Here e is the element of the dual cone C_+^* satisfying the property

$$\omega(e) = 1 \quad \forall \omega \in \mathsf{C}. \tag{1.35}$$

Observation 1.2 *The automorphism of* C *are also automorphism of the cone* C_+[12]. *Moreover, being the bases of* C_+ *and* C_+^* *the same,* $\mathsf{Aut}(\mathsf{C})$ *are automorphism of the dual cone too. The set* $\mathsf{Aut}(\mathsf{C}^*)$ *in general strictly include* $\mathsf{Aut}(\mathsf{C})$*. We can denote by* G_{C^*} *the set of generators of* C^* *defined as follows*

$$\mathrm{Extr}\, \mathsf{C}^* = \mathsf{Aut}(\mathsf{C})\, \mathsf{G}_{C^*}. \tag{1.36}$$

and naturally

$$\mathsf{C}^* = \mathsf{Co}(\mathsf{Aut}(\mathsf{C})\, \mathsf{G}_{C^*}). \tag{1.37}$$

Observation 1.3 *Notice that all the elements in a cone can be obtained by multiplying an element in its base by a positive scalar. For this reason Eq. (1.32) and (1.33) are respectively equivalent to*

$$\mathsf{C}_+^* = \{a \in \mathcal{L}(\mathsf{C}_+) | \, a(\omega) \geqslant 0 \; \forall \omega \in \mathsf{C}_+\}, \tag{1.38}$$

$$a, b \in \mathsf{C}_+^* \qquad a \leqslant b \;\Leftrightarrow\; \omega(a) \leqslant \omega(b) \quad \forall \omega \in \mathsf{C}_+. \tag{1.39}$$

Because of the convex nature of C*, Eqs. (1.32) and (1.33) are even equivalent to the following two*

$$\mathsf{C}_+^* = \{a \in \mathcal{L}(\mathsf{C}_+) | \, a(\omega) \geqslant 0 \; \forall \omega \in \mathsf{Aut}(\mathsf{C})\, \mathsf{G}_\mathsf{C}\}, \tag{1.40}$$

$$a, b \in \mathsf{C}_+^* \qquad a \leqslant b \;\Leftrightarrow\; \omega(a) \leqslant \omega(b) \quad \forall \omega \in \mathsf{Aut}(\mathsf{C})\, \mathsf{G}_\mathsf{C}; \tag{1.41}$$

In the same way also the definition of e is equivalent to

$$\omega(e) = 1 \quad \forall \omega \in \mathsf{Aut}(\mathsf{C})\, \mathsf{G}_\mathsf{C}. \tag{1.42}$$

[11] In the following we will use the same letter to denote elements in a convex or conic hull of a given set. This is the same to say that we will use the same letters for elements in a convex set and in the cone based on it.

[12] see Ssec.1.2.2 for the definition of cone automorphism

cone preserving transformation
positive transformation
cone-automorphism
cone-isomorphism

Observation 1.4 *Notice that regarding* C^* *as a set of functional over* C *we get*

$$a(\omega_1) = a(\omega_2) \quad \forall a \in \mathsf{C}^* \qquad \Leftrightarrow \qquad \omega_1 = \omega_2, \tag{1.43}$$

which means that C^* *separate the elements of* C*. Vice versa* C *separate the elements of* C^**, in fact regarding* C *as a set of functionals over* C^* *we have*

$$\omega(a_1) = \omega(a_2) \quad \forall \omega \in \mathsf{C} \qquad \Leftrightarrow \qquad a_1 = a_2. \tag{1.44}$$

In conclusion C *and* C^* *are said to separate each other.*

1.2.2 Cone preserving transformations

A transformations $\mathscr{A}$ over the convex set C is said to be a linear **cone preserving transformation**, or shortly **positive transformation**, if it is a linear on C_+ and

$$\mathscr{A}\omega \in \mathsf{C}_+ \qquad \forall \omega \in \mathsf{C}_+. \tag{1.45}$$

As usual this condition is equivalent to

$$\mathscr{A}\omega \in \mathsf{C}_+ \qquad \forall \omega \in \mathsf{C} \tag{1.46}$$

or, which is the same,

$$\mathscr{A}\omega \in \mathsf{C}_+ \qquad \forall \omega \in \mathsf{Aut}(\mathsf{C})\,\mathsf{G}_{\mathsf{C}}. \tag{1.47}$$

These transformations send the cone C_+ into a new convex set (which is still a cone) lying inside itself. Eq.(1.46) explicitly define the positive transformations of a convex set C as the linear transformations of C_+ sending all the elements in C in the cone C_+. In this way we achieve a set of positive transformations of C more general than the positive linear ones, the set we achieve is the set of all positive affine transformations of C (see the block representation of transformations in the following subsection). In the following we will denote by $\mathfrak{L}_+(\mathsf{C})$[13] the set of positive linear transformation of the cone C_+.Naturally $\mathfrak{L}_+(\mathsf{C})$ is a cone too. There are two special kind of positive transformations, the automorphisms and the contractions.

Automorphisms. A transformation $\mathscr{A}$ in $\mathfrak{L}_+(\mathsf{C})$ is said to be a **cone-automorphism** if it is a **cone-isomorphism**[14] sending the cone C_+ onto the same cone. Obviously a C_+ automorphism is a C automorphism and vice versa. We have already denoted the set

[13]The symbol $\mathfrak{L}$ here stays for "local". The reason for this notation is that in the following chapters we will be interested in transformations which preserve not only the local convex cone but also the bipartite one (concepts such as local and bipartite system are subject of Chap. 2). These transformations are usually known as completely positive maps and we reserve the symbol $\mathfrak{T}_+(\mathsf{C})$ for them. Obviously $\mathfrak{T}$ stays for "transformations".(see also footnote 9).

[14]We say that two cones C_+^1 and C_+^2 are isomorphic (denoted as $\mathsf{C}_+^1 \simeq \mathsf{C}_+^1$), if there exists a one-to-one linear map between $Span_{\mathbb{R}}(\mathsf{C}_+^1)$ and $Span_{\mathbb{R}}(\mathsf{C}_+^2)$ that is cone-preserving. We will call such map the isomorphism between the cones. It send extremal rays of C_+^1 to extremal rays of C_+^2, and positive linear combinations to positive linear combinations.

contraction

of automorphism as $\mathsf{Aut}(\mathsf{C}) = \mathsf{Aut}(\mathsf{C}_+)$[15]. Notice that it is also $\mathsf{Aut}(\mathsf{C}) = \mathsf{Aut}(\mathsf{C}_+^*)$ and all its elements, according to *observation 1.2* , are also automorphism for the truncated cone C^*, while in general $\mathsf{Aut}(\mathsf{C}^*) \supseteq \mathsf{Aut}(\mathsf{C})$. On the other hand, as will be clarified in the following, the only automorphism in $\mathsf{Aut}(\mathsf{C})$ are of some practical utility in contractions classification.

Contractions. Denoting by $\widehat{\mathsf{C}}_+$ the truncation of C_+ given by

$$\widehat{\mathsf{C}}_+ = \{\omega \in \mathsf{C}_+ | \omega(e) \leqslant 1\}, \tag{1.48}$$

a transformation $\mathscr{A}$ in $\mathfrak{L}_+(\mathsf{C})$ is said to be a **contraction** if

$$\mathscr{A}\omega \in \widehat{\mathsf{C}}_+ \qquad \forall \omega \in \mathsf{C} \tag{1.49}$$

and we will denote by $\widehat{\mathfrak{L}}_+(\mathsf{C})$ the set of all C-contractions. Naturally the automorphism are also special kind of contractions because they send the convex set C into itself and then in the truncated cone $\widehat{\mathsf{C}}_+$.

1.2.3 Block representation for affine transformations of a convex set

In this subsection we introduce an affine-space representation for the contractions of a given convex set. Considering an $(n-1)$-dimensional convex set C we get $\dim \mathsf{C}_+ = \dim \mathsf{C}_+^* = \dim \mathsf{C}^* = n$. It's always possible to find in C^* a set $\{l_i\}$ separating for C. We say that a set of functionals over C is separating for it if[16]

$$l_i(\omega_1) = l_i(\omega_2)\ \forall l_i \in \{l_i\} \qquad \Rightarrow \qquad \omega_1 = \omega_2, \tag{1.50}$$

namely the set of values $\{l_i(\omega)\}$ uniquely identify ω. It is also possible to find a minimal separating set $\{l_i\}_{i=1,\ldots,n}$[17]. At the same time we can take a minimal set $\{\lambda_i\}_{i=1,\ldots,n}$ in C separating for C^*. In terms of these two minimal sets one can expand (in a unique way) any $a \in \mathsf{C}^*$ and any $\omega \in \mathsf{C}$ as follows

$$a = \sum_{j=1}^{n} \lambda_j(a) l_j, \qquad \omega = \sum_{j=1}^{n} l_j(\omega)\lambda_j. \tag{1.51}$$

Instead of using the minimal separating sets $\{l_i\}_{i=1,\ldots,n}$ and $\{\lambda_i\}_{i=1,\ldots,n}$ it will be convenient to adopt canonical basis for $\mathsf{C}_{\mathbb{R}}^*$ and $\mathsf{C}_{\mathbb{R}}$ embedded in into the euclidean space $\mathbb{R}^n$. In order to keep the notation we will again denote these orthonormal basis as $\{l_i\}_{i=1,\ldots,n}$ and $\{\lambda_i\}_{i=1,\ldots,n}$, with[18]

$$(l_i, \lambda_j) = l_i(\lambda_j) = \lambda_j(l_i) = \delta_{ij} \tag{1.52}$$

[15]This set can be both a finite or an infinite set and, in the last case, it can be both a countable or uncountable set.

[16]Obviously a set $\{l_i\}$ such that all elements in C^* can be written as a linear combination of l_i is a separating set according to *observation1.4*

[17]The number of elements in a minimal separating set for C is $(n+1)$ if C is $(n-1)$-dimensional, indeed in order to separate C, the set $\{l_i\}$ must give the same linear span of the whole set C^*. But a base for $\mathsf{Span}_{\mathbb{R}}(\mathsf{C}^*) = \mathsf{C}_{\mathbb{R}}$ has at least (n) elements.

[18]Naturally the couple (l_i, l_j) represent the ordinary scalar product in $\mathbb{R}^n$.

Bloch vector

In the following the symbols l and λ will stay for the whole sets[19] $\{l_i\}$ and $\{\lambda_i\}$. Moreover, it turns out to be convenient to choose a minimal separating set $l = \{l_i\}$ with $l_n = e$. Correspondingly λ_n became the element of C which act as a functional over an $a \in \mathsf{C}^*$ extracting its component along e. Using a Minkowskian notation we write

$$l \doteq (\boldsymbol{l}, e),\ \lambda \doteq (\boldsymbol{\lambda}, \chi), \qquad \lambda \cdot l \doteq \sum_j \lambda_j l_j = \boldsymbol{\lambda} \cdot \boldsymbol{l} + \chi e, \tag{1.53}$$

and for any $a \in \mathsf{C}^*$ and any $\omega \in \mathsf{C}$ we write

$$(a, \omega) = \omega(a) = a(\omega) = l(\omega) \cdot \lambda(a) := \sum_{i=1}^{N} l_i(\omega)\lambda_i(a) \equiv \boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a)e(\omega). \tag{1.54}$$

The vectors $l(\omega)$ and $\lambda(a)$ give a representation of convex sets C_+ and C_+^*. Such representation is *faithful* if l and λ are minimal separating sets. In particular the representation given by orthonormal basis is certainly *faithful*.
In the following we will consider a faithful representation induced by two minimal separating sets l and λ with $l_n = e$. Then the functional relations between the two cones are given by an ordinary scalar product in a space isomorphic to $\mathbb{R}^n$. In particular the cones-duality relations in Eq. (1.38) becomes

$$\mathsf{C}_+^* = \{\lambda(a) |\ \lambda(a) \cdot l(\omega) \geqslant 0\ \forall\ l(\omega) \in \mathsf{C}_+\}. \tag{1.55}$$

For the elements in C we have $\omega(e) = 1$ and we say that they are normalized because their representative last component $l_n(\omega)$ is equal to one. Given a normalized ω, the vector $l(\omega)$ is denoted **Bloch vector** of ω. If $\mathscr{A}$ is a linear positive map on C_+ then we can achieve its Bloch representative by the relation

$$l_i(\mathscr{A}\omega) = \sum_{k=1,\dots,n} A_{ik} l_k(\omega) \ \Rightarrow\ l(\mathscr{A}\omega) = A l(\omega). \tag{1.56}$$

From the positivity of $\mathscr{A}$ we get

$$e(\mathscr{A}\omega) \geqslant 0 \qquad \forall \omega \in \mathsf{C}_+ \tag{1.57}$$

namely, from Eq. (1.56),

$$e(\mathscr{A}\omega) = \sum_{k=1,\dots,n} A_{nk} l_k(\omega) \geqslant 0 \qquad \forall \omega \in \mathsf{C}_+. \tag{1.58}$$

The product of the vector $l(\omega)$ representing $\omega \in \mathsf{C}_+$ with the last row of the matrix representing $\mathscr{A}$ must be positive for every ω. Eq. (1.55) says us that there exist an $a \in \mathsf{C}_+^*$ such that the last row of a linear positive transformations in Block representation is $\lambda(a)$. In particular if $\mathscr{A}$ is a contraction we get

$$0 \leqslant e(\mathscr{A}\omega) \leqslant 1 \ \Rightarrow\ 0 \leqslant \sum_{k=1,\dots,n} A_{nk} l_k(\omega) \leqslant 1 \qquad \forall \omega \in \mathsf{C}_+ \tag{1.59}$$

[19] The introduction of the canonical basis $\{l_i\}$ and $\{\lambda_i\}$ allows the representation of the elements in the convex sets previously defined as vectors in $\mathbb{R}^n$. In particular each element of the canonical basis (l_i or λ_i) is a vector in $\mathbb{R}^{n+1}$. The symbols l and λ stay for the vectors having respectively as components the vectors l_i and λ_i.

and a will be an element of C^*. In both cases,

$$\lambda_i(a) = A_{ni} \tag{1.60}$$

and

$$e(\mathscr{A}\omega) = \sum_{k=1,\dots,n} A_{nk} l_k(\omega) = \lambda(a)\cdot l(\omega) = \boldsymbol{\lambda}(a)\cdot \boldsymbol{l}(\omega) + \chi(a)e(\omega). \tag{1.61}$$

We will denote by $[a]$ the equivalence class of positive transformations $\mathscr{A}$ whose Bloch representation has $\lambda(a)$ as last row. According to the last observations we get In conclusion form the two relations

$$\begin{aligned} &\boldsymbol{l}(\mathscr{A}\omega) = \bar{\boldsymbol{A}}\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A}) \qquad \boldsymbol{k}(\mathscr{A}) = \boldsymbol{l}(\mathscr{A}\chi), \\ &e(\mathscr{A}\omega) = \boldsymbol{\lambda}(a)\cdot \boldsymbol{l}(\omega) + \chi(a)e(\omega). \end{aligned} \tag{1.62}$$

The Bloch form of a linear positive transformation result as in Fig.1.2.3.

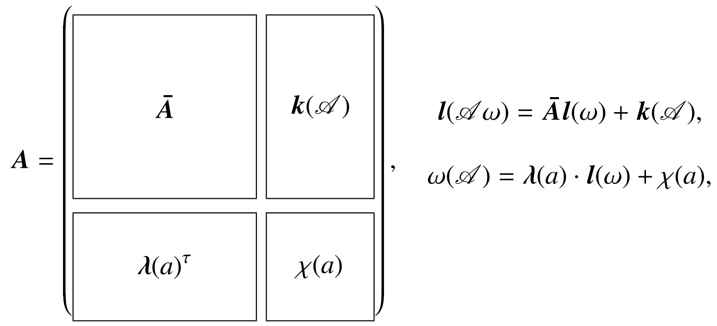


Figure 1.2: Matrix representation of a linear positive transformation $\mathcal{A}$ for a convex cone C_+. The last row represents the an element a in the convex dual cone C_+^* . It gives the transformation of the n-component of the Bloch vector $e(\mathscr{A}\omega) \equiv \omega(\mathscr{A}) = \boldsymbol{\lambda}(a)\cdot \boldsymbol{l}(\omega) + \chi(a)$, namely the transformation of the component giving the normalization of the vector $l(\omega)$. The matrix $\bar{\boldsymbol{A}}$ represent the linear component of the affine transformation of the Bloch vector $\boldsymbol{l}(\omega)$ corresponding to the operation of $\mathscr{A}$, while the column $\boldsymbol{k}(\mathscr{A})$ give the constant translation.

1.2.4 Block representation consequence.

Easy geometrical representation

In block representation the convex structures in exam acquires an interesting pictorial view. We have embedded the cones C_+ and C_+^* in the euclidean space $\mathbb{R}^{n+1}$ and it's possible to embedded the two cones in the same space. Then the functional relations between the two cones era given by an ordinary scalar product in $\mathbb{R}^{n+1}$. Now the the two separating set l and λ are the same canonical base in $\mathbb{R}^{n+1}$. It is always possible to embed a convex set C having dimension n in the Euclidean space $\mathbb{R}^{n+1}$ putting it in the

hyperplane
linear fractional map
perspective map

hyperplane at $x_{n+1} = 1$. Then every ω in C, whose representative in $\mathbb{R}^n$ is $\boldsymbol{l}(\omega)$, get the form

$$l(\omega) = \begin{bmatrix} \boldsymbol{l}(\omega) \\ 1 \end{bmatrix}, \tag{1.63}$$

where $\boldsymbol{l}(\omega) \in \mathbb{R}^n$. The cone C_+ based on C is now the set of open rays in $\mathbb{R}^{n+1}$ given by

$$\mathsf{C}_+ = \{t(\boldsymbol{l}(\omega), 1) \,|\, t > 0,\ \omega \in \mathsf{C}\} \tag{1.64}$$

where the last component of the rays takes positive values. Notice that the cone C_+ is a subspace of the cone

$$\{t(\boldsymbol{v}, 1) \,|\, t > 0,\ \boldsymbol{v} \in \mathbb{R}^n\}. \tag{1.65}$$

The element $e \in \mathsf{C}^*$ in this representation is simply the canonical vector l_{n+1}, in fact

$$\omega(e) = \boldsymbol{l}(\omega) \cdot \lambda(e) + \lambda_{n+1}(e) = 1\ \forall \omega \in \mathsf{C} \Rightarrow e = l_{n+1}, \tag{1.66}$$

namely

$$\lambda(e) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}. \tag{1.67}$$

It is easy to verify that the dual cone C_+^*

$$\mathsf{C}_+^* = \{\lambda(a) \in \mathbb{R}^{n+1} \,|\, \lambda(a) \cdot l(\omega) \geqslant 0\ \forall\ l(\omega) \in \mathsf{C}_+\}, \tag{1.68}$$

in this representation is

$$c_+^* = \{t(\boldsymbol{\lambda}(a), 1) \,|\, t > 0,\ \boldsymbol{\lambda}(a)\boldsymbol{l}(\omega) + 1 \leqslant 0,\ \forall \omega \in \mathsf{C}\}. \tag{1.69}$$

Naturally we can assume the Bloch representation for the positive transformations of the cone C_+. We know that the representative matrix of the linear positive transformation is as in Fig.1.2.3 which is exactly the shape of the affine component of a linear-fractional map in Eq. (1.23). In fact the projective representation of a **linear fractional map** describes correctly the action of a positive linear map on the convex C. Consider a map $\mathscr{A} \in \mathfrak{L}_+(\mathsf{C})$ and its block form $\boldsymbol{A}$. It send an element $\omega \in \mathsf{C}$ in the point

$$l(\omega) \to l(\mathscr{A}\omega) = \boldsymbol{A}l(\omega) = \left(\bar{\boldsymbol{A}}\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A})\,,\ \boldsymbol{\lambda}(a)^\tau \boldsymbol{l}(\omega) + \chi(a)\right) \tag{1.70}$$

which is still in C_+ because $\mathscr{A} \in [a]$ with $a \in \mathsf{C}^*$. Then by scaling or renormalizing so that the last component is one, we yield the point

$$\left(\frac{\bar{\boldsymbol{A}}\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{\lambda}(a)^\tau \boldsymbol{l}(\omega) + \chi(a)}\,,\ 1\right) = \left(\mathscr{P} \circ \mathscr{A}\, l(\omega)\,,\ 1\right), \tag{1.71}$$

where $\mathscr{P}$ is the **perspective map**. In conclusion if want to achieve the normalized image vector of $\omega \in \mathsf{C}$ under the action of a linear positive map $\mathscr{A}$ we get the following result

$$\begin{aligned} l(\omega) \to l(\omega_{\mathscr{A}}) &= \mathscr{P} \circ \mathscr{A}\, l(\omega) \\ \boldsymbol{l}(\omega_{\mathscr{A}}) &= \frac{\bar{\boldsymbol{A}}\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{\lambda}(a)^\tau \boldsymbol{l}(\omega) + \chi(a)}. \end{aligned} \tag{1.72}$$

Naturally if $\mathscr{A}$ is a contraction $\mathscr{A} \in [a]$ with $a \in \mathsf{C}^*$ and the image of each element in C under this map will lay in the truncation of C_+ (see Eq. (1.48)) given by

$$\widehat{\mathsf{C}_+} = \{t(\boldsymbol{l}(\omega), 1) \,|\, 0 < t < 1,\ \omega \in \mathsf{C}\} \tag{1.73}$$

. An example of the action of a contraction on a convex set C is given in Fig.

Extremal contractions

The problem of classify the contractions of a convex set obviously reduces to the problem of identify the extremal contractions. By convex combinations of elements in $\mathrm{Extr}\,\widehat{\mathfrak{L}}_+(\mathsf{C})$ the whole set $\widehat{\mathfrak{L}}_+(\mathsf{C})$ can be achieved. the following propositions will be useful.

Proposition 1.1 *All the contractions in the convex set $\widehat{\mathfrak{L}}_+(\mathsf{C})$ are represented in block form by a matrix with an element of C^* as last row. In different case they will not be contractions.*

Proof. By definition of block representation.■

Proposition 1.2 *For a convex set C the following relation holds*

$$\textit{if } a \in \mathrm{Extr}(\mathsf{C}^*) \textit{ then } \mathscr{A} \in \mathrm{Extr}(\widehat{\mathfrak{L}}_+(\mathsf{C}))\ \forall \mathscr{A} \in \mathrm{Extr}[a] \tag{1.74}$$

Proof. . Also the proof of this statement is a trivial consequence of block representation. If $\mathscr{A}$ is in the equivalence class $[a]$ then it's block representation is a matrix having the vector $\lambda(a)$ as last row. $a \in \mathrm{Extr}(\mathsf{C}^*)$ and then it couldn't be expressed as any convex combination of elements in C^+. As a consequence of Prop.1.1 there will not exist a set of contractions combining convexely to give $\mathscr{A}$ because the same convex combination of their last rows in block representation would give $\lambda(a)$. The only case in which it is possible is when the convex combination is among elements in the same equivalent class $[a]$ but this is excluded by the hypothesis $\mathscr{A} \in \mathrm{Extr}[a]$.■

Observation 1.5 *At firs sight it seem that all the contractions in $\widehat{\mathfrak{L}}_+(\mathsf{C})$ could be derived from the extremal contractions in the equivalence classes $[a]$ for each $a \in \mathsf{G}_{\mathsf{C}^*}$. We can symbolically denote these equivalence classes by $[\mathsf{G}_{\mathsf{C}^*}]$. We know that the set $\mathrm{Extr}(\mathsf{C}^*)$ is achieved as $\mathsf{Aut}(\mathsf{C})\,\mathsf{G}_{\mathsf{C}^*}$. The automorphism in the group $\mathsf{Aut}(\mathsf{C})$ are extremal contractions in the equivalence class $[e]$ which obviously is included in $[\mathsf{G}_{\mathsf{C}^*}]$. Whence we need only the extremal contractions $\mathrm{Extr}[\mathsf{G}_{\mathsf{C}^*}]$ in order to generate the whole set $\mathrm{Extr}[\mathrm{Extr}(\mathsf{C}^*)]$. This only shows that all the extremal contractions in the equivalence classes of the extremal elements of C^* are achievable from the once in the equivalence classes of the generators of C^*. Moreover, from Prop.1.2, all these transformations are in $\widehat{\mathfrak{L}}_+(\mathsf{C})$. Although it seem very intuitive, there is no garantie that the contractions in $\mathrm{Extr}[\mathrm{Extr}(\mathsf{C}^*)]$ are the only extremals of the convex set $\widehat{\mathfrak{L}}_+(\mathsf{C})$. In general this is false and we get a simple example from the case in which C is an n-dimensional ball. We will investigate in depth this situation in Part II in the context of a particular class of probabilistic models called "Spin-factors".*

Proposition 1.3 *Given a convex set* C *we the following relation holds*

$$\text{Extr}[\text{Extr}(\mathsf{C}^*)] = \text{Extr}[\text{Extr}(\mathsf{Aut}(\mathsf{C})\,\mathsf{G}_{C^*})] = \mathsf{Aut}(\mathsf{C})\,\text{Extr}(\mathsf{G}_{C^*}). \tag{1.75}$$

Proof.

tests
events
outcomes

Chapter 2

Probabilistic theories

As pointed out in the introduction to this thesis the original part of this work concerns the development of some concrete probabilistic theories in order to investigate their properties and compare them with the particular features of quantum mechanics. The general framework of the probabilistic theories is defined in the original article [D'A08] whose title is:"***Probabilistic theories: What is special about quantum mechanics***". Here the main contents of such work are reported in order to make possible the understanding of the following chapter. The reader will be advised if some result is not already included in the original article.

2.1 C*-algebra representation of probabilistic theories

2.1.1 Tests and states

A probabilistic operational framework is a collection of **tests**[1] $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ each being a complete collection $\mathbb{A} = \{\mathscr{A}_i\}$, $\mathbb{B} = \{\mathscr{B}_j\}$, $\mathbb{C} = \{\mathscr{C}_k\}$, ... of mutually exclusive **events** $\mathscr{A}_i, \mathscr{B}_j, \mathscr{C}_k, \ldots$ occurring probabilistically[2]; events that are mutually exclusive are often called **outcomes**. Naturally the set of events in a test is said complete because if the test

[1] The present notion of test corresponds to that of **experiment** of Ref. [D'A07b]. Quoted from the same reference: "An experiment on an object system consists in making it interact with an apparatus, which will produce one of a set of possible events, each one occurring with some probability. The probabilistic setting is dictated by the need of experimenting with partial *a priori* knowledge about the system (and the apparatus). In the logic of performing experiments to predict results of forthcoming experiments in similar preparations, the information gathered in an experiment will concern whatever kind of information is needed to make predictions, and this, by definition is the *state* of the object system at the beginning of the experiment. Such information is gained from the knowledge of which transformation occurred, which is the "outcome" signaled by the apparatus."

[2] Also A. Rényi [R0́7] calls our test "experiment". More precisely, he defines an experiment $\mathbb{A}$ as the pair $\mathbb{A} = (\mathfrak{X}, \mathcal{A})$ made of the *basic space* $\mathfrak{X}$—the collection of outcomes—and of the σ-algebra of events $\mathcal{A}$. Here, to decrease the mathematical load of the framework, we conveniently identify the experiment with the basic space only, and consider a different σ-algebra (*e.g.* a coarse graining) as a new test made of new mutually exclusive events. Indeed, since we are considering only discrete basic spaces, we can put basic space and σ-algebra in one-to-one correspondence, by taking $\mathcal{A} = 2^{\mathfrak{X}}$—the power set of $\mathfrak{X}$—and, viceversa, $\mathfrak{X}$ as the collection of the minimal intersections of elements of $\mathcal{A}$.

singleton test
channel
deterministic
union
refinement
coarse-graining
state
state-preparations
cascade
composite event

is performed the resultant outcome must be included in the set of events which define the test. On the other hand the same event can occur in different tests, with occurrence probability independent on the test[3]. A **singleton test**—also called a **channel**—$\mathbb{D} = \{\mathscr{D}\}$ is **deterministic**: it represents a non-test, *i.e.* a free evolution. In these tests there is only a possible event which occur with probability one, that's the reason for nominate the correspondent event a deterministic event. The **union** $\mathscr{A} \cup \mathscr{B}$ of two events corresponds to the event in which either $\mathscr{A}$ or $\mathscr{B}$ occurred, but it is unknown which one. A **refinement** of an event $\mathscr{A}$ is a set of events $\{\mathscr{A}_i\}$ occurring in some test such that $\mathscr{A} = \cup_i \mathscr{A}_i$. The experiment itself $\mathbb{A}$ can be regarded as the deterministic event corresponding to the complete union of its outcomes, and when regarded as an event it will be denoted by the different notation $\mathscr{D}_{\mathbb{A}}$. The opposite event of $\mathscr{A}$ in $\mathbb{A}$ will be denoted as $\overline{\mathscr{A}} := \complement_{\mathbb{A}}\mathscr{A}$.[4] The union of events transforms a test $\mathbb{A}$ into a new test $\mathbb{A}'$ which is a **coarse-graining** of $\mathbb{A}$, *e.g.* $\mathbb{A} = \{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\}$ and $\mathbb{A}' = \{\mathscr{A}_1, \mathscr{A}_2 \cup \mathscr{A}_3\}$. Vice-versa, we will call $\mathbb{A}$ a **refinement** of $\mathbb{A}'$.

A very operational definition of state is connected to the probability rule for any possible event in the system. Thus the **state** ω describing the preparation of the system is the probability rule $\omega(\mathscr{A})$ for any event $\mathscr{A} \in \mathbb{A}$ occurring in any possible test $\mathbb{A}$.[5]As previously noticed any test can be regarded as a deterministic test, in fact, for each test $\mathbb{A}$ and for each probability rule Ω we have the completeness relation

$$\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1 \tag{2.1}$$

States themselves are considered as special tests: the **state-preparations**. In fact one can imagine a test

$$\mathbb{S} = \{\mathscr{S}_i\}, \tag{2.2}$$

whose events $\mathscr{S}_i$ are in one to one correspondence to all the possible states ω_i. The occurrence of an event $\mathscr{S}_i$ means the state reduction $\omega(\mathscr{S}_i)\omega_i$ or, which is the same, the state ω_i is prepared starting from a generic state ω with a probability $\omega(\mathscr{S}_i)$

2.1.2 Cascading, conditioning and transformations

One of the most crucial features of a probabilistic theory is the idea of transformation. Given a particular configuration, which means a particular probability rule for all the possible events, it is reasonable the possibility of evolve in another configuration. At the same time we are interested in the probability of such evolution and we also can think about a multiple evolution. All these situation are well described in the present context.

The **cascade** $\mathbb{B} \circ \mathbb{A}$ of two tests $\mathbb{A} = \{\mathscr{A}_i\}$ and $\mathbb{B} = \{\mathscr{B}_j\}$ is the new test with events $\mathbb{B} \circ \mathbb{A} = \{\mathscr{B}_j \circ \mathscr{A}_i\}$, where $\mathscr{B} \circ \mathscr{A}$ denotes the **composite event** $\mathscr{A}$ "followed by" $\mathscr{B}$.

[3]This means that given an event $\mathscr{A}$ which is both in the tests $\mathbb{A}$ and $\mathbb{B}$ the probability of occurrence is the same in the two tests. It only depends on the probability rule defined in the following.

[4]By adding the intersection of events, one builds up the full *Boolean algebra of events* (see *e.g.* Ref. [RÓ7]).

[5]By definition the state is the collection of the variables of a system whose knowledge is sufficient to make predictions. In the present context, it allows one to predict the results of tests, whence it is the probability rule for all events in any conceivable test.

conditional state

This definition of cascade between test allows us to introduce the first postulate of the probabilistic theory landscape. The composition of events must satisfy the following

Postulate NSF: No-signaling from the future. *The marginal probability* $\sum_{\mathscr{B}_j\in\mathbb{B}}\omega(\mathscr{B}_j\circ\mathscr{A})$ *of any event* $\mathscr{A}$ *is independent on test* $\mathbb{B}$*, and is equal to the probability with no test* $\mathbb{B}$*, namely*

$$\sum_{\mathscr{B}_j\in\mathbb{B}}\omega(\mathscr{B}_j\circ\mathscr{A}) =: f(\mathbb{B},\mathscr{A})\equiv\omega(\mathscr{A}),\quad\forall\mathbb{B},\,\mathscr{A},\,\omega. \tag{2.3}$$

NSF is part of the definition itself of test-cascade, however, we treat it as a separate postulate, since it corresponds to the **choice of the arrow of time**.[6] The interpretation of the test-cascade $\mathbb{B}\circ\mathbb{A}$ is that "test $\mathbb{A}$ can influence test $\mathbb{B}$ but not vice-versa."[7] Postulate NSF allows one to define the conditioned probability

$$p(\mathscr{B}|\mathscr{A}) = \frac{\omega(\mathscr{B}\circ\mathscr{A})}{\omega(\mathscr{A})} \tag{2.4}$$

of event $\mathscr{B}$ occurring conditionally on the previous occurrence of event $\mathscr{A}$. It also guarantees that the probability of $\mathscr{B}$ remains independent of the test $\mathbb{B}$ when conditioned. In fact for each test $\mathbb{B}$ including $\mathscr{B}_j$ as particular event, we get from Eq. (2.3)

$$\sum_{\mathscr{B}_j\in\mathbb{B}} p(\mathscr{B}_j|\mathscr{A}) = \sum_{\mathscr{B}_j\in\mathbb{B}}\frac{\omega(\mathscr{B}_j\circ\mathscr{A})}{\omega(\mathscr{A})} = \frac{\omega(\mathscr{A})}{\omega(\mathscr{A})} = 1,\quad\forall\mathbb{B},\,\mathscr{A},\,\omega. \tag{2.5}$$

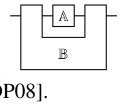
Conditioning sets a new probability rule corresponding to the notion of **conditional state** $\omega_{\mathscr{A}}$, which gives the probability that an event occurs knowing that event $\mathscr{A}$ has occurred with the system prepared in the state ω, namely

$$\omega_{\mathscr{A}} := \frac{\omega(\cdot\circ\mathscr{A})}{\omega(\mathscr{A})} \tag{2.6}$$

where the central dot "·" denotes the location of the pertinent variable. We can now regard the event $\mathscr{A}$ as transforming with probability $\omega(\mathscr{A})$ the state ω to the (unnormalized) state $\mathscr{A}\omega$[8] given by

$$\mathscr{A}\omega := \omega(\cdot\circ\mathscr{A}). \tag{2.7}$$

[6] Postulate NSF is not just a Kolmogorov consistency condition for marginals of a joint probability. In fact, even though the marginal over test $\mathbb{B}$ in Eq. (2.3) is obviously the probability of $\mathscr{A}$, such probability in principle depends on the test $\mathbb{B}$, since the joint probability generally depends on it. And, indeed, the marginal over entry $\mathscr{A}$ does generally depend on the past test $\mathbb{A}\ni\mathscr{A}$. Such asymmetry of the joint probability under marginalization over future or past tests represents *the choice of the arrow of time*. Of course one could have assumed the opposite postulate of no-signaling from the past, considering conditioning from the future instead, thus reversing the arrow of time. Postulate NSF introduces conditioning from tests, and is part of the definition itself of temporal cascade-tests. The need of considering NSF as a Postulate has been noticed for the first time by Masanao Ozawa (private communication with the Author of the article [D'A08]).

[7] One could also defined more general cascades not in time, *e.g.* the circuit diagram 

This would have given rise to a probabilistic version of the quantum comb theory of Ref. [CDP08].

[8] This is the same as the notion of *quantum operation* in QM, which gives the conditioning $\omega_{\mathscr{A}} = \mathscr{A}\omega/\mathscr{A}\omega(\mathscr{I})$, or, in other words, the analogous of the quantum Schrödinger picture evolution of states.

identity transformation
system
mono-systemic evolution

Therefore, the notion of cascade and Postulate NSF entail the identification:

event≡transformation,

which in turn implies the equivalence:

evolution≡tate-conditioning.[9]

Notice that from an operational point of view a transformation $\mathscr{A}$ is completely specified by all the joint probabilities in which it is involved, whence, it is univocally given by the probability rule $\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$ for all states ω. This is equivalent to specify both the conditional state $\omega_{\mathscr{A}}$ and the probability $\omega(\mathscr{A})$ for all possible states ω, due to the identity

$$\mathscr{A}\omega = \omega(\mathscr{A})\omega_{\mathscr{A}}. \tag{2.8}$$

In particular the **identity transformation** denoted by $\mathscr{I}$ is completely specified by the rule

$$\mathscr{I}\omega = \omega \qquad \forall\omega. \tag{2.9}$$

2.1.3 System definition

It's possible to define a system (think about a local one) starting from the concept of test. All the object which define a system are operationally achievable from a collection of tests as follows.

The system as a collection of tests with operational closure

In a pure Copenhagen spirit we will identify a **system** S with a collection of **tests** $\mathsf{S} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$, the collection being operationally closed under:

- coarse-graining,
- convex combination,
- conditioning,
- cascading

and it includes all states as special tests too. Closure under cascading is equivalent to consider **mono-systemic evolution**, *i.e.* in which there are only tests for which the output system is the same of the input one.[10] The operator has always the option of

[10]In this work are not already considered more generally tests in which the output system is different from the input one, in which case the system is no longer closed under test-cascade, and, instead, there are cascades of tests from different systems. This would give more flexibility to the axiomatic approach, and could be useful for proving some theorems related to multipartite systems made of different systems. The fact that there are different systems would impose constraints to the cascades of tests, corresponding to allow only some particular words made of the "alphabet" $\mathbb{A}, \mathbb{B}, \ldots$ of tests, and the system would then correspond to a "language" (see Ref.[CDP09] for a similar framework) In the following we will give a possible definition of subsystem which require the "transition" from a system to a smaller one or viceversa. On the other hand the idea of subsystem is less general then the extension from the present theory to a theory allowing tests with output system different from the input one. In the definition of a subsystem is not strictly necessary to introduce a physical map from the system to the subsystem and viceversa.

convex combination
operational closure
convex monoid
conditioning-equivalent
probabilistically equivalent

performing repeated tests, along with (randomly) alternating tests—say $\mathbb{A}$ and $\mathbb{B}$—in different proportions—say p and $1-p$ ($0 < p < 1$)—thus achieving the test

$$\mathbb{C}_p = p\mathbb{A} + (1-p)\mathbb{B} \tag{2.10}$$

which is the **convex combination** of tests $\mathbb{A}$ and $\mathbb{B}$, and is given by

$$\mathbb{C}_p = \{p\mathscr{A}_1, p\mathscr{A}_2, \ldots, (1-p)\mathscr{B}_1, (1-p)\mathscr{B}_2, \ldots\}, \tag{2.11}$$

where $p\mathscr{A}$ is the same event as $\mathscr{A}$, but occurring with a probability rescaled by p. Since we will consider always the closure under all operator's options (this is our **operational closure**), we will take the system also to be closed under such convex combination.

The transformations of a system. In the following we will denote the set of all possible transformations/events by $\mathfrak{T}(\mathsf{S})$, $\mathfrak{T}$ for short. The convex structure of S entails a convex structure for $\mathfrak{T}$, whereas the cascade of tests entails the composition of transformations. The latter, along with the existence of the identity transformation $\mathscr{I}$, gives to $\mathfrak{T}$ the structure of **convex monoid**.

The states of a system. In particular, the set of all states of the system[11] is closed under convex combinations and under conditioning. In fact given two different probability rules given by two different states—say ω_1 and ω_2, a convex combination of them

$$p\omega_1 + (1-p)\omega_2 \qquad 0 < p < 1 \tag{2.12}$$

is obviously a new probability rule given by the probability rule ω_1 or ω_2 in the proportion dictated by p. The closure under conditioning is still obvious according to the definition in Eq. (2.6). We will denote by $\mathfrak{S}(\mathsf{S})$ ($\mathfrak{S}$ for short) the convex set of all possible states of system S. We will often use the colloquialism "for all possible states ω" meaning $\forall\omega \in \mathfrak{S}(\mathsf{S})$, and we will do similarly for other operational objects.

2.1.4 Effects

The notion of effect is tightly connected to the transformations, in fact an effect is an equivalence class of transformations. From the notion of conditional state we can achieve two complementary types of equivalences for transformations, the *conditional* and the *probabilistic* equivalence.

- The transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **conditioning-equivalent** when

$$\omega_{\mathscr{A}_1} = \omega_{\mathscr{A}_2} \qquad \forall\omega \in \mathfrak{S}, \tag{2.13}$$

  namely when they produce the same conditional state for all prior states ω.

- The transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **probabilistically equivalent** when

$$\omega(\mathscr{A}_1) = \omega(\mathscr{A}_2) \qquad \forall\omega \in \mathfrak{S}, \tag{2.14}$$

  namely when they occur with the same probability for all prior states.

[11] At this stage such set not necessarily contains all *in-principle* possible states. The extension will be done later, after defining effects which are another fundamental component in the system structure.

fully-equivalent
deterministic effect

Transformations fully-equivalent. Since operationally a transformation $\mathscr{A}$ is completely specified by the probability rule $\mathscr{A}\omega$ for all states, it follows that two transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ are **fully-equivalent** (*i.e.* operationally indistinguishable) when $\mathscr{A}_1\omega = \mathscr{A}_2\omega$ for all states ω. We will identify two equivalent transformations, and denote the equivalence simply as $\mathscr{A}_1 = \mathscr{A}_2$. From identity (2.8) it follows that two transformations are equivalent if and only if they are both conditioning and probabilistically equivalent. An important operational relation is then:

two indistinguishable transformations are the same transformation,

A probabilistic equivalence class of transformations defines an **effect**[12]. In the following we will denote effects with lowercase letters $a, b, c, \ldots$ and denote by $[\mathscr{A}]_{\text{eff}}$ the probabilistic equivalence class of transformations containing the transformation $\mathscr{A}$. We will write $\mathscr{A} \in a$ meaning that "the transformation $\mathscr{A}$ belongs to the equivalence class a", or "$\mathscr{A}$ has effect a[13]". We will also write "$\mathscr{A} \in [\mathscr{B}]_{\text{eff}}$ to say that "$\mathscr{A}$ is probabilistically equivalent to $\mathscr{B}$".

Effects as variable of states. Since by definition

$$\omega(\mathscr{A}) = \omega([\mathscr{A}]_{\text{eff}}), \tag{2.15}$$

hereafter we will legitimately write the variable of the state as an effect, *e.g.* $\omega(a)$. The **deterministic effect** will be denoted by e, corresponding to $\omega(e) = 1$ for all states ω.

The effects of a system. We know from the previous definition of system that the set of transformations $\mathfrak{T}$ is closed under convex combination. The effects are equivalence classes of transformations and then they inherits a convex structure from that of transformations. We will denote the set of effects for a system S as $\mathfrak{E}(\mathsf{S})$, or just $\mathfrak{E}$ for short.

States and effects separate each others. By the same definition of state—as probability rule for transformations—states are separated by effects (whence also by transformations[14]), and, conversely, effects are separated by states. Transformations are separated by states in the sense that $\mathscr{A} \neq \mathscr{B}$ iff $\mathscr{A}\omega \neq \mathscr{B}\omega$ for some state. As a consequence, it may happen that the introduction of a new state via some new preparation (such as introducing additional systems) will separate two previously indiscriminable transformations, in which case we will include the new state (and all convex combination with it) in $\mathfrak{S}(\mathsf{S})$ (see also footnote 11), and we will complete the system S

[12] This is the same notion of "effect" introduced by Ludwig [Lud85]

[13] In the following it will result to be convenient think about effects and states as vectors in some space. In general the transformations in the probabilistic equivalence class a, where a is an effect, belong to a space different from the space of effects. For this reason we will use the notation $[a]$ to denote the equivalence class of transformation rather than the simple symbol a. Naturally given a transformation $\mathscr{A}$ having effect a we get $[\mathscr{A}]_{\text{eff}} = [a]$.

[14] In fact, $\mathscr{A}\omega \neq \mathscr{A}\zeta$ for $\mathscr{A} \in \mathfrak{T}$ means that there exists an effect c such that $\mathscr{A}\omega(c) \neq \mathscr{A}\zeta(c)$, whence the effect $c \circ \mathscr{A}$ will separates the same states.

test-compatible transformations
total coarse-graining

accordingly[15]. We will end with $\mathfrak{S}(\mathsf{S})$ separating $\mathfrak{T}(\mathsf{S})$ and $\mathfrak{E}(\mathsf{S})$, and $\mathfrak{E}(\mathsf{S})$ separating $\mathfrak{S}(\mathsf{S})$.

Heisenberg and Schrödinger pictures. The identity $\omega_{\mathscr{A}}(\mathscr{B}) \equiv \omega_{\mathscr{A}}([\mathscr{B}]_{\text{eff}})$ implies that $\omega(\mathscr{B} \circ \mathscr{A}) = \omega([\mathscr{B}]_{\text{eff}} \circ \mathscr{A})$ for all states ω, leading to the chaining rule $[\mathscr{B}]_{\text{eff}} \circ \mathscr{A} = [\mathscr{B} \circ \mathscr{A}]_{\text{eff}}$, corresponding to the "Heisenberg picture" evolution in terms of transformations acting on effects. Notice that transformations act on effects from the right, inheriting the composition rule of transformations ($\mathscr{B} \circ \mathscr{A}$ means "$\mathscr{A}$ followed by $\mathscr{B}$"). Notice also that $e \circ \mathscr{A} \in [\mathscr{I} \circ \mathscr{A}]_{\text{eff}} = a$. It follows that for $\mathscr{D}$ deterministic one has $\mathscr{D} \in e$, whence $\mathscr{D} \circ \mathscr{A} \in [\mathscr{A}]_{\text{eff}}$.

Consistently, in the "Schrödinger picture", we have $\mathscr{B}\omega(\cdot \circ \mathscr{A}) = \omega(\cdot \circ \mathscr{B} \circ \mathscr{A})$, corresponding to $(\mathscr{B} \circ \mathscr{A})\omega = \omega(\cdot \circ \mathscr{B} \circ \mathscr{A})$. Also, we will use the unambiguous notation $\mathscr{B}\omega(a) = [\mathscr{B}\omega](a)$, whence $\mathscr{B}\omega(a) = \omega(a \circ \mathscr{B})$, and $\omega(a) = \mathscr{A}\omega(e)$, $\forall \mathscr{A} \in a$.

2.1.5 Linear structures for transformations states and effects.

In some circumstances it's possible to give an operational definition of the multiplication by a scalar and the addition operation for transformation (and consequently for effects). This definition is easily extended to give a linear structure to the space of transformation and effect. The operational starting point is as follows.

Addition. Transformations $\mathscr{A}_1$ and $\mathscr{A}_2$, for which one has the bound

$$\omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \leqslant 1, \ \forall \omega \in \mathfrak{S} \tag{2.16}$$

can in principle occur in the same test, and we will call them **test-compatible transformations**. For test-compatible transformations one can operationally define their addition $\mathscr{A}_1 + \mathscr{A}_2$ via the probability rule

$$(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1\omega + \mathscr{A}_2\omega, \tag{2.17}$$

were we remind that $\mathscr{A}\omega := \omega(\cdot \circ \mathscr{A})$. Therefore the sum of two test-compatible transformations is just the union-event $\mathscr{A}_1 + \mathscr{A}_2 = \mathscr{A}_1 \cup \mathscr{A}_2$, with the two transformations regarded as belonging to the same test.[16] For any test $\mathbb{A}$ we can define its **total coarse-graining** as the deterministic transformation

$$\mathscr{D}_{\mathbb{A}} = \sum_{\mathscr{A}_i \in \mathbb{A}} \mathscr{A}_i. \tag{2.18}$$

[15] Adding for example the events corresponding to the states preparation for the new states.

[16] The probabilistic class of $\mathscr{A}_1 + \mathscr{A}_2$ is given by

$$\omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2), \quad \forall \omega \in \mathfrak{S},$$

whereas the conditional class is given by

$$\omega_{\mathscr{A}_1 + \mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)}\omega_{\mathscr{A}_1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)}\omega_{\mathscr{A}_2}, \quad \forall \omega \in \mathfrak{S}.$$

atomic event
no-restriction hypothesis for states

Now we can trivially extend the addition rule (2.17) to any set of (generally non test-compatible) transformations, and to subtraction of transformations as well. Notice that the composition "$\circ$" is distributive with respect to addition "+". In fact given two transformations $\mathscr{A}_1$ and $\mathscr{A}_2$, in general non test-compatible after the extension, and another transformation $\mathscr{B}$ we get

$$\mathscr{B} \circ (\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{B}\mathscr{A}_1\omega + \mathscr{B}\mathscr{A}_2\omega \qquad \forall \omega \in \mathfrak{S} \tag{2.19}$$

and then

$$\mathscr{B} \circ (\mathscr{A}_1 + \mathscr{A}_2) = \mathscr{B} \circ \mathscr{A}_1 + \mathscr{B} \circ \mathscr{A}_2 \qquad \forall \mathscr{A}_1, \mathscr{A}_2, \mathscr{B} \in \mathfrak{T}. \tag{2.20}$$

Multiplication by a scalar and cone of transformations. We can operationally define the multiplication $\lambda\mathscr{A}$ of a transformation $\mathscr{A}$ by a scalar $0 \leqslant \lambda \leqslant 1$ by the rule

$$\omega(\cdot \circ \lambda\mathscr{A}) = \lambda\omega(\cdot \circ \mathscr{A}), \tag{2.21}$$

namely $\lambda\mathscr{A}$ is the transformation conditioning-equivalent to $\mathscr{A}$, but occurring with rescaled probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$—as happens in the convex combination of tests. It follows that for every couple of transformations $\mathscr{A}$ and $\mathscr{B}$ the transformations $\lambda\mathscr{A}$ and $(1-\lambda)\mathscr{B}$ are test-compatible for $0 \leqslant \lambda \leqslant 1$, consistently with the convex closure of the system S. By extending the definition (2.21) to any positive λ, we then introduce the cone $\mathfrak{T}_+$ of transformations. We will call an event $\mathscr{A}$ **atomic event** if it has no nontrivial refinement in any test, namely if it cannot be written as $\mathscr{A} = \sum_i \mathscr{A}_i$ with $\mathscr{A}_i \neq \lambda_i\mathscr{A}$ for some i and $0 < \lambda_i < 1$. Notice that the identity transformation is not necessarily atomic.[17] The set of extremal rays of the cone $\mathfrak{T}_+$—denoted by $\mathsf{Erays}(\mathfrak{T}_+)$—contains the atomic transformations.

The cones of states and effects. The notions of (i) test-compatibility, (ii) sum, and (iii) multiplication by a scalar, are naturally inherited from transformations to effects via probabilistic equivalence, and then to states via duality. Remember in fact that $\mathfrak{S}$ is the dual of $\mathfrak{E}$. Correspondingly to the cone of transformation, we can introduce the cone of effects $\mathfrak{E}_+$, and, by duality, we extend the cone of states $\mathfrak{S}_+$[18] to the dual cone of $\mathfrak{E}_+$, completing the set of states $\mathfrak{S}$ to the new cone-base of $\mathfrak{S}_+$ made of all positive linear functionals over $\mathfrak{E}_+$ normalized at the deterministic effect, namely all in-principle legitimate states (in parallel we complete the system S with the corresponding state-preparations). We call such a completion of the set of states the **no-restriction hypothesis for states**, corresponding to the **state–effect duality**, namely the convex cones of states $\mathfrak{S}_+$ and of effects $\mathfrak{E}_+$ are dual each other.[19] The construction of the cones start as usual from the tests. In fact given a system, which is a collection of test,

[17] For example, the identity transformation is refinable in classical abelian probabilistic theory, where states are of the form $\varrho = \sum_l p_l |l\rangle\langle l|$, with $\{|l\rangle\}$ a complete orthonormal basis and $\{p_l\}$ a probability distribution. Here the identity transformation is given by $\mathscr{I} = \sum_k |k\rangle\langle k| \cdot |k\rangle\langle k|$, $\{|k\rangle\}$, which is refinable into rank-one projection maps.. In III we will show how for each classical theory has a simplex local structure and the only atomic transformations are of local nature, namely they will not lead to a violation of the CHSH inequality.

[18] which is the cone base on $\mathfrak{S}$

[19] In infinite dimensions one also takes the closure of cones.

partial ordering
dimension of a system
internal state

we also have the set of effects and the relative cone. Then by duality the cone of states is generated and the set of states is the base of such cone. The state cone $\mathfrak{S}_+$ introduce a natural **partial ordering** relation[20] $\geqslant$ over states and over effect (via duality), and one has $a \in \mathfrak{E}$ iff $0 \leqslant a \leqslant e$. Thus the convex set $\mathfrak{E}$ is a **truncation of the cone** $\mathfrak{E}_+$[21], whereas $\mathfrak{S}$ **is a base for the cone** $\mathfrak{S}_+$[22] defined by the normalization condition $\omega \in \mathfrak{S}$ iff $\omega \in \mathfrak{S}_+$ and $\omega(e) = 1$. In the following it will be useful also to express the probability rule $\omega(a)$ also in its dual form $a(\omega) = \omega(a)$, with the effect acting on the state as a linear functional.

linear complex and real spans of $\mathfrak{S}$,$\mathfrak{E}$ and $\mathfrak{T}$. By extending to any real scalar λ Eq. (2.21) we build the linear real span

$$\mathfrak{T}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathfrak{T}) \tag{2.22}$$

In the same way extending the multiplication to any complex scalar the linear complex span is achieved

$$\mathfrak{T}_{\mathbb{C}} = \mathsf{Span}_{\mathbb{C}}(\mathfrak{T}) \tag{2.23}$$

. The *Cartesian decomposition* $\mathfrak{T}_{\mathbb{C}} = \mathfrak{T}_{\mathbb{R}} \oplus i\mathfrak{T}_{\mathbb{R}}$ holds, *i.e.* each element $\mathscr{A} \in \mathfrak{T}_{\mathbb{C}}$ can be uniquely written as $\mathscr{A} = \mathscr{A}_R + i\mathscr{A}_I$, with $\mathscr{A}_R, \mathscr{A}_I \in \mathfrak{T}_{\mathbb{R}}$.[23] Analogously, also for effects and states we define $\mathfrak{E}_{\mathbb{F}}, \mathfrak{S}_{\mathbb{F}}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. The state–effect duality implies the linear space identifications $\mathfrak{S}_{\mathbb{F}} \equiv \mathfrak{E}_{\mathbb{F}}$.

Now we can uniquely define the **dimension of a system**. Thanks to the identifications between $\mathfrak{S}_{\mathbb{F}}$ and $\mathfrak{E}_{\mathbb{F}}$ and to the identity of the dimension of a convex cone with that of its complex and real spans, in the following without ambiguity, we will simply write

$$\dim(\mathsf{S}) := \dim[\mathfrak{S}_+(\mathsf{S})] \equiv \dim[\mathfrak{E}_+(\mathsf{S})]. \tag{2.24}$$

Moreover, if there is no confusion, with some abuse of terminology we will simply refer by "states," "effects," and "transformations" to the respective generalized versions that are elements of the cones, or of their real and complex linear spans. Naturally the elements in the linear spans are not all true states, effects and transformations of the 'physical system.

Note that the cones of states and effects contain the origin, *i.e.* the null vector of the linear space. For the cone of states one has that $\omega = 0$ iff $\omega(e) = 0$ (since for any effect a one has $0 \leqslant \omega(a) \leqslant \omega(e) = 0$, namely $\omega(a) = 0$). On the other hand, the hyperplane which truncates the cone of effects giving the physical convex set $\mathfrak{E}$ is conveniently characterized using any **internal state** ϑ—*i.e.* a state that can be written as the convex combination of any state with some other state—by using the following lemma

[20] See also Ssec. 1.2.1

[21] $\mathfrak{E}$ is exactly the **truncated dual cone** of $\mathfrak{S}_+$ as explained in Subs. 1.2.1

[22] We remind the reader that a set $\mathsf{B} \subset \mathsf{C}$ of a cone C in a vector space V is called *base* of C if $0 \notin \mathsf{B}$ and for every point $u \in \mathsf{C}, u \neq 0$, there is a unique representation $u = \lambda v$, with $v \in \mathsf{B}$ and $\lambda > 0$. Then, one has that $u \in \mathsf{C}$ spans an extreme ray of C iff $u = \lambda v$, where $\lambda > 0$ and v is an extreme point of B (see Ref.[Bar02]).

[23] Note that the elements $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$ can in turn be decomposed à la Jordan as $\mathscr{T} = \mathscr{T}_+ - \mathscr{T}_-$, with $\mathscr{T}_\pm \in \mathfrak{T}_+$. However, such a decomposition is generally not unique. According to a theorem of Béllissard and Jochum [BI78] the Jordan decomposition of the elements of the real span of a cone (with $\mathscr{T}_\pm$ orthogonal in $\mathfrak{T}_{\mathbb{R}}$ Euclidean space) is unique if and only if the cone is self-dual.

observable
informationally complete observable
separating set of states

Lemma 2.1 *For any $a \in \mathfrak{E}_+$ one has $a = 0$ iff $\vartheta(a) = 0$ and $a = e$ iff $\vartheta(a) = 1$, with ϑ any internal state.*

Proof. For any state ω one can write $\vartheta = p\omega + (1-p)\omega'$ with $0 \leqslant p \leqslant 1$ and $\omega' \in \mathfrak{S}$. Then one has $\vartheta(a) = 0$ iff $\omega(a) = 0$ $\forall\omega \in \mathfrak{S}$, that is iff $a = 0$. Moreover, one has $\vartheta(a) = 1$ iff $\omega(a) = 1$ $\forall\omega \in \mathfrak{S}$, *i.e.* $a = e$. ■

2.1.6 Observables and informational completeness

A very common object of quantum mechanics is the observable. Anyway this last one is not a primitive element of a system namely it can be easily defined from the events (and then by classes of events which are the effects) of the system which are a true operational primitive objects. [24] An **observable** $\mathbb{L}$ is a complete set of effects $\mathbb{L} = \{l_i\}$ summing to the deterministic effect as

$$\sum_{l_i \in \mathbb{L}} l_i = e, \tag{2.25}$$

namely l_i are the effects of the events of a test. But an observable is not a test, is a set of effects not of events and then is something more general than a test. One could think about an observable as a class of tests $\mathbb{A}, \mathbb{A}', \ldots$, which are collection of events $\{\mathscr{A}_i\}, \{\mathscr{A}'_i\}, \ldots$, having the same set of effects $\{l_i\}$. An observable $\mathbb{L} = \{l_i\}$ is named **informationally complete observable** for S when each effect can be written as a real linear combination of l_i, namely

$$\mathsf{Span}_{\mathbb{R}}(\mathbb{L}) = \mathfrak{E}_{\mathbb{R}}(\mathsf{S}). \tag{2.26}$$

When the effects of $\mathbb{L}$ are linearly independent the informationally complete observable is named *minimal*. Clearly, since $\mathfrak{E}$ is separating for states, **any informationally complete observable separates states**, that is using an informationally complete observable we can reconstruct also any state $\omega \in \mathfrak{S}(\mathsf{S})$ from the set of probabilities $\omega(l_i)$. The existence of a minimal informationally complete observable constructed from the set of available tests is guaranteed by the following Theorem:

Theorem 2.1 *:***Existence of minimal informationally complete observable.** *It is always possible to construct a minimal informationally complete observable for* S *out of a set of tests of* S.

For the proof see Ref. [D'A07a].

Symmetrically to the notion of informationally complete observable we have the notion of **separating set of states** $\mathbb{S} = \{\omega_i\}$, in terms of which one can write any state as a real linear combination of the states $\{\omega_i\}$, namely $\mathfrak{S}_{\mathbb{R}}(\mathsf{S}) = \mathsf{Span}_{\mathbb{R}}(\mathbb{S})$. Regarded as a test $\mathbb{S} = \{\mathscr{S}_i\} \in \mathsf{S}$ the set of states $\{\omega_i\}$ correspond to the state-reduction $\mathscr{S}_i\omega = \omega(\mathscr{S}_i)\omega_i$, $\forall\omega \in \mathfrak{S}$. The test in Eq. (2.2) is an example of separating set of states but in general it is not the minimal one because it contains all the possible states of a system.

[24] In fact all the axiomatization of quantum mechanics based on the idea of observables are not really operational axiomatizations.

natural norm
natural distance

Observation 2.1 *In the following we will take a fixed minimal informationally complete observable* $\mathbb{L} = \{l_i\}$ *as a* **reference test**, *with respect to which all basis-dependent representations will be defined. In Sec. 2.5 we will introduce the Block representation for transformations in a probabilistic theory. Such representations, it is not unique, depends on the chosen minimal informationally complete observable.*

2.1.7 Banach structures

It's easy to give a Banach structure to the linear spaces $\mathfrak{S}_{\mathbb{R}}$ and $\mathfrak{E}_{\mathbb{R}}$ introducing a norm with respect to which they are complete. On states $\omega \in \mathfrak{S}$ introduce the **natural norm**

$$|\omega| = \sup_{a \in \mathfrak{E}} \omega(a), \tag{2.27}$$

which extends to the whole linear space $\mathfrak{S}_{\mathbb{R}}$ as

$$|\omega| = \sup_{a \in \mathfrak{E}} |\omega(a)|. \tag{2.28}$$

Then, we can introduce the dual norm on effects

$$|a| := \sup_{\omega \in \mathfrak{S}_{\mathbb{R}}, |\omega| \leqslant 1} |\omega(a)|, \tag{2.29}$$

and then on transformations

$$|\mathscr{A}| := \sup_{b \in \mathfrak{E}_{\mathbb{R}}, |b| \leqslant 1} |b \circ \mathscr{A}|. \tag{2.30}$$

Closures in norm (for mathematical convenience) make $\mathfrak{E}_{\mathbb{R}}$ and $\mathfrak{S}_{\mathbb{R}}$ a **dual Banach pair**. An algebra of maps over a Banach space inherits the norm induced by that of the Banach space on which it acts. It is then easy to prove that the closure of the algebra under such norm is a Banach algebra. Thus $\mathfrak{T}_{\mathbb{R}}$ becomes a real **Banach algebra**. Therefore, all operational quantities can be mathematically represented as elements of such Banach spaces.

The natural distance as a metric

Natural distance between states. One can define a **natural distance** between states $\omega, \zeta \in \mathfrak{S}$ as follows

$$d(\omega, \zeta) := \sup_{l \in \mathfrak{E}} l(\omega) - l(\zeta). \tag{2.31}$$

This definition of distance define a metric on $\mathfrak{S}$ whose extension over $\mathfrak{S}_{\mathbb{R}}$ is the metric induced by the natural norm in Eq. (2.28). The following lemma shows our assertion.

Lemma 2.2 *The function (2.31) is a metric on* $\mathfrak{S}$, *and is bounded as* $0 \leqslant d(\omega, \zeta) \leqslant 1$.

Proof. For every effect l, $e - l$ is also a effect because $0 \leqslant \omega(e - l) \leqslant 1 \forall \omega \in \mathfrak{S}$, whence

$$\begin{aligned} d(\omega, \zeta) &= \sup_{l \in \mathfrak{E}} (l(\omega) - l(\zeta)) = \sup_{l' \in \mathfrak{E}} ((e - l')(\omega) - (e - l')(\zeta)) \\ &= \sup_{l' \in \mathfrak{E}} (l'(\zeta) - l'(\omega)) = d(\zeta, \omega), \end{aligned} \tag{2.32}$$

isometric transformation

that is d is symmetric. On the other hand, $d(\omega,\zeta)=0$ implies that $\zeta=\omega$, since the two states must give the same probabilities for all transformations. Finally, one has

$$\begin{aligned} d(\omega,\zeta) &= \sup_{l\in\mathfrak{E}}(l(\omega)-l(\theta)+l(\theta)-l(\zeta)) \\ &\leq \sup_{l\in\mathfrak{E}}(l(\omega)-l(\theta))+\sup_{l\in\mathfrak{E}}(l(\theta)-l(\zeta)) = d(\omega,\theta)+d(\theta,\zeta), \end{aligned} \tag{2.33}$$

that is it satisfies the triangular inequality, whence d is a metric. By construction, the distance is bounded as $d(\omega,\zeta)\leqslant 1$, since the maximum value of $d(\omega,\zeta)$ is achieved when $l(\omega)=1$ and $l(\zeta)=0$. ■

The natural distance (2.32) is extended to a metric over $\mathfrak{S}_{\mathbb{R}}$ as $d(\omega,\zeta)=|\omega-\zeta|$ with $|\cdot|$ the norm over $\mathfrak{S}_{\mathbb{R}}$.

Natural distance between effects. Analogously we can also define define the natural distance between effects as

$$d(a,b) := \sup_{\omega\in\mathfrak{S}}|\omega(a-b)|^{25}. \tag{2.34}$$

The module in the definition of the natural distance over effects is necessary. In fact the term $\omega(a-b)$ can be always negative if, *e.g.* , $b=e$. This could not happen in Eq. (2.31) because, as already noticed, if l is an effect also $(e-l)$ is an effect.

Isometric transformations

A deterministic transformation $\mathscr{U}$ is called **isometric transformation** if it preserves the distance between states, namely

$$d(\mathscr{U}\omega,\mathscr{U}\zeta)\equiv d(\omega,\zeta), \qquad \forall\omega,\zeta\in\mathfrak{S}. \tag{2.35}$$

The following lemma establishes some relations between automorphism and isometric map for the set of states and effects.

Lemma 2.3 *In finite dimensions, all the following properties of a transformation are equivalent: (a) it is isometric for $\mathfrak{S}$; (b) it is isometric for $\mathfrak{E}$; (c) it is automorphism of $\mathfrak{S}$; (d) it is automorphism of $\mathfrak{E}$.*

Proof. By definition a transformation of the convex set (of states or effects) is a linear map of the convex set in itself. A linear isometric map of a set in itself is isometric on the linear span of the set[26] (Recall that the natural distance between states has been extended to a metric over the whole $\mathfrak{S}_{\mathbb{R}}$). In finite dimensions an isometry on a normed linear space is diagonalizable [KR70]. Its eigenvalues must have unit modulus, otherwise it would not be isometric. It follows that it is an orthogonal transformation, and

[25] It is easy to check that such distance satisfies the trangular inequality

[26] Interestingly, the Mazur-Ulam Theorem states that any surjective isometry (not necessarily linear) between real normed spaces is affine. Therefore, even if nonlinear, it would map convex subsets to convex subsets (From Ssec. 1.1.5 we know that the affine maps preserve convexity).

Adjoint
scalar product

since it maps the set into itself, it must be a linear automorphism of the set. Therefore, an isometric transformation of a convex set is an automorphism of the convex set[27]

Now, automorphisms of $\mathfrak{S}$ are isometric for $\mathfrak{E}$, since

$$\begin{aligned} d(a\circ\mathscr{U}, b\circ\mathscr{U}) &= \sup_{\omega\in\mathfrak{S}}|\omega((a-b)\circ\mathscr{U})| = \sup_{\omega\in\mathfrak{S}}|(\mathscr{U}\omega)(a-b)| \\ &= \sup_{\omega\in\mathscr{U}\mathfrak{S}}|\omega(a-b)| = \sup_{\omega\in\mathfrak{S}}|\omega(a-b)| = d(a,b), \end{aligned} \tag{2.36}$$

and, similarly, automorphisms of $\mathfrak{E}$ are isometric for $\mathfrak{S}$, since

$$\sup_{a\in\mathfrak{E}}[\omega(a\circ\mathscr{U}) - \zeta(a\circ\mathscr{U})] = \sup_{a\in\mathfrak{E}\circ\mathscr{U}}[\omega(a)-\zeta(a)] = d(\omega,\zeta). \tag{2.37}$$

Therefore, automorphisms of $\mathfrak{S}$ are isometric for $\mathfrak{E}$, whence, for the first part of the proof, they are automorphisms of $\mathfrak{E}$, whence they are isometric for $\mathfrak{S}$. ■

The **physical automorphisms** play the role of unitary transformations in QM.

Corollary 2.1 :Wigner theorem. *A transformation of states is inverted by another transformations if and only if send pure states to pure states and is isometric.*

2.1.8 The C* algebra of transformations

We have introduced a Banach spaces structure on the spaces $\mathfrak{S}_\mathbb{R}$ and $\mathfrak{E}_\mathbb{R}$ and consequently a Banach algebra structure in the set of transformations $\mathfrak{T}_\mathbb{R}$. It's possible to represent the transformations as elements of $\mathfrak{T}_\mathbb{C}$ regarded as a complex C*-algebra introducing the * involution, namely the **Adjoint**. We can summarize the construction in three steps.

- $\mathfrak{T}_\mathbb{C}$ are by definition linear transformations of effects, making an associative subalgebra
$$\mathfrak{T}_\mathbb{C} \subseteq \mathsf{Lin}(\mathfrak{E}_\mathbb{C}) \tag{2.38}$$
of the matrix algebra over $\mathfrak{E}_\mathbb{C}$.

- **Adjoint** and **norm** can be easily defined in terms of any chosen **scalar product** $(\cdot,\cdot)$ over $\mathfrak{E}_\mathbb{C}$, with the adjoint defined as
$$(a\circ\mathscr{A}^\dagger, b) = (a, b\circ\mathscr{A}), \tag{2.39}$$
and the norm as
$$|\mathscr{A}| = \sup_{a\in\mathfrak{E}_\mathbb{C}} |a\circ\mathscr{A}|/|a|, \tag{2.40}$$
with $|a| = \sqrt{(a,a)}$. (Notice that these norms are different from the "natural norms" defined in Subsect. 2.1.7. These norm are the norm induced by the chosen scalar product over $\mathfrak{E}_\mathbb{C}$.)

- We can then extend the complex linear space $\mathfrak{T}_\mathbb{C}$ by adding the adjoint transformations and taking as usual the norm-closure. We will denote such extension with the same symbol $\mathfrak{T}_\mathbb{C}$, which is now a C*-algebra.

[27]For a convex set, an automorphism must send the set to itself keeping the convex structure, whence it must be a one-to-one map that is linear on the span of the convex set.

system independence

A particular construction. Upon reconstructing $\mathfrak{E}_\mathbb{C}$ and $\mathfrak{T}_\mathbb{C}$ from the original real spaces via the Cartesian decomposition

$$\mathfrak{E}_\mathbb{C} = \mathfrak{E}_\mathbb{R} \oplus i\mathfrak{E}_\mathbb{R} \qquad \mathfrak{T}_\mathbb{C} = \mathfrak{T}_\mathbb{R} \oplus i\mathfrak{T}_\mathbb{R}, \tag{2.41}$$

and introducing the scalar product on $\mathfrak{E}_\mathbb{C}$ as the sesquilinear extension of a real symmetric scalar product $(\cdot,\cdot)_\mathbb{R}$ over $\mathfrak{E}_\mathbb{R}$, the adjoint of a real element $\mathscr{A} \in \mathfrak{T}_\mathbb{R}$ is just the transposed matrix $\mathscr{A}^\tau$ with respect to a real basis orthonormal for $(\cdot,\cdot)_\mathbb{R}$, and for a general $\mathscr{A} = \mathscr{A}_R + i\mathscr{A}_I \in \mathfrak{T}_\mathbb{C}$ we get

$$\mathscr{A}^\dagger := \mathscr{A}_R{}^\tau - i\mathscr{A}_I{}^\tau \tag{2.42}$$

A natural choice of matrix representation for $\mathfrak{T}_\mathbb{R}$ is given by its action over a minimal informational complete observable $\mathbb{L} = \{l_i\}$. The real scalar product now is

$$(\cdot,\cdot)_\mathbb{R} := (\cdot,\cdot)_\mathbb{L} \tag{2.43}$$

where $\mathbb{L}$ is declared as orthonormal. Upon expanding $[l_i \circ \mathscr{A}]_{\text{eff}}$ again over $\mathbb{L} = \{l_i\}$ one has the matrix representation of a real transformation

$$l_i \circ \mathscr{A} = \sum_j \mathscr{A}_{ij} l_j. \tag{2.44}$$

Using the fact that $\mathbb{L}$ is state-separating, we can write the probability rule as the pairing $\omega(a) = (\omega, a)_\mathbb{R}$ between $\mathfrak{E}_\mathbb{R}$ and $\mathfrak{S}_\mathbb{R}$ (and analogously for their complex spans).[28]

Observation 2.2 *In this way we see that for every probabilistic theory one can always represent transformations/events as elements of the C^*-algebra $\mathfrak{T}_\mathbb{C}$ of matrices acting on the linear space of complex effects $\mathfrak{E}_\mathbb{C}$. Conversely, given (1) a C^*-algebra $\mathfrak{T}_\mathbb{C}$, (2) the cone of transformations $\mathfrak{T}_+$, and (3) the vector $e \in \mathfrak{E}_\mathbb{C}$ representing the deterministic effect, we can rebuild the full probabilistic theory by constructing the cone of effects as the orbit $\mathfrak{E}_+ = e \circ \mathfrak{T}_+$, and taking the cone of states $\mathfrak{S}_+$ as the dual cone of $\mathfrak{E}_+$.*[29]

2.2 Multipartite systems

2.2.1 Dynamical independence and composition of systems

Dynamical independence. A purely dynamical notion of **system independence** coincides with the possibility of performing local tests. To be precise, we will call systems S_1 and S_2 **independent** if it is possible to perform their tests as **local tests**, *i.e.* in such

[28] Adding the two postulates (a) the existence of dynamically independent systems, (b) the existence of faithful symmetric bipartiten states, it's possible to introduce The specific C*-algebra in Ref. [D'A07a] possessed operational notions of adjoint and of scalar product over effects. Such construction is briefly reviewed in the following after definitions of faithful bipartite systems and faithful states.

[29] The "orbit" $e \circ \mathfrak{T}_+$ is defined as the set: $e \circ \mathfrak{T}_+ := \{e \circ \mathscr{A} | \mathscr{A} \in \mathfrak{T}_+\}$.

local tests
bipartite system

a way that for every joint state of S_1 and S_2[30] the transformations on S_1 commute with transformations on S_2, namely[31]

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}, \ \forall \mathscr{A}^{(1)} \in \mathbb{A}^{(1)}, \forall \mathscr{B}^{(2)} \in \mathbb{B}^{(2)}. \tag{2.45}$$

It's important to observe that the tests obviously don't commute if executed on the same system in sequence. On the other hand all couple of tests must be independently performed on a couple of system which is the notion of dynamical independence.

Composition of systems. The local tests comprise the Cartesian product $\mathsf{S}_1 \times \mathsf{S}_2$ which is closed under cascade. We will close this set also under convex combination, coarse-graining and conditioning, making it a "system", and denote such a system with the same symbol

$$\mathsf{S}_1 \times \mathsf{S}_2, \tag{2.46}$$

and call **local tests** all tests in $\mathsf{S}_1 \times \mathsf{S}_2$. We now **compose** the two systems S_1 and S_2 into the **bipartite system**

$$\mathsf{S}_1 \odot \mathsf{S}_2 \tag{2.47}$$

by adding the local tests into the new system $\mathsf{S}_1 \odot \mathsf{S}_2$ as

$$\mathsf{S}_1 \odot \mathsf{S}_2 \supseteq \mathsf{S}_1 \times \mathsf{S}_2 \tag{2.48}$$

and closing under cascading, coarse-graining and convex combination. We call the tests in $\mathsf{S}_1 \odot \mathsf{S}_2 \setminus \mathsf{S}_1 \times \mathsf{S}_2$ **non-local**, and we will extend the local/non-local nomenclature to the pertaining transformations. In the following for identical systems we will also use the notation $\mathsf{S}^{\odot N} = \mathsf{S} \odot \mathsf{S} \odot \ldots \odot \mathsf{S}$ (N times), and $\mathfrak{Z}^{\odot N} := \mathfrak{Z}(\mathsf{S}^{\odot N})$ to denote N-partite sets/spaces, with $\mathfrak{Z} = \mathfrak{S}, \mathfrak{S}_+, \mathfrak{S}_{\mathbb{R}}, \mathfrak{S}_{\mathbb{C}}, \mathfrak{E}, \mathfrak{E}_+, \ldots$

Consider a set of local transformations acting on different systems of a multioartite system. Since the local transformations commute according to the notion of dynamical independence, we will just put them in a string, as

$$(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) := \mathscr{A}^{(1)} \circ \mathscr{A}^{(2)} \circ \mathscr{A}^{(3)} \circ \ldots \tag{2.49}$$

(convex combinations and coarse graining will be sums of strings). Clearly, given a joint state Ω the probability $\Omega(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots)$ is independent of the time ordering of transformations, it is just a function only of the effects

$$\Omega(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) = \Omega([\mathscr{A}]_{\text{eff}}, [\mathscr{B}]_{\text{eff}}, [\mathscr{C}]_{\text{eff}}, \ldots), \tag{2.50}$$

namely the joint effect corresponding to local transformations is made of (sums of) local effects

$$[(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots)]_{\text{eff}} \equiv ([\mathscr{A}]_{\text{eff}}, [\mathscr{B}]_{\text{eff}}, [\mathscr{C}]_{\text{eff}}, \ldots). \tag{2.51}$$

[30] a joint state intuitively is a probability rule for the joint tests of the two systems.

[31] The present definition of independent systems is purely dynamical, in the sense that it does not involve statistical requirements, *e.g.* the existence of factorized states. This, however, is implied by the mentioned no-restriction hypothesis for states.

factorized states
non local state
marginal state

But in the bipartite system there will be tests more general than the local ones and the embedding of local tests $\mathsf{S}_1 \times \mathsf{S}_2$ into the bipartite system $\mathsf{S}_1 \odot \mathsf{S}_2$ implies that

$$\mathfrak{T}_{\mathbb{F}}(\mathsf{S}_1 \odot \mathsf{S}_2) \supseteq \mathfrak{T}_{\mathbb{F}}(\mathsf{S}_1) \otimes \mathfrak{T}_{\mathbb{F}}(\mathsf{S}_2) \qquad \mathfrak{E}_{\mathbb{F}}(\mathsf{S}_1 \odot \mathsf{S}_2) \supseteq \mathfrak{E}_{\mathbb{F}}(\mathsf{S}_1) \otimes \mathfrak{E}_{\mathbb{F}}(\mathsf{S}_2) \tag{2.52}$$

both for real and complex spans $\mathbb{F} = \mathbb{R}, \mathbb{C}$. On the other hand, since local tests include local state-preparation (or, otherwise, because of the no-restriction hypothesis for states) the set of bipartite states $\mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ always includes the **factorized states**, *i.e.* those corresponding to factorized probability rules *e.g.* $\Omega(a, b) = \omega_1(a)\omega_2(b)$ for local effects a and b. In parallel with local transformations and effects, we will denote factorized states as strings $\Omega = (\omega_1, \omega_2, \ldots)$, *e.g.* $(\omega_1, \omega_2)(a, b) = \omega_1(a)\omega_2(b)$. Then, closure under convex combination implies that

$$\mathfrak{S}_{\mathbb{F}}(\mathsf{S}_1 \odot \mathsf{S}_2) \supseteq \mathfrak{S}_{\mathbb{F}}(\mathsf{S}_1) \otimes \mathfrak{S}_{\mathbb{F}}(\mathsf{S}_2) \tag{2.53}$$

for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. A bipartite state in $\mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ could produce a probability rule which is not factorizable over two probability rules for local tests of the two systems. This is the idea of **non local state** which is included in the present probabilistic structure.

Marginal state. For N systems in the joint state Ω, we define the **marginal state** $\Omega|_n$ of the n-th system the probability rule for any local transformation $\mathscr{A}$ at the n-th system, with all other systems untouched, namely

$$\Omega|_n(\mathscr{A}) \doteq \Omega(\mathscr{I}, \ldots, \mathscr{I}, \underbrace{\mathscr{A}}_{n\text{th}}, \mathscr{I}, \ldots). \tag{2.54}$$

Clearly, since the probability for local transformations depends only on their respective effects, the marginal state is equivalently defined as

$$\Omega|_n(a) \doteq \Omega(e, \ldots, e, \underbrace{a}_{n\text{th}}, e, \ldots) \quad \text{for } a \in \mathfrak{E}. \tag{2.55}$$

Observation 2.3 *It readily follows that the marginal state $\Omega|_n$ is independent of any deterministic transformation—i.e. any test—that is performed on systems different from the nth: this is exactly the general statement of the* **no-signaling** *or* **acausality of local tests**. *Therefore, the present notion of dynamical independence directly implies no-signaling. The definition in Eq. (2.54) can be trivially extended to unnormalized states.*[32,33]

[32] Notice that any generally unnormalized state is zero iff the joint state is zero, since $\Omega(e, e, \ldots, e) = \Omega_n(e) = 0$.

[33] The present notion of dynamical independence is indeed so minimal that it can be satisfied not only by the quantum tensor product, but also by the quantum direct sum [D'A06]. (Notice, however, that an analogous of the Tsirelson's Theorem [SW08] for transformations in finite dimensions would imply a representation of dynamical independence over the tensor-product of effects.) In order to extract only the tensor product an additional assumption is needed. As shown in Refs. [D'A06, D'A07a] two possibilities are either postulating the existence of bipartite states that are dynamically and preparationally faithful, or postulating the local observability principle. Here we will consider the former as a postulate, and derive the latter as a theorem.

In the following we will use the following identities

dynamically faithful
preparationally faithful

$$\Psi|_2(a) = \Psi(e, a) = \Psi(e, e \circ \mathscr{A}) = (\mathscr{I}, \mathscr{A})\Psi(e, e), \ \forall \mathscr{A} \in a. \tag{2.56}$$

2.2.2 Faithful states.

Given two systems S_1 and S_2 the faithfulness of a bipartite state in $\mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ concerns the relation between the cones of transformation for the two system—say $\mathfrak{T}_+(\mathsf{S}_1)$ and $\mathfrak{T}_+(\mathsf{S}_2)$—and the bipartite cone of states $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$. We can distinguish two different kind of faithfulness.

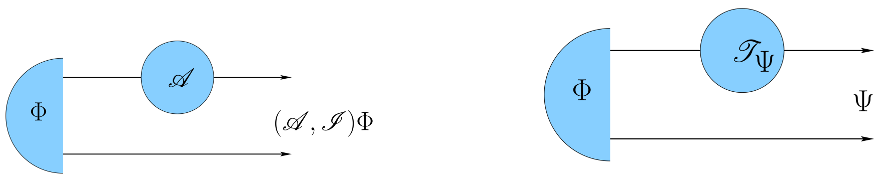


Figure 2.1: Illustration of the notions of dynamically (left figure) and preparationally (right figure) faithful state for a bipartite system. A bipartite state Φ is dynamically faithful with respect to system S_1 when the output state $(\mathscr{A}, \mathscr{I})\Phi$ is in one-to-one correspondence with the local transformation $\mathscr{A}$ on system S_1, whereas it is preparationally faithful with respect to S_1 if every bipartite state Ψ can be achieved as $\Psi = (\mathscr{T}_\Psi, \mathscr{I})\Phi$ via a local transformation $\mathscr{T}_\Psi$ on S_1.

- A bipartite state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ is **dynamically faithful** with respect to S_1 when the output state $(\mathscr{A}, \mathscr{I})\Phi$ is in one-to-one correspondence with the local transformation $\mathscr{A}$ on system S_1, that is the cone-homomorphism[34] $\mathscr{A} \leftrightarrow (\mathscr{A}, \mathscr{I})\Phi$ from $\mathfrak{T}_+(\mathsf{S}_1)$ to $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$ is a monomorphism.[35] Equivalently the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ extends to an injective linear map between the linear spaces $\mathfrak{T}_\mathbb{R}(\mathsf{S}_1)$ to $\mathfrak{S}_\mathbb{R}(\mathsf{S}_1 \odot \mathsf{S}_2)$ preserving the partial ordering relative to the spanning cones, and this is true also in the inverse direction on the range of the map. Notice that no physical transformation $\mathscr{A} \neq 0$ "annihilates" Φ, *i.e.* giving $(\mathscr{A}, \mathscr{I})\Phi = 0$. Substantially Φ is dynamically faithful with respect to S_1 when each different transformation in $\mathfrak{T}_+(\mathsf{S}_1)$ is mapped to a different state in $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$ by the cone homomorphism $\mathscr{A} \leftrightarrow (\mathscr{A}, \mathscr{I})\Phi$. If two transformations are distinguishable in $\mathfrak{T}_+(\mathsf{S}_1)$ then their images in $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$ are still distinguishable. In other word the state Φ is faithful because the homomorphism induced by it doesn't loose information about the input, namely $\mathfrak{T}_+(\mathsf{S}_1)$.

- A bipartite state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ is called **preparationally faithful** with respect to S_1 if every bipartite state Ψ can be achieved as $\Psi = (\mathscr{T}_\Psi, \mathscr{I})\Phi$ by a local

[34] A cone-homomorphism between cones C_1 and C_2 is a linear map between $\mathsf{Span}_\mathbb{R}(\mathsf{C}_1)$ and $\mathsf{Span}_\mathbb{R}(\mathsf{C}_2)$ which sends elements of C_1 to elements of C_2, but not necessarily vice-versa, preserving the cone structure. This means that the image of C_1 under the homomorphism is still a cone in C_2.

[35] This means that $(\mathscr{A}_1, \mathscr{I})\Phi = (\mathscr{A}_2, \mathscr{I},)\Phi$ iff $\mathscr{A}_1 = \mathscr{A}_2$, or, in other words, $\forall \mathscr{A} \in \mathfrak{T}_\mathbb{R}$: $(\mathscr{A}, \mathscr{I})\Phi = 0 \Longleftrightarrow \mathscr{A} = 0$.

transformation $\mathscr{T}_\Psi \in \mathfrak{T}_+(\mathsf{S}_1)$. This means that the cone-homomorphism $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ from $\mathfrak{T}_+(\mathsf{S}_1)$ to $\mathfrak{S}_+(\mathsf{S}_1 \odot \mathsf{S}_2)$ is an epimorphism. Equivalently, the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ extends to a surjective linear map between the linear spaces $\mathfrak{T}_\mathbb{R}(\mathsf{S}_1)$ to $\mathfrak{S}_\mathbb{R}(\mathsf{S}_1 \odot \mathsf{S}_2)$ preserving the partial ordering relative to the spanning cones.

In simple words, a dynamically faithful state keeps the imprinting of a local transformation on the output, *i.e.* from the output we can recover the transformation. On the other hand, a preparationally faithful state allows to prepare any desired joint state (probabilistically) by means of local transformations. In fact, given a preparationally faithful state Φ the probability of achieving the state Ψ from the transformation $\mathscr{T}_\Psi$ is equal to $\Phi(e \circ \mathscr{T}_\Psi, e)$. Dynamical and preparational faithfulness correspond to the properties of being *separating* and *cyclic* for the C^*-algebra of transformations.

Now we report important theorem exploring some of the main consequences of the definitions of dynamically and preparationally faithful states for a bipartite system. In particular are proved very interesting features satisfied by bipartite systems having identical local systems. In Part II we will construct some concrete probabilistic models and such construction will draw heavily on this theorem.

Theorem 2.2 *The following assertions hold:*

1. *Any state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ that is preparationally faithful with respect to S_1 is dynamically faithful with respect to S_2.*
2. *For identical systems in finite dimensions any state Φ that is preparationally faithful with respect to a system is also dynamically faithful with respect to the same system, and one has the cone-isomorphism*[36] $\mathfrak{T}_+(\mathsf{S}) \simeq \mathfrak{S}_+(\mathsf{S}^{\odot 2})$*. Moreover, a local transformation on Φ produces an output pure (unnormalized) bipartite state iff the transformation is atomic.*
3. *If there exists a state of $\mathsf{S}_1 \odot \mathsf{S}_2$ that is preparationally faithful with respect to S_1, then* $\dim(\mathsf{S}_1) \geqslant \dim(\mathsf{S}_2)$.
4. *If there exists a state of $\mathsf{S}_1 \odot \mathsf{S}_2$ that is preparationally faithful with respect to both systems, then one has the cone-isomorphisms $\mathfrak{E}_+(\mathsf{S}_1) \simeq \mathfrak{S}_+(\mathsf{S}_2)$ and $\mathfrak{E}_+(\mathsf{S}_2) \simeq \mathfrak{S}_+(\mathsf{S}_1)$.*
5. *If for two identical systems there exists a state that is preparationally faithful with respect to both systems, then one has the cone-isomorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$ (weak self-duality).*
6. *If the state $\Phi \in \mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$ is preparationally faithful with respect to S_1, for any invertible transformation $\mathscr{A} \in \mathfrak{T}_+(\mathsf{S}_1)$ also the (unnormalized) state $(\mathscr{A}, \mathscr{I})\Phi$*

[36] We say that two cones C_1 and C_2 are isomorphic (denoted as $\mathsf{C}_1 \simeq \mathsf{C}_2$), if there exists a one-to-one linear mapping between $\mathsf{Span}_\mathbb{R}(\mathsf{C}_1)$ and $\mathsf{Span}_\mathbb{R}(\mathsf{C}_2)$ that is cone-preserving in both directions. We will call such a map a cone-isomorphism between the two cones. Such a map will send extremal rays of C_1 to extremal rays of C_2, and positive linear combinations to positive linear combinations, and the same is true for the inverse map.

is preparationally faithful with respect to the same system. In particular, it will be a faithful state for any physical automorphism of $\mathfrak{S}(\mathsf{S}_1)$.[37]

7. *For identical systems in finite dimensions, for* Φ *preparationally faithful with respect to both systems, the state* $\chi := \Phi(e, \cdot)$ *is cyclic in* $\mathfrak{S}_+(\mathsf{S})$ *under* $\mathfrak{T}_+(\mathsf{S})$*, and the observables* $\mathbb{L} = \{l_i\}$ *of* S_2 *are in one-to-one correspondence with the ensemble decompositions* $\{\rho_i\}_{i=1}^{|\mathbb{L}|}$ *of* χ*, with* $\rho_i := \Phi(l_i, \cdot)$*, and* χ *is an internal state.*

Proof.

1. Introduce the map $\omega \mapsto \mathscr{T}_\omega$ where for every $\omega \in \mathfrak{S}(\mathsf{S}_2)$ one chooses a local transformation $\mathscr{T}_\omega$ on S_1 such that $(\mathscr{T}_\omega, \mathscr{I})\Phi|_2 = \omega$. This is possible because Φ is preparationally faithful with respect to S_1. One has $\mathscr{A}\omega = (\mathscr{T}_\omega, \mathscr{A})\Phi|_2 = (\mathscr{T}_\omega, \mathscr{I})(\mathscr{I}, \mathscr{A})\Phi|_2\ \forall \omega \in \mathfrak{S}(\mathsf{S}_2)$. Therefore, from $(\mathscr{I}, \mathscr{A})\Phi$ one can recover the action of $\mathscr{A}$ on any state ω by first applying $(\mathscr{T}_\omega, \mathscr{I})$ and then take the marginal, *i.e.* one recovers $\mathscr{A}$ from $(\mathscr{I}, \mathscr{A})\Phi$, which is another way of saying that $\mathscr{A} \mapsto (\mathscr{I}, \mathscr{A})\Phi$ is injective, namely Φ is dynamically faithful with respect to S_2.

2. Denote by $\Phi \in \mathfrak{S}^{\odot 2}$ a state that is preparationally faithful with respect to S_1. Since the linear map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ from $\mathfrak{T}_\mathbb{R}$ to $\mathfrak{S}_\mathbb{R}^{\odot 2}$ is surjective, one has $\dim(\mathfrak{T}_\mathbb{R}) \geqslant \dim(\mathfrak{S}_\mathbb{R}^{\odot 2})$. However, one has also $\dim(\mathfrak{T}_\mathbb{R}) \leqslant \dim(\mathfrak{S}_\mathbb{R}^{\odot 2})$ since $\mathfrak{T}_\mathbb{R} \subseteq \mathsf{Lin}(\mathfrak{S}_\mathbb{R}) \simeq \mathfrak{S}_\mathbb{R}^{\otimes 2} \subseteq \mathfrak{S}_\mathbb{R}^{\odot 2}$, whence $\dim(\mathfrak{T}_\mathbb{R}) = \dim(\mathfrak{S}_\mathbb{R}^{\odot 2})$, and, having null kernel, the map is also injective, whence Φ is dynamically faithful with respect to S_1. Since now the state Φ is both preparationally and dynamically faithful with respect to the same system S_1, it follows that the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ establishes the cone-isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$. Since the faithful state establishes the cone-isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$, it maps extremal rays of $\mathfrak{T}_+$ to extremal rays of $\mathfrak{S}_+^{\odot 2}$ and vice-versa, that is $\mathscr{A} \in \mathsf{Erays}(\mathfrak{T}_+)$ iff $(\mathscr{A}, \mathscr{I})\Phi \in \mathsf{Erays}(\mathfrak{S}_+^{\odot 2})$.

3. For Φ preparationally faithful with respect to S_1, consider the cone homomorphism $a \mapsto \omega_a := \Phi(a, \cdot)$ which associates an (un-normalized) state $\omega_a \in \mathfrak{S}_+(\mathsf{S}_2)$ to each effect $a \in \mathfrak{E}_+(\mathsf{S}_1)$. The extension to a linear map $a \mapsto \omega_a$ between the linear spaces $\mathfrak{E}_\mathbb{R}(\mathsf{S}_1)$ and $\mathfrak{S}_\mathbb{R}(\mathsf{S}_2)$ preserves the cone structure, and is surjective, since Φ is preparationally faithful with respect to S_1 (whence every bipartite state, and in particular every marginal state, can be obtained from a local effect). Remembering the definition of dimension for a system the bound $\dim(\mathsf{S}_1) \geqslant \dim(\mathsf{S}_2)$ follows from surjectivity.

4. Similarly to the proof of item (1), consider the map $\lambda \mapsto \mathscr{T}_\lambda$ where for every marginal state $\lambda \in \mathfrak{S}(\mathsf{S}_1)$ one chooses a local transformation $\mathscr{T}_\lambda$ on S_2 such that $(\mathscr{I}, \mathscr{T}_\lambda)\Phi|_1 = \lambda$ (Φ is preparationally faithful with respect to S_2). Then, one has

$$\forall \lambda \in \mathfrak{S}(\mathsf{S}_1), \quad \lambda(a) = (\mathscr{I}, \mathscr{T}_\lambda)\Phi(a, e) = \Phi(a, \mathscr{T}_\lambda) = \omega_a(\mathscr{T}_\lambda). \tag{2.57}$$

[37] One may be tempted to consider all automorphisms of $\mathfrak{S}(\mathsf{S}_1)$, instead of just the physical ones. However, there is no guarantee that any automorphism will be also an automorphism of bipartite states when applied locally. This is the case of QM, where the transposition is an automorphism of $\mathfrak{S}(\mathsf{S}_1)$, nevertheless is not a local automorphism of $\mathfrak{S}(\mathsf{S}_1 \odot \mathsf{S}_2)$.

It follows that $\omega_a = \omega_b$ implies that $\lambda(a) = \lambda(b)$ for all states $\lambda \in \mathfrak{S}(\mathsf{S}_1)$, that is $a = b$, whence the homomorphism $a \mapsto \omega_a$ which is surjective (since Φ is preparationally faithful) is also injective, *i.e.* is bijective, and since it maps elements of $\mathfrak{E}_+(\mathsf{S}_1)$ to elements of $\mathfrak{S}_+(\mathsf{S}_2)$ and, vice-versa, to each element of $\mathfrak{S}_+(\mathsf{S}_2)$ it corresponds an element of $\mathfrak{E}_+(\mathsf{S}_1)$ (Φ is preparationally faithful), it is a cone-isomorphism. We then have the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}_1) \simeq \mathfrak{S}_+(\mathsf{S}_2)$. The cone-isomorphism $\mathfrak{E}_+(\mathsf{S}_2) \simeq \mathfrak{S}_+(\mathsf{S}_1)$ follows by exchanging the two systems.

5. According to point (4) one has the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}_1) \simeq \mathfrak{S}_+(\mathsf{S}_2) \simeq \mathfrak{S}_+(\mathsf{S}_1)$.

6. Obvious, by definition of preparationally faithful state.

7. According to (4) $\omega_a := \Phi(a, \cdot)$ establishes the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}) \simeq \mathfrak{S}_+(\mathsf{S})$. On the other hand, since the state is both preparationally and dynamically faithful for either systems, then for any transformation $\mathscr{T}$ on the first system there exists a unique transformation $\mathscr{T}'$ on the other system giving the same output state (see also the definition of the "transposed" transformation with respect to a dynamically faithful state in the following). Therefore, since any effect a can be written as $a = e \circ \mathscr{T}_a$ for any $\mathscr{T}_a \in a$, one has $\omega_a = \Phi(e \circ \mathscr{T}_a, \cdot) = \Phi(e, \cdot \circ \mathscr{T}_a') = \mathscr{T}_a' \chi$. The observable-ensemble correspondence and the fact that χ is an internal state are both immediate consequence of the fact that $\omega_a := \Phi(a, \cdot)$ is a cone-isomorphism. ■

Operational definition of the transposed of a transformation. For a symmetric bipartite state Φ of two identical systems that is preparationally faithful for one system—hence, according to Theorem 2.2, is both dynamically and preparationally faithful with respect to both systems—one can define operationally the **transposed** $\mathscr{T}'$ of a transformation $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$ through the identity

$$\Phi(a, b \circ \mathscr{T}) = \Phi(a \circ \mathscr{T}', b), \tag{2.58}$$

i.e. $(\mathscr{T}', \mathscr{I})\Phi = (\mathscr{I}, \mathscr{T})\Phi$, namely, operationally the transposed $\mathscr{T}'$ of a transformation $\mathscr{T}$ is the transformation which will give the same output bipartite state of $\mathscr{T}$ if operated on the twin system. It is easy to verify (using symmetry of Φ) that $\mathscr{T}'' = \mathscr{T}$ and that $(\mathscr{B} \circ \mathscr{A})' = \mathscr{A}' \circ \mathscr{B}'$. This construction of the transposed transformation is very interesting because is a strong example of operational definition of a mathematical structure over the set of tests. Given a preparationally faithful state Φ, which is the same to say, given a probability rule for our experiments, we can "make an experiment to find the transposition of a transformation" and then define it as the result of that experiment. Naturally in practice isn't so easy to do such operation but it's certainly possible to do it. This ensures the operational nature of the mathematical properties involved in transposition. A mathematical set of rules is reproduced by an experiment.

Upon the definition of the principal objects involved in a probabilistic theory we can formulate the main postulate for the probabilistic theories landscape.

PFAITH

Figure 2.2: Illustration of the notions of transposed of a transformation for a symmetric dynamically and preparationally faithful state.

Postulate PFAITH: Existence of a symmetric preparationally-faithful pure state. *For any couple of identical systems, there exist a symmetric (under permutation of the two systems) pure state that is preparationally faithful.*

Theorem 2.2 guarantees that such a state is both dynamically and preparationally faithful, and with respect to both systems, as a consequence of symmetry.[38] Postulate PFAITH thus guarantees that to any system we can adjoin an ancilla and prepare a pure state which is dynamically and preparationally faithful with respect to our system. This is operationally crucial in guaranteeing the preparability of any quantum state for any bipartite system using only local transformations, and to assure the possibility of experimental calibrability of tests for any system. Notice that it would be impossible, even in principle, to calibrate transformations without a dynamically faithful state, since any set of input states $\{\omega_n\} \in \mathsf{S}'$ that is "separating" for transformations $\mathfrak{T}(\mathsf{S}')$ is equivalent to a bipartite state $\Phi = \sum_n \omega_n \otimes \lambda_n \in \mathfrak{S}(\mathsf{S}' \odot \mathsf{S}'')$ which is dynamically faithful for S', with the states $\{\lambda_n\}$ working just as "flags" representing the "knowledge" of which state of the set $\{\omega_n\}$ has been prepared. Notice that in QM every maximal Schmidt-number entangled state of two identical systems is both preparationally and dynamically faithful for both systems. In classical mechanics, on the other hand, a state of the form $\Phi = \sum_l |l\rangle\langle l| \otimes |l\rangle\langle l|$ with $\{|l\rangle\}$ complete orthogonal set of states (see footnote 17) will be both dynamically and preparationally faithful, however, being not pure, it would require a (possibly unlimited) sequence of preparations. In Chap. 6 we will construct a classical probabilistic theory and we will show that is such situation is impossible to find a pure preparationally and dinamically faithful state. All the states having this properties for a classical theory, namely a theory satisfying the CHSH inequality, are mixed states.

On the mathematical side, instead, according to Theorem 2.2 Postulate PFAITH restricts the theory to the weakly self-dual scenario (*i.e.* with the cone-isomorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$), and in finite dimensions one also has the cone-isomorphism $\mathfrak{T}_+(\mathsf{S}) \simeq \mathfrak{S}_+(\mathsf{S}^{\odot 2})$. In Part II we will construct models assuming the PFAITH postulate and in finite dimension then weak self-duality should be satisfied by all of them.

In addition, one also has the following very useful lemma.

Local observability principle First let us formulate the fundamental lemma:

[38] In fact, upon denoting by $\mathscr{T}_\Psi$ the local transformation such that $(\mathscr{T}_\Psi, \mathscr{I})\Phi = \Psi$, one has $(\mathscr{I}, \mathscr{T}_\Psi)(\mathscr{I}, \mathscr{S})\Phi = (\mathscr{I}, \mathscr{T}_\Psi \circ \mathscr{S})\Phi = (\mathscr{I}, \mathscr{T}_{\mathscr{S}\Psi})\Phi = \Psi$, $\mathscr{S}$ denoting the transformation swapping the two systems.

Lemma 2.4 *For finite dimensions Postulate PFAITH implies that the linear space of transformations is full, i.e.* $\mathfrak{T}_{\mathbb{F}} = \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}})$*. Moreover, one has* $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ *and* $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ *for* $\mathbb{F} = \mathbb{R}, \mathbb{C}$*, that is bipartite states and effects are cones spanning the tensor products* $\mathfrak{S}_{\mathbb{F}}^{\otimes 2}$ *and* $\mathfrak{E}_{\mathbb{F}}^{\otimes 2}$*, respectively.*

Proof. In the following we restrict to finite dimensions, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$ denoting either the real or the complex fields, respectively. According to item (2) of Theorem 2.2, for two identical systems the existence of a state that is preparationally faithful with respect to either one of the two systems implies $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_{\mathbb{F}}(\mathsf{S})$. Since transformations act linearly over effects one has $\mathfrak{T}_{\mathbb{F}} \subseteq \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}}) \simeq \mathfrak{E}_{\mathbb{F}}^{\otimes 2}$, whence $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_{\mathbb{F}}(\mathsf{S}) \subseteq \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$. However, by local-test embedding one also has $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \supseteq \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$, whence $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$, which implies that $\mathfrak{T}_{\mathbb{F}} = \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}})$. Finally, by state-effect duality one also has $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}^{\otimes 2}(\mathsf{S})$. ■

The previous lemma is of great importance because encapsulates the local observability principle. The equality

$$\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}, \tag{2.59}$$

proved in the Lemma is anything but trivial. It say us that every bipartite effect can be written as an element of the linear space $\mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ which is the tensor product of the linear spans of the local cones of effect $(\mathfrak{E}_{+}(\mathsf{S}))$. Then the local observability principle follows as a corollary of the Lemma 2.4:

Corollary 2.2 :Local observability principle *For every composite system there exist informationally complete observables made of local informationally complete observables.*

Proof. A joint observable made of local observables $\mathbb{L} = \{l_i\}$ on S_1 and $\mathbb{L} = \{m_j\}$ on S_2 is of the form $\mathbb{L} \times \mathbb{L} = \{(l_i, m_j)\}$. Then, by definition, the statement of the corollary is $\mathfrak{E}_{\mathbb{R}}(\mathsf{S}^{\odot 2}) \subseteq \mathsf{Span}_{\mathbb{R}}(\mathbb{L} \times \mathbb{L}) = \mathfrak{E}_{\mathbb{R}}^{\otimes 2}(\mathsf{S})$, which is true according to Lemma 2.4. ■

Observation 2.4 *Operationally, the Local Observability Principle plays a crucial role, since it reduces enormously experimental complexity, by guaranteeing that only local (although jointly executed) tests are sufficient to retrieve a complete information of a composite system, including all correlations between the components. This principle reconciles holism with reductionism in a non-local theory, in the sense that we can observe a holistic nature in a reductionistic way, i.e. locally.*

Observation 2.5 *In practice the local observability principle is equivalent to say that all transformations (or bipartite effect according to the isomorphism* $\mathfrak{T}_{\mathbb{F}} \simeq \mathfrak{E}_{\mathbb{F}}^{\odot 2}$*) can be represented by a matrix taking as orthonormal bases of the spaces the local informationally complte observables we have been speaking about in the Corollary.*

Observation 2.6 *Notice that, in finite dimension and taking* $\mathbb{F} = \mathbb{R}$*, it's possible to embed the space* $\mathfrak{T}_+(\mathsf{S})$ *into an Euclidean space* $\mathbb{R}^{n^2}$ *where* $n^2 = \dim(\mathsf{S}^{\odot 2}) = \dim(\mathsf{S}^{\otimes 2}) = \dim(\mathfrak{T}_+(\mathsf{S})) = \dim(\mathfrak{T}_\mathbb{R}(\mathsf{S})) = \dim(\mathsf{Lin}(\mathfrak{E}_\mathbb{R}))$*. Naturally* $\mathbb{R}^{n^2}$ *is isomorphic to both the linear spaces* $\mathfrak{T}_\mathbb{R}(\mathsf{S})$ *and* $\mathsf{Lin}(\mathfrak{E}_\mathbb{R})$ *which are the same linear space. At the same time* $\mathfrak{E}_+(\mathsf{S})$ *can be embedded in the Euclidean* $\mathbb{R}^n$ *where* $n = \dim(\mathsf{S}) = \dim(\mathfrak{E}_+(\mathsf{S})) = \dim(\mathfrak{E}_\mathbb{R}(\mathsf{S}))$*. Clearly* $\mathfrak{E}_\mathbb{R}$ *is isomorphic to* $\mathbb{R}^n$*. The same argumentation hold for the set of states and for the bipartite set of states and effects. In Part II we will study some probabilistic theories embedded in Euclidean spaces to make them concrete as quantum mechanics.*

In addition to Lemma 2.4 and to the local observability principle, Postulate PFAITH has a long list of remarkable consequences for the probabilistic theory, which are given by the following theorem(In the original work Ref. [D'A08] the following theorem contains one more statement concerning the bit commitment).

Theorem 2.3 *If PFAITH holds, the following assertions are true*

1. *The identity transformation is atomic.*
2. *One has* $\omega_{a\circ\mathscr{A}'} = \mathscr{A}\omega_a$*, or equivalently* $\mathscr{A}\omega = \Phi(a_\omega \circ \mathscr{A}', \cdot)$*, where* $\mathscr{A}'$ *denotes the transposed of* $\mathscr{A}$ *with respect to* Φ*.*
3. *The transpose of a physical automorphism of the set of states is still a physical automorphism of the set of states.*
4. *The marginal state* χ *is invariant under the transpose of a channel (deterministic transformation) whence, in particular, under a physical automorphism of the set of states.*

Proof.

1. According to Theorem 2.2-2, the map $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{I})\Phi$ establishes the cone-isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$, whence mapping extremal rays of $\mathfrak{T}_+$ to extremal rays of $\mathfrak{S}_+^{\odot 2}$ and vice-versa it maps the state Φ itself (which is pure) to the identity, which then must be atomic.
2. Immediate definition of the transposition with respect to the dynamically faithful state Φ.
3. Point (2) establishes that the transposed of a state-automorphism is an effect automorphism, which, due to the cone-isomorphism, is again a state-automorphism (see also footnote 37).
4. For deterministic $\mathscr{T}$ one has $\mathscr{T}'\chi = \Phi(e, \cdot \circ \mathscr{T}') = \Phi(e \cdot \mathscr{T}, \cdot) = \Phi(e, \cdot) = \chi$. The last statement follows from (3) (see also footnote 37). ■

Observation 2.7 *Notice that atomicity of identity occurs in QM, whereas it is not true in a classical probabilistic theory (see Footnote 17). In classical mechanics one can gain information on the state without making disturbance thanks to non-atomicity of* bilinear form

the identity transformation. According to Theorem 2.3-1 the need of disturbance for gaining information is a consequence of the purity of the preparationally faithful state, whence disturbance is the price to be payed for the reduction of the preparation complexity.

Scalar product over effects induced by a symmetric faithful state In this is briefly reviewed the construction in Ref. [D'A07a] of a scalar product over $\mathfrak{E}_{\mathbb{C}}$ via a symmetric faithful state, along with the corresponding operational definition of "transposed" and "complex conjugation"—with the composition of the two giving the adjoint. Naturally all these constructions are operational because arise from a state of the bipartite system.

According to Theorem 2.2-2, for two identical systems in finite dimensions any state that is preparationally faithful with respect to a system is also dynamically faithful with respect to the same system. Moreover, according to Postulate PFAITH, there always exists such a state, say Φ, which is symmetric under permutation of the two systems. The state Φ is then a symmetric real **bilinear form** over $\mathfrak{E}_{\mathbb{R}}$, whence it provides a non-degenerate scalar product over $\mathfrak{E}_{\mathbb{R}}$ via its Jordan form

$$\forall a, b \in \mathfrak{E}_{\mathbb{R}}, \quad {}_{\Phi}(b|a)_{\Phi} := |\Phi|(b, a) = (\Phi_+ - \Phi_-)(a, b) = \Phi(\varsigma_{\Phi}(b), a), \tag{2.60}$$

where ς_Φ is the involution[39]

$$\varsigma_\Phi = \pi_+ - \pi_-, \tag{2.61}$$

$\pi_\pm$ denoting the orthogonal projectors over the positive (negative) eigenspaces of the symmetric bilinear form, or, explicitly,

$$\varsigma_\Phi(a) := \sum_j \Phi(a, \tilde{f}_j)\tilde{f}_j \tag{2.62}$$

and $\{\tilde{f}_j\}$ is the canonical Jordan basis.[40] Notice that the Jordan form is representation-dependent—*i.e.* it is defined through the reference test $\mathbb{L} = \{l_i\}$—whereas its signature—*i.e.* the difference between the numbers of positive and negative eigenvalues—will be a property of the system S, and will generally depend on the specific probabilistic theory. The action of the involution $\varsigma_\Phi$correspond to a generalized transformation $\mathscr{Z}_\Phi \in \mathfrak{T}_{\mathbb{R}}$ according to the relation

$$\varsigma_\Phi(a) = a \circ \mathscr{Z}_\Phi{}^{41}. \tag{2.63}$$

Now the involution ς_Φ can be extended over $\mathscr{T}_{\mathbb{R}}$. Given a transformation $\mathscr{T} \in \mathfrak{T}_{\mathbb{R}}$ we define

$$a \circ \varsigma_\Phi(\mathscr{T}) := \varsigma_\Phi(\varsigma_\Phi(a) \circ \mathscr{T}), \tag{2.64}$$

from which

$$\varsigma_\Phi(\mathscr{T}) = \mathscr{Z}_\Phi \circ \mathscr{T} \circ \mathscr{Z}_\Phi.{}^{42} \tag{2.65}$$

[39]Notice that $\varsigma_\Phi \circ \varsigma_\Phi = \mathscr{I}$, namely ς_Φ is an involution.

[40]In the diagonalizing orthonormal basis one has $s_j\delta_{ij} = \Phi(\tilde{f}_i, \tilde{f}_j) = |\lambda_j|^{-1}\Phi(f_i, f_j)$, $s_j = \pm 1$, $\tilde{f}_j = f_j/\sqrt{|\lambda_j|}$.

[41]The explicit form of $\mathscr{Z}_\Phi$ can be obtained in terms of $\{\tilde{f}_j\}$ by equating Eqs. (2.62) and (2.63).

[42]Notice that from $\varsigma_\Phi \circ \varsigma_\Phi = \mathscr{I}$ follows $\mathscr{Z}_\Phi \circ \mathscr{Z}_\Phi = \mathscr{I}$ and then $\varsigma_\Phi(\mathscr{B} \circ \mathscr{A}) = \varsigma_\Phi(\mathscr{B}) \circ \varsigma_\Phi(\mathscr{A})$ according to the property of an involution.

Naturally for the identity transformation we have $\varsigma(\mathscr{I}) = \mathscr{X} \circ \mathscr{X} = \mathscr{I}$. Corresponding to a symmetric faithful bipartite state Φ and for a fixed orthonormal basis $\mathbb{L} = \{l_j\}$ one has the generalized transformation $\mathscr{T}_\Phi$, given by

$$a \circ \mathscr{T}_\Phi := \sum_k \Phi(l_k, a) l_k, \tag{2.66}$$

and in terms of the symmetric scalar product corresponding to $\mathbb{L} = \{l_j\}$ $(\cdot,\cdot)_\mathbb{L}$ introduced in Subsection 2.1.8, one has

$$(a, b \circ \mathscr{T}_\Phi)_\mathbb{L} = (a \circ \mathscr{T}_\Phi, b)_\mathbb{L} = \Phi(a, b). \tag{2.67}$$

Using the dynamical and preparational faithfulness of Φ we have defined operationally the transposed $\mathscr{T}'$ of a transformation $\mathscr{T} \in \mathfrak{T}_\mathbb{R}$. Such "operational" transposed is related to the transposed $\tilde{\mathscr{C}}$ under the scalar product $(\cdot,\cdot)_\mathbb{L}$ as $\mathscr{C}' = \mathscr{T}_\Phi \circ \tilde{\mathscr{C}} \circ \mathscr{T}_\Phi^{-1}$. It is easy to check that $\tilde{\mathscr{X}}_\Phi = \mathscr{X}_\Phi = \mathscr{X}'_\Phi$.

On the complex linear span $\mathfrak{T}_\mathbb{C}$ one can introduce a scalar product as the sesquilinear extension of the real symmetric scalar product $(\cdot,\cdot)_\Phi$ over $\mathfrak{E}_\mathbb{R}$ via the complex conjugation $\eta(\mathscr{T}) = \mathscr{T}_R - i\mathscr{T}_I$, $\mathscr{T}_{R,I} \in \mathfrak{T}_\mathbb{R}$, and the adjoint for the sesquilinear scalar product is then given by

$$\mathscr{T}^\dagger = \mathscr{X}_\Phi \circ \eta(\mathscr{T}') \circ \mathscr{X}_\Phi = |\mathscr{T}_\Phi| \circ \eta(\tilde{\mathscr{T}}) \circ |\mathscr{T}_\Phi|^{-1}, \tag{2.68}$$

namely $\mathscr{T}^\dagger = \mathscr{X}_\Phi \circ \mathscr{T}' \circ \mathscr{X}_\Phi$ on real transformations $\mathscr{T} \in \mathfrak{T}_\mathbb{R}$. The Jordan involution ς thus plays the role of a complex conjugation on $\mathfrak{T}_\mathbb{R}$, which must be anti-linearly extended to $\mathfrak{T}_\mathbb{C}$.

The faithful state Φ becomes a cyclic and separating vector of a GNS representation by noticing that $(\mathscr{A}^{(2)}\Phi)(\eta\varsigma b, a) = {}_\Phi(b, a \circ \mathscr{A})_\Phi$,[43]

2.3 Exploring Postulate FAITH and Postulate PURIFY

In this section we are exploring two additional postulates of a probabilistic theory: Postulate FAITHE—the existence of a faithful effect—and Postulate PURIFY—the existence of a purification for every state. The Postulate FAITH is not exactly the dual of PFAITH, which is a corollary of the last one, the Postulate introduced in the following is stronger of the dual version of PFAITH and imply the possibility of achieve teleportation. As we will see, the two new postulates make the probabilistic theory closer and closer to Quantum Mechanics. On the other hand it isn't still Quantum Mechanics and in Part II a counterexample to this will be found. Making stronger the following Postulate PURIFY, as we will better investigate, also this counterexample will be excluded from the probabilistic theories landscape.

Preparationally faithful effects. Before introducing the new postulates we can show how the dual version of Postulate PFAITH is a consequence of this one. We have

[43]The action of the algebra of generalized transformations on the first system corresponds to the transposed representation $(\mathscr{A}^{(1)}\Phi)(\eta\varsigma b, a) = \Phi(\eta\varsigma b \circ \mathscr{A}, a) = \Phi(\eta\varsigma b, a \circ \mathscr{A}') = (\mathscr{A}'^{(2)}\Phi)(\eta\varsigma b, a)$.

FAITHE

defined a preparationally faithful state for a couple of identical systems. In the same way, given a couple of identical systems $\mathsf{S}^{\odot 2}$, we will say that the bipartite effect $F \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ is a symmetric pure preparationally faithful effect if every bipartite effect A can be achieved as $A = F(\mathscr{T}_A, \mathscr{I})$ by a local transformation $\mathscr{T}_A \in \mathfrak{T}_+(\mathsf{S})$. Or, which is the same, the cone homomorphism $\mathscr{A} \to F(\mathscr{A}, \mathscr{I})$ from $\mathfrak{T}_+(\mathsf{S})$ to $\mathfrak{E}_+(\mathsf{S}^{\odot 2})$ is an epimorphism.

Lemma 2.5 *If Postulate PFAITH holds then for any couple of identical systems, there exist also a symmetric (under permutation of the two systems) pure effect that is preparationally faithful.*

Proof. From Postulate PFAITHthere exist a pure preparatonally faithful state $\Phi \in \mathfrak{S}(\mathsf{S}^{\odot 2})$. Moreover, considering the quadripartite system $\mathsf{S}^{\odot 4}$, Postulate PFAITH ensure the existence of a state $\Omega_{quad} \in \mathfrak{S}(\mathsf{S}^{\odot 4})$ which is a symmetric pure state preparationally faithful with respect to the bipartite system $\mathsf{S}^{\odot 2}$. According to Theorem 2.2-5 the state Ω_{quad} provide the isomorphism $\mathfrak{E}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{S}(\mathsf{S}^{\odot 2})$ and then we can take the effect in $\mathfrak{E}(\mathsf{S}^{\odot 2})$ such that

$$\Omega_{quad}(F, \cdot, \cdot) = \beta\Phi \qquad 0 < \beta \leqslant 1. \tag{2.69}$$

Since Φ is pure then also F is a pure effect according to the definition of isomorphism between cones. Now we have to show that F is preparationally faithful too. From the faithfulness of Φ, for each $\Psi \in \mathfrak{S}(\mathsf{S}^{\odot 2})$ there exist a local transformation $\mathscr{T}_\Psi$ such that $(\mathscr{T}_\Psi, \mathscr{I})\Phi = \Psi$, namely, according to Eq. (2.69),

$$\forall\Psi \in \mathfrak{S}(\mathsf{S}^{\odot 2}) \qquad \exists\mathscr{T}_\Psi \in \mathfrak{T}_+(\mathsf{S}) \text{ such that } \Big((\mathscr{T}_\Psi, \mathscr{I}), \mathscr{I}_{bip}\Big)\Omega_{quad}(F, \cdot, \cdot) = \beta\Psi, \tag{2.70}$$

or, which is the same,

$$\forall\Psi \in \mathfrak{S}(\mathsf{S}^{\odot 2}) \qquad \exists\mathscr{T}_\Psi \in \mathfrak{T}_+(\mathsf{S}) \text{ such that } \Omega_{quad}\Big(F \circ (\mathscr{T}_\Psi, \mathscr{I}), \cdot, \cdot\Big) = \beta\Psi. \tag{2.71}$$

According to the cone isomorphism $\mathfrak{E}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{S}(\mathsf{S}^{\odot 2})$ and denoting by A_Ψ the effect

$$A_\Psi = F \circ (\mathscr{T}_\Psi, \mathscr{I}) \tag{2.72}$$

we get the faithfulness of F. ■

2.3.1 FAITHE: a postulate on a faithful effect

As previously mentioned, Postulate FAITHE is stronger than the dual version of Postulate PFAITH in Lemma 2.5:

Postulate FAITHE: Existence of a faithful effect achieving probabilistic inversion of a faithful state. *There exist a bipartite effect $F \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ achieving the inverse of the isomorphism $a \mapsto \omega_a := \Phi(a, \cdot)$. More precisely*

$$F_{23}(\omega_a)_2 = F_{23}\Phi_{12}(a, \cdot) = \alpha a_3, \quad 0 < \alpha \leqslant 1. \tag{2.73}$$

teleportation

Notice that, since Φ establishes an isomorphism between the cones of states and effects, there must exist a generalized effect $F \in \mathfrak{E}_{\mathbb{R}}^{\otimes 2}$ satisfying Eq. (2.73), but we are not guaranteed that it is a physical, *i.e.* $F \in \mathfrak{E}_{+}(\mathsf{S}^{\odot 2})$. Moreover the preparationally faithful effect whose existence is ensured by Lemma 2.5 in general doesn't achieve the inverse of the isomorphism in Postulate FAITH. Tshe last one assumes something stronger than the simply faithfulness of the effect F.

Let's denote by $\hat{F} = \alpha^{-1} F$ the rescaled effect in the cone. Eq. (2.73) can be rewritten in different notation as follows

$$\hat{F}(\omega_a, \cdot) = \hat{F}(\Phi(a, \cdot), \cdot) = a \tag{2.74}$$

$$\Phi(a_\omega, \cdot) = \Phi(\hat{F}(\omega, \cdot), \cdot) = \omega. \tag{2.75}$$

[One needs to be careful with the notation in the multipartite case, *e.g.* in Eq. (2.75) $\Phi(\hat{F}(\omega, \cdot), \cdot) = \omega$ is actually a state, since $\hat{F}(\omega, \cdot)$ is an effect, etc.] Both faithful state Φ and faithful effect F can be used to express the state-effect pairing, namely

$$\zeta(b) = \Phi(a_\zeta, b) = \hat{F}(\omega_b, \zeta), \qquad a_\zeta := \hat{F}(\zeta, \cdot), \quad \omega_b := \Phi(b, \cdot), \tag{2.76}$$

or, substituting

$$\zeta(b) = \Phi(\hat{F}(\zeta, \cdot), b) = \hat{F}(\Phi(b, \cdot), \zeta). \tag{2.77}$$

Eq. (2.73) can also be rewritten as follows

$$F_{23}\Phi_{12} = \alpha\, \mathbf{Swap}_{13}, \tag{2.78}$$

where $\mathbf{Swap}_{ij}$ denotes the transformation swapping S_i with S_j. In Fig. 2.3 Postulate FAITHE is illustrated graphically. Eq. (2.78) means that using the state Φ and the

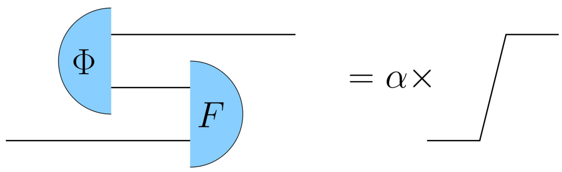


Figure 2.3: Illustration of Postulate FAITHE.

effect F one can achieve probabilistic **teleportation** of states from S_2 to S_4. In fact, one has

$$F_{23}\omega_2\Phi_{34} = F_{23}\Phi_{12}(a_\omega, \cdot)\Phi_{34} = \alpha\Phi_{14}(a_\omega, \cdot) = \alpha\omega_4. \tag{2.79}$$

Using the last identity we can also see that Postulate FAITHE is also equivalent to the identity

$$F_{23}\Phi_{12}\Phi_{34} = \alpha\Phi_{14}, \tag{2.80}$$

which by linearity is extended from local effects to all effects, in virtue of $\mathfrak{E}^{\odot 2} = \mathfrak{E}^{\otimes 2}$. With equivalent notation we can write

$$(\Phi, \Phi)(\cdot, F, \cdot) = \alpha\Phi. \tag{2.81}$$

Observation 2.8 *At first sight it seems that the existence of an effect F such that $F_{23}\Phi_{12}\Phi_{34} = \alpha\Phi_{14}$ could be derived directly from PFAITH. Indeed, according to Lemma 2.4 for finite dimensions and identical systems we have $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ and $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Moreover, according to Theorem 2.2-4 the map $a \mapsto \omega_a = \Phi(a, \cdot)$, for Φ symmetric preparationally faithful achieves the cone-isomorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$, whence for the bipartite system one has $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_+(\mathsf{S}^{\odot 2})$. This leads one to think that it should be possible to achieve a preparationally faithful state for $\mathsf{S}^{\odot 4}$ as thegproduct $\Phi_{12}\Phi_{34}$. However, this is not necessarily true. In fact, the map $\mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2} \ni E \mapsto \Omega_E = E_{23}\Phi_{12}\Phi_{34}$ is a linear bijection between $\mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ and $\mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$ [since $\mathsf{Span}_{\mathbb{F}}\{\Phi_{12}(\cdot, a)\Phi_{34}(b, \cdot) | a, b \in \mathfrak{E}\} = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2} = \mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2})$], is cone-preserving, it sends separable effects to separable states, whence it sends non-separable effects to non-separable states (since it is one-to-one). However, it doesn't necessarily achieve the cone-isomorphism $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_+(\mathsf{S}^{\odot 2})$, since it is not necessarily true that any bipartite state Ω is the mapped of a bipartite effect E_Ω (we remember that a cone-isomorphism is a bijection that preserves the cone* in both directions*). If by chance this would be the case—i.e. $E \mapsto \Omega_E$ is a cone-isomorphism for $\mathsf{S}^{\odot 2}$—then this means that there exists an effect $F \in \mathfrak{E}(\mathsf{S}^{\odot 2})$—such that $\Omega_F = \alpha\Phi$, with* $0 < \alpha \leqslant 1$.

On the other hand Postulate FAITH implies the statement of Lemma 2.5. Precisely, if FAITH holds then the effect F is also completely faithful, in the sense that the correspondence $F_{\mathscr{A}} := F \circ (\mathscr{A}', \mathscr{I}) \Longleftrightarrow \mathscr{A}$ is bijective (in finite dimensions). In fact one has

$$[F \circ (\mathscr{A}', \mathscr{I})]_{23}(\Phi, \Phi) = \alpha(\mathscr{A}, \mathscr{I})\Phi, \tag{2.82}$$

and since Φ is dynamically faithful (it is symmetric preparationally faithful and then it is also dynamically faithful according to Theorem 2.2-2), the correspondence $F_{\mathscr{A}} := F \circ (\mathscr{A}', \mathscr{I}) \Longleftrightarrow \mathscr{A}$ is one-to-one and surjective, whence it is a bijection (in finite dimensions). It is also easy to see that $[F \circ (\mathscr{A}', \mathscr{I})] = F \circ (\mathscr{I}, \mathscr{A})$, since

$$\begin{aligned}[F \circ (\mathscr{I}, \mathscr{A})]_{23}(\Phi, \Phi) &= F_{23}(\Phi, (\mathscr{A}, \mathscr{I})\Phi) = F_{23}(\Phi, (\mathscr{I}, \mathscr{A}')]\Phi) \\ &= \alpha(\mathscr{I}, \mathscr{A}')\Phi = \alpha(\mathscr{A}, \mathscr{I})\Phi = [F \circ (\mathscr{A}', \mathscr{I})]_{23}(\Phi, \Phi),\end{aligned} \tag{2.83}$$

whence transposition can be equivalently defined with respect to the faithful effect F. The bijection $F_{\mathscr{A}} := F \circ (\mathscr{I}, \mathscr{A}) \Longleftrightarrow \mathscr{A}$ is cone-preserving in both directions, since to every transformation it corresponds an effect, and to each effect $A \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ it corresponds a transformation, since

$$A_{23}(\Phi, \Phi) = \Omega_A = (\mathscr{T}_{\Omega_A}, \mathscr{I})\Phi =: (\mathscr{T}_A, \mathscr{I})\Phi. \tag{2.84}$$

Therefore, the map $\mathscr{A} \mapsto F_{\mathscr{A}}$ realizes the cone-isomorphism $\mathfrak{E}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_+(\mathsf{S})$ which is just the composition of the weak-selfduality and of the isomorphism $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_+(\mathsf{S})$ due to PFAITH. However, as mentioned in footnote 2.8, the map

$$\mathfrak{E}_+(\mathsf{S}^{\odot 2}) \ni A \mapsto \Omega_A := A_{23}(\Phi, \Phi) \in \mathfrak{S}(\mathsf{S}^{\odot 2}), \tag{2.85}$$

is bijective between $\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2})$ and $\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2})$, but it does not realize the cone-isomorphism $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_+(\mathsf{S}^{\odot 2})$, since it is not surjective over $\mathfrak{E}_+(\mathsf{S}^{\odot 2})$. Indeed, for $A \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ physical effect, one has $A_{23}(\Phi, \Phi) = (\mathscr{T}_A, \mathscr{I})\Phi$ with $\mathscr{T}_A \in \mathfrak{T}(\mathsf{S})$ physical transformation. However, there is no guarantee that, vice-versa, a physical transformation always has a corresponding physical effect, *e.g.* for the identity transformation in Eq. (2.80). It also follows that any bipartite observable $\mathbb{A} = \{A_l\}$ leads to the **totally depolarizing channel** $\mathscr{T}_{(e,e)}\omega = \chi$, $\forall \omega \in \mathfrak{S}$.[44] Using the faithfulness of F it is possible to achieve probabilistically any transformation on a state ω by performing a joint test on the system interacting with an ancilla, *i.e.* $(\omega\Phi)(F_{\mathscr{A}'}, \cdot) = \alpha\mathscr{A}\omega$ (for Stinespring-like dilations in an operational context see Ref. [CDP09]).

More about the constant α. Notice that the number $0 < \alpha \leqslant 1$ is the probability of achieving teleportation

$$\alpha = (F_{23}\omega_2\Phi_{34})(e). \tag{2.86}$$

From the last equation seem that α depends on the local state ω, but this is not the case. Indeed it depends only on F (and then on the faithful state Φ inverted by F), since it is given by

$$\alpha \equiv \alpha_F = [F_{23}\Phi_{12}\Phi_{34}](e, e). \tag{2.87}$$

The maximum value maximized over all bipartite effects

$$\alpha(\mathsf{S}) = \max_{A \in \mathfrak{E}(\mathsf{S}^{\odot 2})} \{(\Phi, \Phi)(e, A, e)\} \tag{2.88}$$

is a property of the system S only, and depends on the particular probabilistic theory. We will look for this constant in the concrete probabilistic models in Part II.

More on the relation between Postulates PFAITH and FAITHE. Postulate PFAITH guarantees the existence of a symmetric preparationally faithful state for each pair of identical systems $\mathsf{S}^{\odot 2}$. Now, consider the bipartite system $\mathsf{S}^{\odot 2} \odot \mathsf{S}^{\odot 2}$, and denote by Φ_{quad} a symmetric preparationally faithful state for it. The map

$$A \mapsto \Omega_A := \Phi_{quad}(A, \cdot, \cdot) \forall A \in \mathfrak{E}(\mathsf{S}^{\odot 2}) \tag{2.89}$$

establishes the state–effect cone-isomorphism for $\mathsf{S}^{\odot 2}$, whence there must exists an effect A_Φ such that

$$\Phi_{quad}(A_\Phi, \cdot, \cdot) = \beta\Phi, \quad 0 < \beta \leqslant 1. \tag{2.90}$$

Suppose now that the faithful state can be chosen in such a way that it maps separable effects to separable states as follows

$$\Phi_{quad}(\cdot, \cdot, (a, b)) = \gamma(\omega_a, \omega_b) = \gamma\Phi(\cdot, a)\Phi(\cdot, b), \ \gamma > 0. \tag{2.91}$$

Then one has

$$\gamma(A_\Phi)_{13}(\Phi, \Phi) = \Phi_{quad}(A_\Phi, \cdot, \cdot) = \beta\Phi, \tag{2.92}$$

[44] Indeed, one has $\sum_l (A_l)_{23}\omega_2\Phi_{34} = (e, e)_{23}\omega_2\Phi_{34} = \Phi_{12}(a_\omega, e)\Phi_{34}(e, \cdot) = \omega(e)\chi$.

super-faithful

namely, according to Eq. (2.80) one has $\beta^{-1}\gamma A_\Phi \equiv \hat{F}$, which is the effect whose existence is postulated by FAITHE. Notice, however, that the factorization Eq. (2.91) doesn't need to be satisfied. In other words, the automorphism relating the cone-isomorphism induced by Φ_{quad} with another cone-isomorphism that preserves local effects may be unphysical (see also footnote 2.8). One can instead require a stronger version of postulate PFAITH, postulating the existence of a preparationally **super-faithful** symmetric state Φ, also achieving a four-partite preparationally symmetric faithful state Φ_{quad} as $(\Phi,\Phi) = \Phi_{quad}$. A weaker version of such postulate is thoroughly analyzed in Ref. [CDP09], where it is also shown that it leads to Stinespring-like dilations of deterministic transformations.

The case of QM It is a useful exercise to see how the present framework translates in the quantum case, and find which additional constraints can arise from a specific probabilistic theory. For simplicity we consider a maximally entangled state (with all positive amplitudes in a fixed basis) as a preparationally symmetric state Φ. The corresponding marginal state is given by the density matrix $d^{-1}I$, I denoting the identity on the Hilbert space. For the constant α one has $\alpha = d^{-2}$, where d is the dimension of the Hilbert space. A simple calculation shows that the identity $\omega_a = \mathscr{T}_a'\chi$ for $\mathscr{T}_a \in a$[45] translates to[46]

$$\omega_a = \sqrt{\alpha}\varsigma(a), \qquad \Leftarrow \text{in QM} \tag{2.93}$$

where the involution ς of the Jordan form in Eq. Φ (2.60) here is also an automorphism of states/effects, whence identity (2.93) expresses the self-duality of QM. Rewriting Eq. (2.93) in terms of the faithful effect F (which would be an element of a Bell measurement), one obtains[47]

$$(\cdot, F)(\Phi, \cdot) = \sqrt{\alpha}|\Phi|, \qquad \Leftarrow \text{in QM.} \tag{2.94}$$

Another feature of QM is that the preparationally faithful symmetric state Φ is super-faithful, namely $\Phi_{quad} = (\Phi,\Phi)$ is preparationally faithful for $\mathsf{S}^{\odot 4}$.

Observation 2.9 *Notice that is a probabilistic theory admit a super-faithful state Φ, then Postulate PFAITH is automatically satisfied. In fact considering the symmetric faithful quadripartite state $\Phi_{quad} = (\Phi,\Phi)$, according to the isomorphism $\mathfrak{E}_+^{\odot 2} \simeq \mathfrak{S}_+^{\odot 2}$ we can find a bipartite effect $F \in \mathfrak{E}^{\odot 2}$ such that*

$$(\Phi,\Phi)(\cdot, F_\Phi, \cdot) = \alpha\Phi, \tag{2.95}$$

as required by PFAITH.

---

[45] Recall that $\omega_a = \Phi(a,\cdot) = \Phi(e \circ \mathscr{T}_a, \cdot) = \Phi(e, \cdot \circ \mathscr{T}_a') = (\mathscr{I}, \mathscr{T}_a')\Phi(e,\cdot) = \mathscr{T}_a'\chi$

[46] For $\Phi = d^{-1}\sum_{nm}|n\rangle|n\rangle\langle m|\langle m|$ the marginal state is $\chi = d^{-1}I$ and the Jordan involution is the complex conjugation with respect to the orthonormal basis $\{|n\rangle\}$. For quantum operation $\mathscr{T} = \sum_n T_n \cdot T_n^\dagger$ with corresponding effect $a = \sum_n T_n^\dagger T_n$, one has $\mathscr{T}'\chi = d^{-1}\sum_n {T_n}^\tau T_n^* = d^{-1}\sum_n (T_n^\dagger T_n)^* = \sqrt{\alpha}\varsigma(a)$.

[47] In fact, one has $\omega_a := \Phi(a,\cdot) = \sqrt{\alpha}\varsigma(a)$, namely $\Phi(\varsigma(a),\cdot) = \sqrt{\alpha}a$, *i.e.* $|\Phi|(a,\cdot) = \sqrt{\alpha}a$, and using Eq. (2.74) one has $\sqrt{\alpha}\hat{F}(\Phi(a,\cdot),\cdot) = |\Phi|(a,\cdot)$, namely the statement.

2.3.2 PURIFY: a postulate on purifiability of all states

PURIFY
entanglement swapping

Inspired by Quantum Mechanics a good property to be explored is the purifiability of states. In the present section for completeness the consequences of assuming purifiability as a postulate are briefly mentioned. The Postulate is the following for all states, namely:

Postulate PURIFY: Purifiability of states. *For every state ω of* S *there exist a pure bipartite state Ω of* $\mathsf{S}^{\odot 2}$ *having it as marginal state, namely*

$$\forall\omega \in \mathfrak{S}(\mathsf{S}),\ \exists\Omega \in \mathfrak{S}(\mathsf{S}^{\odot 2}) \textit{ pure, such that } \Omega(e,\cdot) = \omega. \tag{2.96}$$

Postulate PURIFY is of great importance because probably leads us very close to Quantum Mechanics. A deep analysis of PURIFY is in Ref. [CDP09], where the following Lemma is proved

Lemma 2.6 *If Postulate PFAITH holds, then Postulate PURIFY implies the following assertions*

1. *Even without assuming purity of the preparationally faithful state Φ, the identity transformation is atomic, and purity of Φ can be derived.*
2. $\mathfrak{S}_+ \equiv \mathsf{Erays}(\mathfrak{T}_+)\chi$, *i.e. each state can be obtained by applying an atomic transformations to the marginal state $\chi := \Phi(e,\cdot)$.*
3. $\mathfrak{E}_+ \equiv e \circ \mathsf{Erays}(\mathfrak{T}_+)$, *i.e. each effect can be achieved with an atomic transformation.*

Points (2) and (3) corresponds to the square-root of states and effects in the quantum case.

Another very interesting result included in the same work Ref. [CDP09] is a stronger version of Postulate PFAITH having the **entanglement swapping** (and then the probabilistic teleportation) as a consequence. We introduce here this result because it is of some relevance in the context of this thesis (see Part II).

2.4 Quantum mechanics as a particular probabilistic theory

All the postulates introduced until us are the most rational features of quantum mechanics. Nevertheless just these properties are not enough to reproduce all the Quantum mechanics features. In the future probably we will be able to achieve the Quantum theory by adding some operational postulate in the general probabilistic theories formulation but at the moment the uniqueness of Quantum mechanics can be find only through a mathematical postulate. That is the content of this section

Atomicity of evolution
Choi-Jamiolkowski isomorphism

The mathematical representation of the operational probabilistic framework derived up to now is completely general for any fair operational framework that allows local tests, test-calibration, and state preparation. These include not only QM and classical-quantum hybrid, but also other non-signaling non-local probabilistic theories such as the *PR-boxes* theories [RP94] which is the first model analysed in Part II. Postulate PFAITH has proved to be remarkably powerful, implying (1) the local observability principle, (2) the tensor-product structure for the linear spaces of states and effects, (3) weak self-duality, (4) realization of all states as transformations of the marginal faithful state $\Phi(e,\cdot)$, (5) locally indistinguishable ensembles of states corresponding to local observables—*i.e.* impossibility of bit commitment—and more. By adding FAITHE one even has teleportation! However, despite all these positive landmarks, it is still unclear if one can derive QM from these principles only. What is then special about QM? The peculiarity of QM among probabilistic operational theories is:

Effects not only can be linearly combined, but also they can be composed each other, so that complex effect make a C*-algebra.

Operationally the last assertion is odd, since *the notion of effect abhors composition!* Therefore, the composition of effects (*i.e.* the fact that they make a C*-algebra, *i.e.* an operator algebra over complex Hilbert spaces) must be derived from additional postulates:

With a single mathematical postulate, and assuming atomicity of evolution, one can derive the composition of effects in terms of composition of atomic events.

One thus is left with the problem of translating the remaining mathematical postulate into an operational one. Let's now expose the two postulates.

Postulate AE: Atomicity of evolution. *The composition of atomic transformations is atomic.*

This postulate is so natural that looks obvious. Indeed, when joining events $\mathscr{A}$ and $\mathscr{B}$ into the event $\mathscr{A} \wedge \mathscr{B}$, the latter is atomic if both $\mathscr{A}$ and $\mathscr{B}$ are atomic. However, even though for atomic events $\mathscr{A}$ and $\mathscr{B}$ the event $\mathscr{C} = \mathscr{B} \circ \mathscr{A}$ is not refinable in the corresponding cascade-test, there is no guarantee that $\mathscr{C}$ is not refinable in any other test. This means that there will exist a set of transformations of the system in exam included in some tests, different from the test $\mathbb{A} \circ \mathbb{B}$, whose convex combination is exactly the transformation $\mathscr{A} \circ \mathscr{B}$. We remember that mathematically atomic events belong to $\mathsf{Erays}(\mathfrak{T}_+)$, the extremal rays of the cone of transformations.

We now state the only mathematical Postulate up to now:

Mathematical Postulate CJ: Choi-Jamiolkowski isomorphism. *The cone of transformations is isomorphic*[48] *to the cone of positive bilinear forms over complex effects [Cho75, Jam72], i.e.* $\mathfrak{T}_+ \simeq \mathsf{Lin}_+(\mathfrak{E}_\mathbb{C})$.

[48]For the definition of cone-isomorphisms, see Footnote 36.

Notice that in terms of a sesquilinear scalar product over complex effects, positive bilinear forms can be regarded as a positive matrices over complex effects, *i.e.* elements of the cone $\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C})$.

The extremal rays $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$ are rank-one positive operators

$$|x\rangle\langle x| \in \mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C})) \tag{2.97}$$

with $x \in \mathfrak{E}_\mathbb{C}$, and the map

$$\pi : x \mapsto \pi(x) := |x\rangle\langle x| \tag{2.98}$$

is surjective over $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$. One has $\pi(xe^{i\phi}) = \pi(x)$, and $\pi^{-1}(|x\rangle\langle x|) = \{e^{i\phi}x\} \subseteq \mathfrak{E}_\mathbb{C}$, *i.e.* the set of complex effects mapped to the same rank-one positive operator is the set of complex effects that differ only by a multiplicative phase factor. We will denote by $|x| \in \mathfrak{E}_\mathbb{C}$ a fixed choice of representative for such an equivalence class,[49] introduce the phase corresponding to such choice as $x =: |x|e^{i\phi(x)}$, and denote by $\mathfrak{E}_\mathbb{C}/\phi$ the set of equivalence classes, or, equivalently, of their representatives. Now, since the representatives $|x| \in \mathfrak{E}_\mathbb{C}/\phi$ are in one-to-one correspondence with the points on $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$, the CJ isomorphism establishes a bijective map between $\mathfrak{E}_\mathbb{C}/\phi$ and $\mathsf{Erays}(\mathfrak{T}_+)$ as follows

$$\tau : \ \mathfrak{E}_\mathbb{C}/\phi \ni |x| \leftrightarrow \tau(|x|) \in \mathsf{Erays}(\mathfrak{T}_+). \tag{2.99}$$

2.4.1 Building up an associative algebra structure for complex effects

The composition of effects can now be defined. Assuming Postulate AE, we can introduce an associative composition between the effects in $\mathfrak{E}_\mathbb{C}/\phi$ via the bijection τ

$$|a||b| := \tau^{-1}(\tau(|a|) \circ \tau(|b|)). \tag{2.100}$$

Notice that, by definition, $|a||b|$ is a representative of an equivalence class in $\mathfrak{E}_\mathbb{C}$, whence $|\,(|a||b|)\,| = |a||b|$. The above composition extends to all elements of $\mathfrak{E}_\mathbb{C}$ by taking

$$ab := |a||b|e^{i\phi(a)}e^{i\phi(b)}, \tag{2.101}$$

and since $|\,(|a||b|)\,| = |a||b|$, one has $|ab| = |a||b|$, and $\phi(ab) = \phi(a) + \phi(b)$. It follows that the extension is itself associative, since

$$\begin{aligned}(ab)c =&|ab||c|e^{i\phi(ab)+i\phi(c)} = |a||b||c|e^{i\phi(a)+i\phi(b)+i\phi(c)}\\ =&|a||bc|e^{i\phi(a)+i\phi(bc)} = a(bc).\end{aligned} \tag{2.102}$$

The composition is also distributive with respect to the sum, since it follows the same rules of complex numbers. We will denote by ι the identity in $\mathfrak{E}_\mathbb{C}/\phi$ when it exists, which also works as an identity for multiplication of effects as in Eq. (2.101). Notice that since the identity transformation $\mathscr{I}$ is atomic, one has $\iota := \tau^{-1}(\mathscr{I}) \in \mathfrak{E}_\mathbb{C}/\phi$ according to Eq. (2.100).

[49] An example of choice of representative is given by $\||x|\rangle := \langle e_{\iota(x)}|\pi(x)|e_{\iota(x)}\rangle^{-\frac{1}{2}}\pi(x)|e_{\iota(x)}\rangle$, namely $|x| := |(x, e_{\iota(x)})|^{-1}(x, e_{\iota(x)})x$, with with $\iota(x) = \min\{i : (x, e_i) \neq 0\}$, for given fixed basis for $\mathfrak{E}_\mathbb{C}$.

2.4.2 Building up a C*-algebra structure over complex effects

We want now to introduce a notion of adjoint for effects in order to equip the algebra in the subsection above with an involution confering it a C* algebra structure. We will do this in two steps:(a) we introduce an antilinear involution on the linear space $\mathfrak{E}_{\mathbb{C}}$; (b) we extend the associative product (2.101) under such antilinear involution.

(a) First we notice that the complex space $\mathfrak{E}_{\mathbb{C}}$ has been constructed as $\mathfrak{E}_{\mathbb{C}} = \mathfrak{E}_{\mathbb{R}} \oplus i\mathfrak{E}_{\mathbb{R}}$ starting from real combinations of physical effects $\mathfrak{E}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathfrak{E}_+)$, *i.e.* one has the unique Cartesian decomposition $x = x_R + ix_I$ of $x \in \mathfrak{E}_{\mathbb{C}}$ in terms of $x_R, x_I \in \mathfrak{E}_{\mathbb{R}}$. We can then define the antilinear *dagger* involution † on $\mathfrak{E}_{\mathbb{C}}$ by taking $x^\dagger = x\ \forall x \in \mathfrak{E}_{\mathbb{R}}$ and $x^\dagger := x_R - ix_I\ \forall x \in \mathfrak{E}_{\mathbb{C}}$. Notice that $\mathfrak{E}_{\mathbb{C}}$ is closed under such involution. Taking the involution of the defining identity $x =: |x|e^{i\phi(x)}$ one has $|x^\dagger| = |x|^\dagger e^{-i\phi(x^\dagger)-i\phi(x)}$ which is consistently satisfied by choosing $|x^\dagger| = |x|^\dagger$ and $\phi(x^\dagger) = -\phi(x)\ \forall x \in \mathfrak{E}_{\mathbb{C}}$ (these identities are satisfied *e.g.* for the choice of representative in Footnote 49).

(b) It's necessary to define the left and right multiplication between effects and the involved effect to extent our algebra. The multiplications $a^\dagger b$ and $ab^\dagger$ are defined via the chosen scalar product over $\mathfrak{E}_{\mathbb{C}}$ as follows[50]

$$\forall c \in \mathfrak{E}_{\mathbb{C}} : \qquad (c, a^\dagger b) := (ac, b), \qquad (c, ab^\dagger) := (cb, a). \tag{2.103}$$

This is possible since the scalar product over $\mathfrak{E}_{\mathbb{C}}$ is supposed to be non-degenerate. It is then easy to verify that one has the identities $(ab)^\dagger = b^\dagger a^\dagger$ and $\iota^\dagger = \iota$.

In this way $\mathfrak{E}_{\mathbb{C}}$ is closed under complex linear combinations, adjoint, and associative composition, and possibly contains the identity element ι, that is it is an associative complex algebra with adjoint, closed with respect to the adjoint. The non-degenerate scalar product on $\mathfrak{E}_{\mathbb{C}}$ in conjunction with the identity, leads to a strictly positive linear form over $\mathfrak{E}_{\mathbb{C}}$, defined as $\Phi = (\iota, \cdot)$, and one has $\Phi(a^\dagger b) = (\iota, a^\dagger b) = (a, b)$.[51] Such form is also a *trace*, *i.e.* it satisfies the identity $\Phi(ba) = \Phi(ab)$, which can be easily verified using definitions (2.103).[52] The complex linear space of the algebra closed with respect to the norm induced by the scalar product makes it a Hilbert space, and the action of the algebra over itself regarded as a Hilbert space makes it an operator algebra.[53] It is a standard result of the theory of operator algebras that the closure of $\mathfrak{E}_{\mathbb{C}}$ under the operator norm (which is guaranteed in finite dimensions) is a C*-algebra. We have therefore built a C*-algebra structure over the complex linear space of effects $\mathfrak{E}_{\mathbb{C}}$. We can introduce the *cyclic representation* [Haa96] given by

$$\Phi(a) = \langle\iota|\pi_\Phi(a)|\iota\rangle, \tag{2.104}$$

[50] The right and left multiplications are just special elements of the algebra $\mathsf{Lin}(\mathfrak{E}_{\mathbb{C}})$, whence their adjoints are definable via the scalar product as usual.

[51] The form is strictly positive since $\Phi(a^\dagger a) = (a, a) \geqslant 0$, with the equal sign only if $a = 0$, since the scalar product is non-degenerate.

[52] One has $\Phi(ab) = (\iota, ab) = (\iota, a(b^\dagger)^\dagger) = (b^\dagger, a)$ and $\Phi(ba) = (\iota, ba) = (\iota, (b^\dagger)^\dagger a) = (b^\dagger, a)$.

[53] This construction is a special case of the Gelfand-Naimark-Segal (GNS) construction[GN43], in which the form Φ is a trace. In the standard GNS construction the form Φ maybe degenerate, *i.e.* one can have $\Phi(a^\dagger a) = 0$ for some $a \neq 0$, and the vectors of the representation are built up as equivalence classes modulo vectors having $\Phi(a^\dagger a) = 0$.

where π_Φ denotes the algebra representation corresponding to Φ.[54] In our case one has $\pi_\Phi(a)|\iota\rangle = |a\rangle$, along with the *trace* property $\langle\iota|\pi_\Phi(a)\pi_\Phi(b)|\iota\rangle = \langle\iota|\pi_\Phi(b)\pi_\Phi(a)|\iota\rangle$. The latter can be actually realized as a trace as $\Phi(a^\dagger b) = \mathrm{Tr}[O(a)^\dagger O(b)]$, via a faithful representation $O : a \mapsto O(a) \in \mathsf{Lin}(\mathsf{H})$ of the algebra $\mathfrak{E}_\mathbb{C}$ as a subalgebra of $\mathsf{Lin}(\mathsf{H})$ of operators over a Hilbert space H with dimension $\dim(\mathsf{H})^2 \geqslant \dim(\mathfrak{E}_\mathbb{C})$. In this way, one has $\pi_\Phi(a) = (O(a) \otimes I)$ with the cyclic vector represented as $|\iota\rangle = \sum_n |n\rangle \otimes |n\rangle$, $\{|n\rangle\}$ being any orthonormal basis for H.

2.4.3 Recovering the action of transformations over effects

In order to complete the mathematical representation of the probabilistic theory, we now need to define the action of the elements of $\mathfrak{T}_\mathbb{C}$ over $\mathfrak{E}_\mathbb{C}$, and to select the cone of physical transformations $\mathfrak{T}_+$. We will show that $\mathfrak{T}_+$ is given by the completely positive linear maps on $\mathfrak{E}_\mathbb{C}$, namely the linear maps of the Kraus form, *i.e.* the set of maps whose extremal elements, the atomic transformations, act on $x \in \mathfrak{E}_\mathbb{C}$ as $x \circ \tau(|a|) = |a|^\dagger x|a| \equiv a^\dagger xa$.

Up to now we have introduced the transformations corresponding to the rank one operator in $\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$ via the map τ defined in Eq. (2.99). We haven't already find the transformations in $\mathfrak{T}_+$ corresponding to the non extremal elements in $\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$. First, notice that the full span $\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$ is recovered from $\mathsf{Erays}(\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C}))$ via the polarization identity

$$|a\rangle\langle b| = \frac{1}{4}\sum_{k=0}^{3} i^k |(a + i^k b)\rangle\langle(a + i^k b)|. \tag{2.105}$$

Correspondingly, we introduce the generalized transformations

$$\tau(b, a) := \frac{1}{4}\sum_{k=0}^{3} i^k \tau(|a + i^k b|) \in \mathfrak{T}_\mathbb{C}. \tag{2.106}$$

The map

$$|a\rangle\langle b| \mapsto \chi(|a\rangle\langle b|) := b^\dagger \cdot a \tag{2.107}$$

is a CJ isomorphism: it represents a bijective map between the cones $\mathsf{Lin}_+(\mathfrak{E}_\mathbb{C})$ and $\mathfrak{T}_+$ which can be extended to a cone-preserving linear bijection between $\mathsf{Lin}(\mathfrak{E}_\mathbb{C})$ and $\mathfrak{T}_\mathbb{C} \equiv \mathsf{Lin}(\mathfrak{E}_\mathbb{C})$.[55] As a consequence of Eq. (2.100), the CJ isomorphism $\tau : |a| \mapsto \tau(|a|)$ will differ from the isomorphism χ by an automorphism $\mathscr{U}$ of the C*-algebra of effects, that is one has $x \circ \tau(|a|) = \mathscr{U}(a^\dagger)x\mathscr{U}(a)$, with $\mathscr{U}(a) = u^\dagger au$ with $uu^\dagger = u^\dagger u = \iota$. It follows that the probabilistic equivalence classes are given by $[\tau(|a|)]_{\mathrm{eff}} = e \circ \tau(|a|) = u^\dagger a^\dagger au$. Notice that $[\tau(\iota)]_{\mathrm{eff}} = u^\dagger\iota^\dagger\iota u = \iota$, that is ι coincides with the deterministic effect $\iota = e$. Complex effects are thus recovered from atomic transformations via the identity $e \circ \tau(e, a) = u^\dagger au$. In Fig 2.4 a flow-diagram is reported summarizing the relevant logical implications of the present operational axiomatic framework for QM.

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[54] This means that $\pi_\Phi(a)\pi_\Phi(b) = \pi_\Phi(ab)$ and $\pi_\Phi(a^\dagger) = \pi_\Phi(a)^\dagger$.

[55] This can be directly checked using the operator algebra representation built over $\mathfrak{E}_\mathbb{C}$, whereas the isomorphism corresponds to the map $O(b^\dagger xa) = \chi(|a\rangle\langle b|)(x) = \mathrm{Tr}_1[(O(x) \otimes I)|a\rangle\langle b|]$, and, reversely, $|a\rangle\langle b| = \chi^{-1}(\tau(b, a)) = (\tau(b, a) \otimes \mathscr{I})(|\iota\rangle\langle\iota|)$.

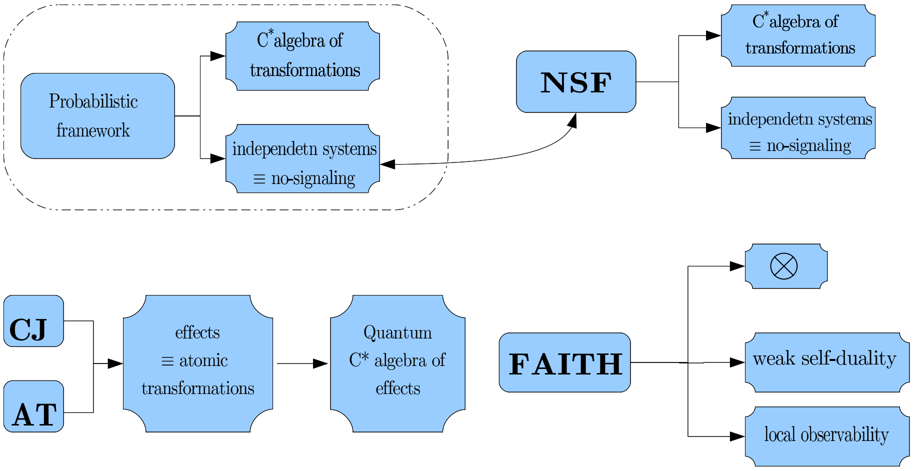


Figure 2.4: Operational axiomatic framework for quantum Mechanics: summary of the relevant logical implications.

2.5 Bloch representation for transformations in a probabilistic theory

In this section we introduce an affine-space representation based on the existence of a minimal informationally complete observable and of a separating set of states. This representation has already been introduced in Ssec. 1.2.3 for the affine transformation of a general convex set. In the following chapters our context will be the probabilistic framework ones and seems to be useful a short specification of Block representation in such context. Such representation generalizes the popular Bloch representation used in Quantum Mechanics.

In terms of a minimal informationally complete observable, denoted by $\{l_j\}$, $i = 1, \ldots, N$, and of a minimal separating set of states $\{\lambda_j\}$, $i = 1, \ldots, N$, one can expand (in a unique way) any effect $a \in \mathfrak{E}$ and state $\omega \in \mathfrak{S}$ as follows

$$a = \sum_{j=1}^{N} \lambda_j(a) l_j, \qquad \omega = \sum_{j=1}^{N} l_j(\omega)\lambda_j. \tag{2.108}$$

Instead of using a minimal informationally complete observable and a minimal set of separating states it is convenient to adopt canonical orthonormal basis $l = \{l_i\}$ and $\lambda = \{\lambda_i\}$ for $\mathfrak{E}_\mathbb{R}$ and $\mathfrak{S}_\mathbb{R}$ embedded into $\mathbb{R}^N$ as Euclidean space, with

$$(l_i, \lambda_j) = l_i(\lambda_j) = \lambda_j(l_i) = \delta_{ij} \tag{2.109}$$

Moreover, it turns out to be convenient to replace one element of the informationally complete observable $l = \{l_j\}$ with the deterministic effect e, and, correspondingly replace one element of the state-basis $\lambda = \{\lambda_j\}$ with the functional giving the determin-

Bloch vector

istic component of the effect. Using a Minkowskian notation we write

$$l \doteq (\boldsymbol{l}, e),\ \lambda \doteq (\boldsymbol{\lambda}, \chi), \qquad \lambda \cdot l \doteq \sum_j \lambda_j l_j = \boldsymbol{\lambda} \cdot \boldsymbol{l} + \chi e, \tag{2.110}$$

and for any effect a and any state ω we write

$$(a, \omega) = \omega(a) = a(\omega) = l(\omega) \cdot \lambda(a) := \sum_{i=1}^{N} l_i(\omega)\lambda_i(a) \equiv \boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a)e(\omega). \tag{2.111}$$

Clearly one can extend the convex sets of effects and states to their complexification by keeping the expansion coefficients as complex.

The vectors $l(\omega)$ and $\lambda(a)$ give a complete description of the (unnormalized) state ω and (unbounded) effect a, thanks to identity (2.111). For normalized state ω $\boldsymbol{l}(\omega)$ is the **Bloch vector** representing the state ω. The representation is *faithful* (*i.e.* one-to-one) for orthonormal basis or, generally, for minimal informationally complete observable.

We now recover the linear transformation describing conditioning. The conditioning is given by

$$\mathscr{A}\omega(b) = \omega(b \circ \mathscr{A}) = b(\mathscr{A}\omega) \tag{2.112}$$

From linearity of transformations one can introduce a matrix $\{M_{jl}(\mathscr{A})\}$, and write

$$l_j(\mathscr{A}\omega) = \omega(l_j \circ \mathscr{A}) = \sum_i M_{ji}(\mathscr{A})l_i(\omega) + M_{j0}(\mathscr{A})e(\omega), \tag{2.113}$$

and, in particular, upon denoting $a := [\mathscr{A}]_{\text{eff}}$, one has

$$e(\mathscr{A}\omega) = \omega(e \circ \mathscr{A}) \equiv \omega(a) = \sum_i M_{0i}(\mathscr{A})l_i(\omega) \equiv \boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a)e(\omega), \tag{2.114}$$

from which we derive the identities

$$\lambda_i(a) \equiv M_{0i}(\mathscr{A}), \qquad \chi(a) = M_{00}(\mathscr{A}). \tag{2.115}$$

The real matrices $M_{ji}(\mathscr{A})$ are a *representation* of the real algebra of generalized transformations $\mathcal{A}$. The last row of the matrix is a representation of the effect a of $\mathscr{A}$ (see Fig. 2.5). In vector notation, for a normalized input state one has

$$\begin{aligned} \boldsymbol{l}(\mathscr{A}\omega) =& \boldsymbol{M}(\mathscr{A})\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A}), \qquad k_j(\mathscr{A}) \doteq \chi(l_j \circ \mathscr{A}), \\ e(\mathscr{A}\omega) =& \boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a), \\ \mathscr{A}\omega(b) =& \boldsymbol{\lambda}(b) \cdot \boldsymbol{l}(\mathscr{A}\omega) + \chi(b)e(\mathscr{A}\omega) \end{aligned} \tag{2.116}$$

The matrix representation of the transformation is synthesized in Fig. 2.5.

Therefore, summarizing we have found the following conditioning transformation

$$\omega \in \mathfrak{S}, \quad \boldsymbol{l}(\omega) \longrightarrow \boldsymbol{l}(\omega_{\mathscr{A}}) = \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a)}, \tag{2.117}$$

with the transformation occurring with probability given by

$$\omega(\mathscr{A}) = \boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a). \tag{2.118}$$

In the following we will use the abbreviate notation $a_j \equiv \lambda_j(a)$ and $\omega_j \equiv l_j(\omega)$.

$$M_{ij}(\mathscr{A}) = \begin{pmatrix} \boldsymbol{M}(\mathscr{A}) & \boldsymbol{k}(\mathscr{A}) \\ \boldsymbol{\lambda}(a)^{\tau} & q(a) \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{l}(\mathscr{A}\omega) &= \boldsymbol{M}(\mathscr{A})\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A}), \\ \omega(\mathscr{A}) &= \boldsymbol{\lambda}(a) \cdot \boldsymbol{l}(\omega) + \chi(a), \\ \boldsymbol{l}(\omega_{\mathscr{A}}) &= \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{\lambda}(\underline{\mathscr{A}}) \cdot \boldsymbol{l}(\omega) + \chi(\underline{\mathscr{A}})}. \end{aligned}$$

Figure 2.5: Matrix representation of the real algebra of transformations $\mathcal{A}$. The last row represents the effect a of the transformation $\mathscr{A}$. It gives the transformation of the zero-component of the Bloch vector $e(\mathscr{A}\omega) \equiv \omega(\mathscr{A}) = \lambda(a) \cdot \boldsymbol{l}(\omega) + \chi(a)$, namely the probability of the transformation. The following rows represent the affine transformation of the Bloch vector $\boldsymbol{l}(\omega)$ corresponding to the operation of $\mathscr{A}$, the first column giving the translation $\boldsymbol{k}(\mathscr{A})$, and the remaining square matrix $\boldsymbol{M}(\mathscr{A})$ the linear Part Overall, the Bloch vector of the state ω is transformed as $\boldsymbol{l}(\mathscr{A}\omega) = \boldsymbol{M}(\mathscr{A})\boldsymbol{l}(\omega) + \boldsymbol{k}(\mathscr{A})$, and the conditioning over the convex set of states is the fractional affine tranformation in figure.

Part II

Probabilistic models

In this part of the thesis some probabilistic models are developed. A probabilistic model is a concrete fulfilment of the abstract idea of probabilistic theory. The quantum mechanics for example is a very particular probabilistic model as talked about in Chap. 2. The peculiarity of the quantum mechanics model could be better understood if a a set of other probabilistic models with whom compare its features would exist. Our task in the following chapters is to begin the construction of such a set of models in order to understand the quantum one. Naturally not all the properties of quantum mechanics will be satisfied by these models and this will allow (in a future) a progressive convergence to a set of properties having quantum mechanics as the only representative probabilistic model.

Conventions Naturally the probabilistic framework in the following chapter is that from Ref. [D'A08] which is also the content of Chap. 2, then we will adopt the notation introduced in that article. In the following we will also denote by $\{\lambda_i\}$ the canonical basis for $\mathfrak{S}_{\mathbb{R}}$ and by $\{l_i\}$ the canonical basis for $\mathfrak{E}_{\mathbb{R}}$ as explained in Sec. (2.5).

CHSH
no-signaling

Chapter 3

Extended Popescu-Rohrlich model

In this probabilistic model the local convex set of states results to be a 2-dimensional polytope, precisily it is a square. There will be a lot of models having this particular set as local convex set of states but our task is to reproduce in the framework of the probabilistic theory the well known models introduced by Popescu and Rohrlich in order to achieve the maximal violation of the **CHSH** inequality compatible with the **no-signaling** postulate. We will better investigate this arguments in Part III. This model is of great interest because it is a good test of the general formulation of probabilistic theories. If it is a good formulation we would be able to described this model as a probabilistic theory.

3.1 Original model: the Popescu-Rohrlich box

The Popescu-Rohrlich original model is that of Ref.[RP95]. The original model is made of a "box" which provide the probability rule for the output given the input. The simplest situation is that in which input and output are both binary. Then such model has two local tests labeled by the variable $x = 0, 1$, with two possible outcomes $i = 0, 1$ for each test and a set of probability rules provided by the box as showed in Fig.3.1.

In the following we will introduce the cones of states and effects and finally we will extend the original model by defining transformation. This will be achieved from a bipartite state of two Popescu-Rohrlich boxes. Notice that the construction of a probabilistic theory for Popescu-Rohrlich boxes is in some way forced by the original model which provides both information about the single system and the composed system. For this reason it is a good test of our probabilistic framework which have to give an easy description for a particular preexisting probabilistic model different from Quantum mechanic.

polytope

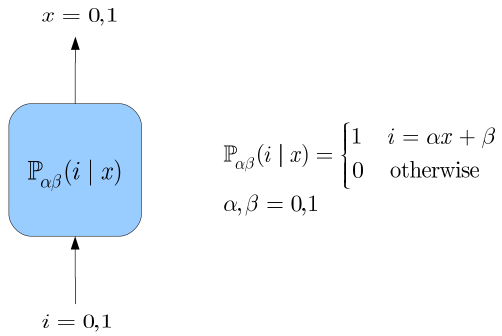


Figure 3.1: The Popescu-Rohrlic box which provide the probability rule $\mathbb{P}_{\alpha\beta}(i|x)$ for the binary output given the binary imput.

3.2 Local convex sets of states and effects

The set of states is the convex hull of the probability rules

$$\alpha, \beta = 0, 1, \qquad \mathbb{P}_{\alpha\beta}(i|x) = \begin{cases} 1, & i = \alpha x \oplus \beta \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

which represent the pure states of the model and than the extremal of the convex set. We will denote the two tests by $\mathbb{A}^{(x)} = \{\mathscr{A}_0^{(x)}, \mathscr{A}_1^{(x)}\}$ with $x = 0, 1$, and, correspondingly, we will denote the effects as $a_0^{(x)}, a_1^{(x)}$, with

$$a_0^{(0)} + a_1^{(0)} = a_0^{(1)} + a_1^{(1)} = e, \tag{3.2}$$

e denoting the deterministic effect. As previously observed the original model doesn't specify the transformations, but only the effects, and here we will extend the model by introducing transformations that are compatible with the given effects. We will denote by $\omega_{\alpha\beta}$ the states corresponding to the probabilities in Eq. (3.1). Clearly, since there are only two linearly independent effects plus the deterministic one, one has $\dim(\mathfrak{E}_+) = \dim(\mathfrak{S}_+) = 3$, and there are only two linearly independent states. Therefore, the convex set of states is the 2-dimensional **polytope** $\mathbb{P}^2$.

It is convenient to represent effects in a three-dimensional vector space with the canonical coordinate along the z-axis corresponding to the deterministic effect e. All the probabilistic models in this chapter follow the canonical geometrical representation described in Ssec.1.2.4. Then a possible representation of the four effects of the two tests is

$$\lambda(e) = \begin{bmatrix}0\\0\\1\end{bmatrix}, \lambda(a_0^{(0)}) = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \lambda(a_1^{(0)}) = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \lambda(a_0^{(1)}) = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \lambda(a_1^{(1)}) = \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}. \quad (3.3)$$

Correspondingly, according to the probability rule in Eq. (3.1), the four pure states will be represented as

$$l(\omega_{00}) := \begin{bmatrix}1\\0\\1\end{bmatrix}, l(\omega_{11}) := \begin{bmatrix}0\\1\\1\end{bmatrix}, l(\omega_{01}) := \begin{bmatrix}-1\\0\\1\end{bmatrix}, l(\omega_{10}) := \begin{bmatrix}0\\-1\\1\end{bmatrix}. \quad (3.4)$$

One can easily verify the application of the state to the effect[1]

$$\mathbb{P}_{\alpha\beta}(i|x) = \omega_{\alpha\beta}(a_i^{(x)}) \equiv l(\omega_{\alpha\beta}) \cdot \lambda(a_i^{(x)}) = \begin{cases}1, & i = \alpha x \oplus \beta\\0 & \text{otherwise.}\end{cases} \quad (3.5)$$

Notice that the third coordinate has been fixed to unit, corresponding to the axis of the cone $\mathfrak{S}_+$. The polytope (see Ssec. 1.1.4) of states is the square

$$\mathfrak{S} = \mathbb{P}^2 = \left\{\lambda(\omega) = \begin{bmatrix}x\\y\\1\end{bmatrix}, \ |x| + |y| \leqslant 1\right\}, \quad (3.6)$$

which is the convex hull of the vectors $\lambda(\omega_{\alpha\beta})$, $\alpha = 0, 1$, $\beta = 0, 1\}$ corresponding to the vertexes. Clearly $\mathfrak{S}_+$ based on $\mathfrak{S}$ is given by

$$\mathfrak{S}_+ = \left\{\lambda(\omega) = \begin{bmatrix}x\\y\\c\end{bmatrix}, \ |x| + |y| \leqslant z, z \geqslant 0\right\}. \quad (3.7)$$

The convex cone of effects is the dual of $\mathfrak{S}_+$

$$\mathfrak{E}_+ = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix}, |x| \leqslant z, \ |y| \leqslant z, \ z \geqslant 0\right\}, \quad (3.8)$$

[1]The above cone representation in $\mathbb{R}^3 = \mathfrak{S}_\mathbb{R} = \mathfrak{E}_\mathbb{R}$ corresponds to the choice of the canonical basis for effects

$$l_1 = \lambda(a_0^{(0)}) - \lambda(a_1^{(1)}), \ l_2 = \lambda(a_0^{(1)}) - \lambda(a_0^{(0)}), \ l_3 = \lambda(e)$$

and for states

$$\lambda_1 = \frac{1}{2}(l(\omega_{00}) - l(\omega_{01})), \ \lambda_2 = \frac{1}{2}(l(\omega_{11}) - l(\omega_{10})), \ \lambda_3 = l(\chi).$$

As specified in Ssec.1.2.4 there is a canonical isomorphism between the real spaces $\mathbb{R}^3 = \mathfrak{S}_\mathbb{R}$ and $\mathfrak{E}_\mathbb{R}$ which are the same Euclidean space $\mathbb{R}^3$. Embedding the two cones in the same space the canonical base for the effects identify with the one of states. Despite that we maintain a different symbol in order to recall the duality relations between the cones. On the other hand the vectors representing effects and states are vectors of the same space and we can use the same letters to specify the components of such vectors.

CHSH inequality

whence the convex set of physical effects is

$$\mathfrak{E} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } \begin{cases} |x| \leqslant z, \ |y| \leqslant z, & z \in [0, \frac{1}{2}] \\ |x| \leqslant 1 - z, \ |y| \leqslant 1 - z, & z \in [\frac{1}{2}, 1] \end{cases} \right\}, \tag{3.9}$$

which correspond to the truncation of $\mathfrak{E}_+$ given by the order prescription $0 \leqslant a \leqslant e$.

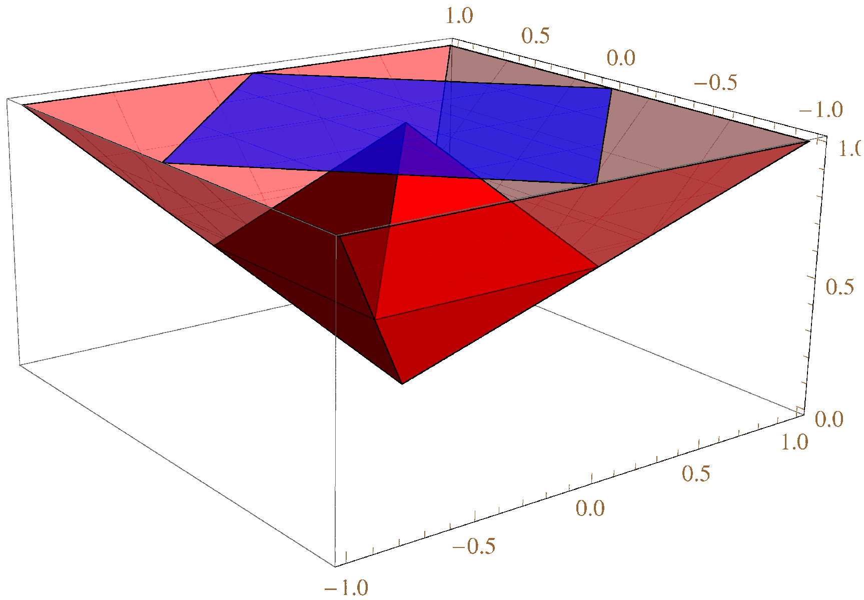


Figure 3.2: The blue square at the top represent the set of states $\mathfrak{S}$. The red transparent cone represent the dual cone of effects $\mathfrak{E}_+$. The solid inside the transparent cone represent the convex set of effects $\mathfrak{E}$ which is the $\mathfrak{E}_+$-truncation given by the condition $a \leqslant e$ where a is a generic effect and e is the deterministic one

3.3 The original bipartite system: the correlated boxes

The core of the original work are the bipartite correlated boxes (see Fig.3.3) defined through the probability

$$\alpha, \beta, \gamma = 0, 1, \qquad \mathbb{P}_{\alpha\beta\gamma}(ij|xy) = \begin{cases} \frac{1}{2}, & i \oplus j = xy \oplus \alpha x \oplus \beta y + \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

The correlations introduced by these probability rules are non local correlations. The main pourpose of Popescu and Rohrlick was in fact to find a model with the maximum violation of the **CHSH inequality** compatible with a the no-signaling postulate. We will investigate non locality in PartIII.

no-signaling

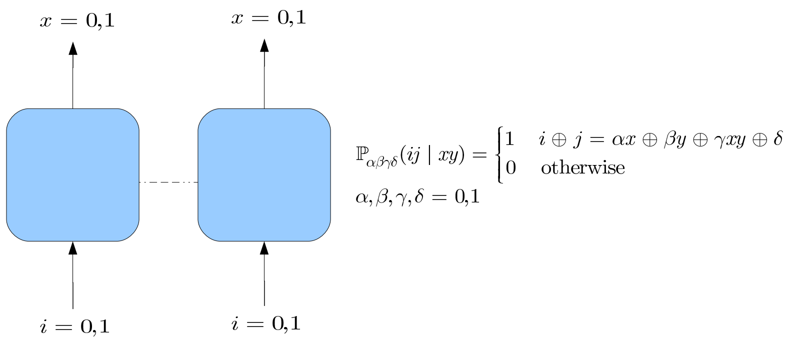


Figure 3.3: The Popescu-Rohrlic correlated boxes which provide the probability rules $\mathbb{P}_{\alpha\beta\gamma}(ij|xy)$ for the binary outputs given the binary imputs.

Dimension of the original bipartite system. The joint probabilities $\mathbb{P}_{\alpha\beta\gamma}(ij|xy)$ form a table of sixteen (2^4) entries, but these are not all independent because of normalization and **no-signaling** conditions imposed by Postulate NSF. Normalization is expressed as

$$\sum_{ij=0,1} \mathbb{P}(ij|xy) = 1 \qquad \forall\, x, y, \tag{3.11}$$

while no-signaling imply the following two relations

$$\sum_{j=0,1} \mathbb{P}(ij|x0) = \sum_{j=0,1} \mathbb{P}(ij|x1) = \mathbb{P}(i|x) \qquad \forall\, i, x \tag{3.12}$$

and

$$\sum_{i=0,1} \mathbb{P}(ij|0y) = \sum_{i=0,1} \mathbb{P}(ij|1y) = \mathbb{P}(j|x) \qquad \forall\, j, y. \tag{3.13}$$

There are four normalization equalities and four equalities from each no-signaling conditions. But looking at (3.12) we advise that for each value of x, the no-signaling condition for one output i can be deduced from normalization and the no-signaling on the other output. The same argument applies for each value of y in (3.13). According to the last observation the constraints given by the twelve equations reduce to only eight. Therefore the independent entries in the table $\mathbb{P}(ij|xy)$ are sixteen minus eight namely there are eight independent entries. The convex set of states for the bipartite original model is then an 8-dimensional polytope.

Vertexes of the original bipartite set of states. The vertexes of the polytope correspond to the pure bipartite states of the composed system and in the original model there are only twenty four pure bipartite states: the eight independent probability rules in Eq. (3.10), which are non local states, plus the sixteen pure states given by the factorization of the local probability rules in Eq. (3.1), which are obviously local states.

3.4 The bipartite system and the faithful state

Denote by $\Phi_{\alpha\beta\gamma}$ the eight states in $\mathfrak{S}_{+}^{\odot 2}$ which correspond to the above joint probability rules in Eq. (3.10) and take one of them—say Φ_{000}—as a pure symmetric preparationally faithful state in our probabilistic framework. Naturally at the end of the construction we will verify the faithfulness of this state. First we have to check that, regarded as a matrix over effects such state is non singular because if it is a preparationally faithful state than it's also an isomorphic map between the cones $\mathfrak{S}_{+}$ and $\mathfrak{E}_{+}$. Indeed we have

$$\Phi = \sum_{ij} \Phi_{ij} \lambda_i \otimes \lambda_j \equiv \mathbf{\Phi} = \{\Phi_{ij}\} = \{\Phi(l_i, l_j)\} \tag{3.14}$$

or, explicitating the effects of the canonical base,

$$\mathbf{\Phi} = \begin{bmatrix} \Phi(a_0^{(0)} - a_1^{(1)}, a_0^{(0)} - a_1^{(1)}) & \Phi(a_0^{(0)} - a_1^{(1)}, a_1^{(0)} - a_0^{(0)}) & \Phi(a_0^{(0)} - a_1^{(1)}, e) \\ \Phi(a_0^{(0)} - a_1^{(1)}, a_1^{(0)} - a_0^{(0)}) & \Phi(a_1^{(0)} - a_0^{(0)}, a_1^{(0)} - a_0^{(0)}) & \Phi(a_1^{(0)} - a_0^{(0)}, e) \\ \Phi(a_0^{(0)} - a_1^{(1)}, e) & \Phi(a_1^{(0)} - a_0^{(0)}, e) & \Phi(e, e) \end{bmatrix}. \tag{3.15}$$

Then from the rules in Eq. (3.10) we get the non singular symmetric matrix

$$\mathbf{\Phi} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{3.16}$$

The cone-isomorphism $\mathfrak{E}_{+} \simeq \mathfrak{S}_{+}$ established by the map $\Phi(a, \cdot) =: \omega_a$ is given by the relation

$$\Phi(l_i, l_j) =: \varphi_i(l_j) \equiv \varphi_i \cdot l_j \implies \mathbf{\Phi} = (\varphi_1, \varphi_2, \varphi_3). \tag{3.17}$$

In the last equation the vectors φ_i are the imagines of the base effects l_i under the map Φ and the isomorphism is explicitly given by $\varphi_i := \Phi(l_i, \cdot)$. One also has

$$\Phi(a_0^{(0)}, \cdot) = \tfrac{1}{2}(\varphi_1 - \varphi_2 + \varphi_3) = \tfrac{1}{2}\omega_{00}, \tag{3.18}$$
$$\Phi(a_1^{(0)}, \cdot) = \tfrac{1}{2}(-\varphi_1 + \varphi_2 + \varphi_3) = \tfrac{1}{2}\omega_{01}, \tag{3.19}$$
$$\Phi(a_0^{(1)}, \cdot) = \tfrac{1}{2}(+\varphi_1 + \varphi_2 + \varphi_3) = \tfrac{1}{2}\omega_{10}, \tag{3.20}$$
$$\Phi(a_1^{(1)}, \cdot) = \tfrac{1}{2}(-\varphi_1 - \varphi_2 + \varphi_3) = \tfrac{1}{2}\omega_{11}. \tag{3.21}$$

Notice that

$$\Phi(e, \cdot) = \chi \tag{3.22}$$

having representative

$$l(\chi) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda_3, \tag{3.23}$$

namely the centre of the polytope[2].

The state Φ is only one of the eight pure bipartite states in $\mathfrak{S}_+^{\odot 2}$. All of them could be assumed as faithful state of the theory as will be better clarified in the following where the corresponding transformations will be achieved. The same arguments which leads to the matrix representation for Φ_{000} can be iterated for each state $\Phi_{\alpha\beta\gamma}$. The eight representative matrix result to be

$$\Phi_{000} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Phi_{001} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Phi_{011} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Phi_{110} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Phi_{100} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \Phi_{010} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Phi_{101} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Phi_{111} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{3.24}$$

Beside these eight non local pure states there are the sixteen local ones given by

$$\omega_{\alpha\beta} \otimes \omega_{\gamma\delta} \qquad \forall\ \alpha, \beta, \gamma, \delta = 0, 1 \tag{3.25}$$

where $\omega\alpha\beta$ and $\omega\gamma\delta$ are the vertexes of $\mathfrak{S}$. The representative of these states are reported in Tab.3.1.

Now by convex combinations of its twenty-four vertexes we can generate the whole set $\mathfrak{S}_+^{\odot 2}$. Indeed the general bipartite state is as follows

$$\Psi = \sum_{\alpha\beta\gamma\delta=0,1} c_{\alpha\beta\gamma\delta} \left(\omega_{\alpha\beta} \otimes \omega_{\gamma\delta}\right) + \sum_{\alpha'\beta'\gamma'=0,1} c_{\alpha'\beta'\gamma'} \Phi_{\alpha'\beta'\gamma'} \tag{3.26}$$

where

$$\sum_{\alpha\beta\gamma\delta=0,1} c_{\alpha\beta\gamma\delta} + \sum_{\alpha'\beta'\gamma'=0,1} c_{\alpha'\beta'\gamma'} = 1. \tag{3.27}$$

3.4.1 Transformations of the bipartite system

The only way to introduce the whole set of completely positive transformations $\mathfrak{T}_+$ compatible, in our probabilistic framework, with the set of bipartite states $\mathfrak{S}_+^{\odot 2}$ introduced in the last subsection, is to assume the cone isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$ induced by

[2] By exploring other probabilistic models we will convince ourselves that it's a general rule the identification between the centre of the convex set of states and the imagine of the deterministic effect under the faithful state isomorphism. Naturally it's not the case when the convex set of states hasn't got a centre.

$\otimes$	ω_{00}	ω_{01}	ω_{10}	ω_{11}
ω_{00}	$\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}-1&0&1\\0&0&0\\-1&0&1\end{bmatrix}$	$\begin{bmatrix}0&-1&1\\0&0&0\\0&-1&1\end{bmatrix}$	$\begin{bmatrix}0&1&1\\0&0&0\\0&1&1\end{bmatrix}$
ω_{01}	$\begin{bmatrix}-1&0&-1\\0&0&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&-1\\0&0&0\\0&-1&1\end{bmatrix}$	$\begin{bmatrix}0&-1&-1\\0&0&0\\0&1&1\end{bmatrix}$
ω_{10}	$\begin{bmatrix}0&0&0\\-1&0&-1\\1&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\1&0&-1\\-1&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&1&-1\\0&-1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&-1&-1\\0&1&1\end{bmatrix}$
ω_{11}	$\begin{bmatrix}0&0&0\\1&0&1\\1&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\-1&0&1\\-1&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&-1&1\\0&-1&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&1&1\\0&1&1\end{bmatrix}$

Table 3.1: The sixteen factorized vertexes of $\mathfrak{S}_+^{\odot 2}$ for the Popescu-Rohrlick model.

the preparationally faithful state Φ according to Postulate PFAITH. The first step is to achieve from the isomorphism an explicit relation between elements in the two cones. Then by this relation the whole set $\mathfrak{T}_+$ could be generated from the set of bipartite states $\mathfrak{S}_+^{\odot 2}$. Let $\mathscr{A}$ be a generic transformation in $\mathfrak{T}_+$. Then take the matrix representation of $\mathscr{A}$ induced by the relation

$$l_i \circ \mathscr{A} =: \sum_k A_{ik} l_k. \tag{3.28}$$

From the isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$ we know that

$$\forall\ \Psi \in \mathfrak{S}_+^{\odot 2} \qquad \exists!\ \mathscr{A} \in \mathfrak{T}_+ \ \text{ such that } \ (\mathscr{I}, \mathscr{A})\Phi = \Psi. \tag{3.29}$$

Matching the last two equations we have

$$(\mathscr{I}, \mathscr{A})\Phi(l_i, l_j) = \Psi(l_i, l_j) \ \Rightarrow\ \Phi(l_i, l_j \circ \mathscr{A}) = \Psi(l_i, l_j) \ \Rightarrow\ \sum_k \Phi_{ik} A_{ik} = \Psi_{ij} \tag{3.30}$$

and then[3]

$$\boldsymbol{\Phi} \boldsymbol{A}^{\tau} = \boldsymbol{\Psi} \qquad \Rightarrow \qquad \boldsymbol{A}^{\tau} = \boldsymbol{\Phi}^{-1}\boldsymbol{\Psi}. \tag{3.32}$$

[3]The inverse of Φ is simply the matrix

$$\boldsymbol{\Phi}^{-1} = \begin{bmatrix}1&-1&0\\-1&-1&0\\0&0&1\end{bmatrix}. \tag{3.31}$$

complete probabilistic model

It's sufficient to find the transformation associated to the pure states in $\mathfrak{S}_+^{\odot 2}$ because they are the extremal elements of $\mathfrak{T}_+$ according to the cones isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$.

First we achieve the transformations corresponding to the non local vertexes $\Phi_{\alpha\beta\gamma}$. The result is summarised in Tab.3.2. The most considerable information written in the table is that the transformation linked to the eight non local pure states are just the eight automorphism of the local set of states $\mathfrak{S}$

$$\mathsf{Aut}(\mathfrak{S}) = \{\mathscr{A}^{\alpha\beta\gamma}\} \text{ such that } (\mathscr{I}, \mathscr{A}^{\alpha\beta\gamma})\Phi = \Phi_{\alpha\beta\gamma} \qquad \alpha, \beta, \gamma = 0, 1. \tag{3.33}$$

According to Theorem 2.3-3 the transposed of the automorphisms are still automorphisms as can be directly verified by the representative in the table. Moreover, as stated by Theorem 2.2-6 the application of the automorphisms to the faithful state Φ produce the eight pure bipartite states of $\mathfrak{S}^{\odot 2}$ which are all pure symmetric preparationally faithful states.

The remaining extremal of $\mathfrak{T}_+$ are the transformation associated to the sixteen states in Tab.3.1. From the explicit isomorphism in Eq.(3.32) we get the required sixteen transformations[4]

$$\begin{bmatrix} c & -c & c \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ c & -c & c \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} -c & c & c \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ -c & c & c \\ -1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} c & c & c \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ c & c & c \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -c & -c & c \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ -c & -c & c \\ -1 & -1 & 1 \end{bmatrix} \tag{3.34}$$
$$c = \pm 1.$$

In conclusion the set $\mathsf{Aut}(\mathfrak{S})$ given in Tab.3.2 plus the set of extremal transformations in Eq. (3.34) generate the whole set $\mathsf{Erays}\mathfrak{T}_+$ and then, by convex combinations, all the completely positive maps in $\mathfrak{T}_+$.

3.4.2 A complete probabilistic theory

In Chap.1 we studied the contraction for a convex set. In this context the main con vex set is obviously the local convex of states $\mathfrak{S}$ which play the role of the generic C in that chapter. As usual we use the symbol $\mathfrak{L}_+$ for the cone of positive transformations namely the transformations which preserve the cone of states $\mathfrak{S}_+$. In general we know that $\mathfrak{T}_+ \subseteq \mathfrak{L}_+$. We can call the Popescu-Rohrlick generalized probabilistic model a **complete probabilistic model** because we can show that

$$\mathfrak{T}_+ = \mathfrak{L}_+. \tag{3.35}$$

[4] These transformations are not contractions but in order to achieve the corresponding contractions is sufficient to perform a rescaling by multiplying by a scalar $\lambda < 1$. We will better clarify this in the following subsection.

state		transformation		action
Φ_{000}	$\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{000}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	identity transformation
Φ_{001}	$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{001}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{\pi}{2}$ clockwise rotation
Φ_{011}	$\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{011}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	π clockwise rotation
Φ_{110}	$\begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{110}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{3\pi}{2}$ clockwise rotation
Φ_{100}	$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{100}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection
Φ_{010}	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{010}$	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection
Φ_{101}	$\begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{101}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection
Φ_{111}	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\mathscr{A}^{111}$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection

Table 3.2: In the first column are showed the matrix representative of the eight bipartite non local vertexes of the polytope $\mathfrak{S}^{\odot 2}$. In the second column we can see the eight corresponding transformations in $\mathfrak{T}_+$ acording to the cones isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$. In the last column the action of the transformations is specified. Notice that the eight transformations corresponding to the non local pure states are exactly the eight automorphism of the set $\mathfrak{S}$ (namely $\mathsf{Aut}(\mathfrak{S})$).

To check this equality we can find out the set $\mathsf{Erays}(\mathfrak{L}_+)$ and compare it with the set $\mathsf{Erays}(\mathfrak{T}_+)$ generated in Ssec.3.4.1.

Observation 3.1 *The shape of the convex set $\mathfrak{S}$ makes easier our task. Indeed in observation 1.5 we advise that* $\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\,\mathsf{G}_{\mathfrak{E}}] \subseteq \mathrm{Extr}(\mathfrak{L}_+)$, *but in this case it's easy to verify that*

$$\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\,\mathsf{G}_{\mathfrak{E}}] = \mathrm{Extr}(\mathfrak{L}_+). \tag{3.36}$$

So we need only the extremal transformations in the equivalence classes $[\mathsf{Aut}(\mathfrak{S})\,\mathsf{G}_{\mathfrak{E}}]$. As generator $\mathsf{G}_{\mathfrak{E}}$ of $\mathfrak{S}$ we can choose the set composed by the deterministic effect, the null effect plus one of the effects in Eq. (3.3). Looking at Fig.3.2 this correspond to take one of the vertex of the square at middle high which clearly could be mapped into the other three vertexes by the automorphism of $\mathfrak{S}$. Our choice is

$$\mathsf{G}_{\mathfrak{E}} = \{e, o, a_0^{(0)}\}. \tag{3.37}$$

Extremal positive transformations in $[a_0^{(0)}]$. According to the block representation of affine transformations we know that a contraction $\mathscr{A}$ in the equivalence class $[a_0^{(0)}]$ must have representative matrix

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{3.38}$$

The positivity condition reads

$$\mathscr{A}\,\omega_{\alpha\beta} \in \mathfrak{S}_+ \qquad \forall\, \alpha, \beta \in \{0, 1\}. \tag{3.39}$$

Remembering the definitions of $\omega_{\alpha\beta}$ and $\mathfrak{S}_+$ the last condition becomes

$$\begin{aligned} &|A_{11} + A_{13}| + |A_{21} + A_{23}| \leqslant 1 \\ &|A_{12} + A_{13}| + |A_{22} + A_{23}| \leqslant 0 \\ &|-A_{11} + A_{13}| + |-A_{21} + A_{23}| \leqslant 0 \\ &|-A_{12} + A_{13}| + |-A_{22} + A_{23}| \leqslant 1 \end{aligned} \tag{3.40}$$

The second and the third bounds fix the equalities $A_{12} = -A_{13}$, $A_{21} = -A_{23}$, $A_{11} = A_{13}$ and $A_{21} = A_{23}$ making the positivity condition as simple as $|A_{11}| + |A_{21}| \leqslant \frac{1}{2}$. We are left with the set of transformations

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & -A_{11} & A_{11} \\ A_{21} & -A_{21} & A_{21} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad |A_{11}| + |A_{21}| \leqslant \frac{1}{2}. \tag{3.41}$$

whose extremal $\mathrm{Extr}[a_0^{(0)}]$ are the four transformations

$$\begin{bmatrix} c & c & c \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ c & c & c \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad c = \pm\frac{1}{2} \tag{3.42}$$

Extremal deterministic transformations. A deterministic transformation $\mathscr{D}$ in $[e]$ has block form

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.43}$$

with positivity ensured by

$$\begin{aligned} &|D_{11} + D_{13}| + |D_{21} + D_{23}| \leqslant 1 \\ &|D_{12} + D_{13}| + |D_{22} + D_{23}| \leqslant 1 \\ &|-D_{11} + D_{13}| + |-D_{21} + D_{23}| \leqslant 1 \\ &|-D_{12} + D_{13}| + |-D_{22} + D_{23}| \leqslant 1. \end{aligned} \tag{3.44}$$

It's easy to convince yourself that the extremals in this set of transformations are exactly the automorphism of $\mathfrak{S}$ namely

$$\mathrm{Extr}[e] = \mathsf{Aut}(\mathfrak{S}). \tag{3.45}$$

According to Prop.1.3 the set $\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\, \mathsf{G}_{\mathfrak{E}}]$ can be achieved by operating the automorphism on $\mathrm{Extr}[\mathsf{G}_{\mathfrak{E}}]$. The Action of $\mathsf{Aut}(\mathfrak{S})$ over the matrixes in Eq. (3.42) (representing $\mathrm{Extr}[a_0^{(0)}]$) generates the sixteen extremal transformations showed in Eq. (3.34). Besides acting over $\mathrm{Extr}[e] = \mathsf{Aut}(\mathfrak{S})$ we naturally get the same group $\mathsf{Aut}(\mathfrak{S})$. In conclusion the whole set $\mathrm{Extr}(\mathfrak{T}_+)$ given in Ssec.3.4.1 has been recovered as $\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\, \mathsf{G}_{\mathfrak{E}}]$ $= \mathrm{Extr}(\mathfrak{L}_+)$.

3.4.3 Teleportation.

To verify if teleportation is achievable in this probabilistic theory, we have to ensure the existence of a bipartite effect F in $\mathfrak{E}^{\odot 2}$ such that

$$F_{23}\Phi_{12}\Phi_{34} = \alpha\Phi_{14} \tag{3.46}$$

where α is peculiar of the theory. In order to satisfy Eq. (3.46) the matrix $\boldsymbol{F}$, which represent F in our working base, must be proportional to $\boldsymbol{\Phi}^{-1}$ than

$$\boldsymbol{F} \propto \boldsymbol{\Phi}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.47}$$

which is equivalent to say

$$F \propto (l_1 \otimes l_1) - (l_1 \otimes l_2) - (l_2 \otimes l_1) - (l_2 \otimes l_2) + (l_3 \otimes l_3). \tag{3.48}$$

We doesn't know the shape of $\mathfrak{E}^{\odot 2}$ but it will suffice a state on which F acts returning a negative value to ensure F not being an effect. It's easy to verify that the application of F to separable states always give positive result

$$F(\omega, \zeta) \geqslant 0 \qquad \forall\, \omega, \zeta \in \mathfrak{S} \tag{3.49}$$

On the other hand exploring among bipartite states we find that

$$F(\Phi_{001}) \propto \Phi_{001}(l_1, l_1) - \Phi_{001}(l_1, l_2) - \Phi_{001}(l_2, l_1) - \Phi_{001}(l_2, l_2) + \Phi_{001}(l_3, l_3) = -1 \tag{3.50}$$

3.4.4 Purifiability

PURIFY

In the Popescu-Rohrlich model Postulate **PURIFY** doesn't hold. In fact there not exist enough pure bipartite states in order to purify all the local states in $\mathfrak{S}$. The only bipartite pure states are the eight maximally correlates states in Eq. (3.10) and having representative in the first column of Tab. (3.2). It's easy to see that the only local state having a purification is the maximally chaotic state χ. In particular all the eight states above purify it, indeed

$$\Phi_{\alpha\beta\gamma}(\cdot, e) = \Phi_{\alpha\beta\gamma}(e, \cdot) = \chi \qquad \forall \alpha, \beta, \gamma = 0, 1. \tag{3.51}$$

Notice also that, according to Theorem 2.3-4 the chaotic state is invariant under isomorphisms

$$\boldsymbol{A}^{\alpha\beta\gamma} l(\chi) = l(\chi) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \forall \alpha, \beta, \gamma = 0, 1. \tag{3.52}$$

In conclusion there are too few pure bipartite states to avoid purification of the theory. This will not be the case in the following probabilistic model.

3.4.5 Weak self duality of the model

The Jordan form of Φ is given by

$$|\boldsymbol{\Phi}| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \tag{3.53}$$

Remember that the strictly positive bilinear form $|\Phi|$ over $\mathfrak{E}_{\mathbb{R}}$ is related to the involution ς_Φ such that

$$|\Phi|(a, b) = \Phi(\varsigma_\Phi(a), b). \tag{3.54}$$

The generalized transformation $\mathscr{Z}_\Phi$ which correspond to the involution ς_Φ has to satisfy

$$a \circ \mathscr{Z}_\Phi = \sum_j \Phi(f_j, a) f_j, \tag{3.55}$$

where $\{f_j\}$ is the canonical Jordan basis. From Eq. (3.55) we can find the matrix representative of $\mathscr{Z}_\Phi$ to be

$$Z_\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \tag{3.56}$$

Observe thath the isomorphism $\mathfrak{E}_+ \simeq \mathfrak{S}_+$ established by the map $\Phi(a, \cdot) =: \omega_a$, is a composition of a counterclockwise $\frac{\pi}{4}$ rotation around the cone axis and a reflection respect to the plain xz. Such composed transformation in exactly the one performed by (3.56) as can be easily checked. Whence

$$\Phi(a, \cdot) \propto \varsigma(a) \tag{3.57}$$

which exprime the weakly self duality of this probabilistic theory.

clock
ball
self-dual

Chapter 4

The two-clocks probabilistic models

As suggested by the name the two-clocks probabilistic models have as local system the **clock**. A clock is a system whose convex set of states is a disc, namely a 2-dimensional **ball** (see Ssec. 1.1.4). Many theories with this local system can be generated. In this chapter we will investigate some of them in order to understand which properties of a probabilistic theory could be satisfied.

4.1 A local self-dual model

Among the different models having the clock as local system we can take the case in which the cone of states and effect are the same, namely the model is **self-dual** at the single system level.

4.1.1 The local convex sets of states and effects

In the usual representation the cone of states is given by

$$\mathfrak{S}_+ = \left\{ l(\omega) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 \leqslant t^2,\ t \geqslant 0 \right\}, \tag{4.1}$$

,while the cone of effects, which is the dual one, given by the same cone

$$\mathfrak{E}_+ = \left\{ \lambda(a) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 \leqslant t^2,\ t \geqslant 0 \right\}, \tag{4.2}$$

namely the theory is pointedly self-dual at a single system level if we embed both cones in the same euclidean space $\mathbb{R}^3$. The deterministic effect in our canonical basis is given as always by

$$\lambda(e) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.3}$$

The set of states $\mathfrak{S} \equiv \mathbb{B}^2$ is the base of the cone $\mathfrak{S}_+$ at $z = 1$, whereas the convex set of effects $\mathfrak{E}$ is the set of points of the cone $\mathfrak{E}_+$ satisfying $e - a \in \mathfrak{E}_+$, namely

$$\mathfrak{S} = \left\{ l(\omega) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 = 1 \right\}, \tag{4.4}$$

$$\mathfrak{E} = \left\{ \lambda(a) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 \leqslant \min(t^2, (1-t)^2),\ t \in [0,1] \right\}. \tag{4.5}$$

Therefore the convex set of effects $\mathfrak{E}$ is made of two truncated cones of height $\frac{1}{2}$ glued together at the basis as in Fig.4.1, with the two vertices given by the zero and the deterministic effect.

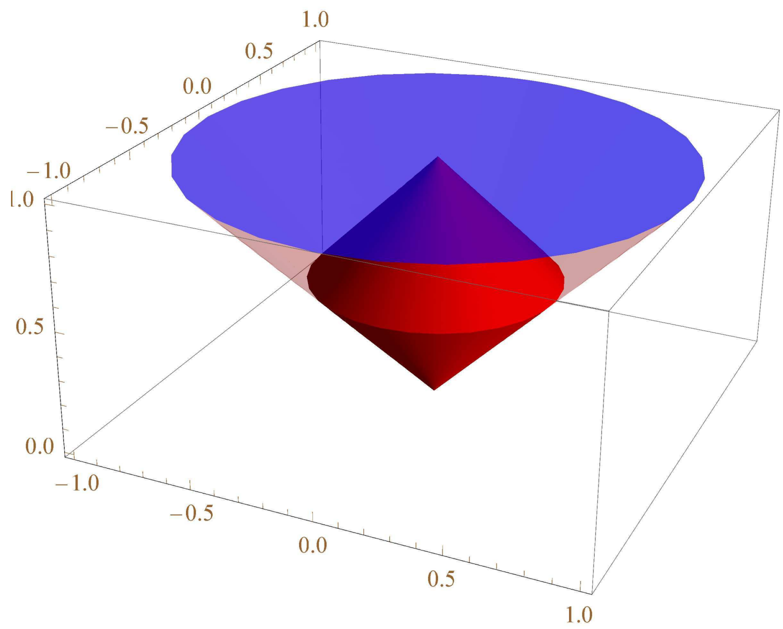


Figure 4.1: The blue disc at the top represent the set of states $\mathfrak{S}$. The red transparent cone represent the cones $\mathfrak{S}_+$ and $\mathfrak{E}_+$. The red solid inside the transparent cone represent the convex set of effects $\mathfrak{E}$ which is the $\mathfrak{E}_+$-truncation given by the condition $a \leqslant e$ where a is a generic effect and e is the deterministic one.

The state-effect pairing is given by

$$\begin{aligned}\forall\omega \in \mathfrak{S}, \quad \omega(a) = a(\omega) = \lambda(a)\cdot l(\omega) &\equiv \boldsymbol{\lambda}(a)\cdot \boldsymbol{l}(\omega) + \chi(a)e(\omega)\\ &= x_1x_2 + y_1y_2 + t_1,\end{aligned} \tag{4.6}$$

with normalization given by $\omega(e) = t$.

4.1.2 The faithful state choice

The structure of the composed system is not fixed by a preexisting model as in the Popescu-Rohrlic case. We can already understand how the structure of the bipartite system is tightly connected to the local system, in fact the faithful state must provide the automorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$ between the local cones of states and effect[1]. Then our possible choices for the faithful state are somehow narrowed. There will be a lot of different theories whose local system is the clock. The local set of states in fact does not identify uniquely the bipartite system and many theories will be generated from the local clock system. Let's introduce now the bipartite functional

$$\Phi(a,b) = \lambda(a)\cdot\lambda(b). \tag{4.7}$$

One can check that it is positive over the cone of effects, but also over its linear span, namely $|\Phi| \equiv \Phi$. We will use Φ to generate the complete convex set of bipartite states, by taking it as a pure preparationally faithful state. Indeed, Φ realizes the cone isomorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$, via the map

$$\omega_a := \Phi(a,\cdot) = a, \tag{4.8}$$

and the deterministic effect corresponds to the state at the center of $\mathfrak{S}$

$$\chi = \Phi(e,\cdot) = \lambda_3, \tag{4.9}$$

in agreement with self-duality, as in Fig. 4.1. In terms of the canonical basis one has $\boldsymbol{\Phi} = I_3$, namely

$$\Phi = \sum_{i=1}^{3} \lambda_i \otimes \lambda_i. \tag{4.10}$$

According to the isomorphism $\mathfrak{S}_+^{\odot 2} \simeq \mathfrak{T}_+$ induced by the faithful state each bipartite state has the same representative matrix of the corresponding transposed transformation. Indeed let Ψ be a state in $\mathfrak{S}_+^{\odot 2}$ and $\mathscr{A}$ the associated transformation in $\mathfrak{T}_+$, then

$$\begin{aligned}\Psi = (\mathscr{I},\mathscr{A})\Phi \Rightarrow \Psi(l_i,l_j) = (\mathscr{I},\mathscr{A})\Phi(l_i,l_j) \Rightarrow \Psi(l_i,l_j) = \Phi(l_i, l_j\circ\mathscr{A})\\ \Rightarrow \Psi_{ij} = \sum_k A_{jk}\Phi_{ik} \equiv \boldsymbol{A}^\tau = \boldsymbol{\Phi}^{-1}\boldsymbol{\Psi} \equiv \boldsymbol{A}^\tau = \boldsymbol{\Psi}.\end{aligned} \tag{4.11}$$

[1] We will better discuss the connection between local and bipartite systems in PartIII.

4.1.3 Positive transformations

In the Popescu-Rohrlich probabilistic theory the bipartite system was given by the original model and we had to generalize the latter by introducing the correct set of transformation in order to respect the isomorphism $\mathfrak{S}_+^{\odot 2} \simeq \mathfrak{T}_+$ induced by the correct symmetric preparationally faithful state. The construction of the two-clocks probabilistic model is not the same. In this case we can choose the bipartite symmetric state Φ in Eq. (4.7) and take it as a preparationally faithful state. Then the cone of bipartite states $\mathfrak{S}_+^{\odot 2}$ will be generated via Eq. (4.11) from the whole set $\mathfrak{T}_+$ of completely positive transformations. We are left with the problem of decide which transformations in $\mathfrak{L}_+$ are completely positive and which are only positive.

As usual, in order to find the set $\mathfrak{L}_+$ we need only its extremals $\mathrm{Extr}(\mathfrak{L}_+)$. We start to find the transformation $\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\,\mathsf{G}_{\mathfrak{E}}]$[2]. The geometry of the convex set of effects represented in fig.(4.1) entails the set $\mathsf{G}_{\mathfrak{E}}$ of $\mathfrak{E}$-generators to be made of a single effect plus the deterministic and the null one. Indeed an effect lying on the circle $x^2+y^2 = 1/2$ will generate via the automorphism of $\mathfrak{S}$ the entire circle. Adding the deterministic and the null effects we get the set $\mathrm{Extr}(\mathfrak{E})$ and then, via convex combinations, the whole set $\mathfrak{E}$. For convenience consider

$$\mathsf{G}_{\mathfrak{E}} = \{e, o, a_0\} \tag{4.12}$$

where a_0 has representative

$$\lambda(a_0) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}. \tag{4.13}$$

Extremal transformations in $[a_0]$**.** The equivalence class of contractions $[a_0]$ is made of the following matrices in block representation

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \tag{4.14}$$

As usual $\mathscr{A}$ is a contraction if and only if

$$\mathscr{A}\omega \in \mathfrak{S}_+ \qquad \forall \omega \in \mathrm{Extr}(\mathfrak{S}). \tag{4.15}$$

where $\mathrm{Extr}(\mathfrak{S})$ can be parametrized as follows

$$l(\omega_\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix}, \qquad \theta \in [0, 2\pi]. \tag{4.16}$$

In particular Eq.(4.15) has to be satisfy for the state

$$l(\omega_\pi) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{4.17}$$

[2] We know from 1.2 that the elements in $\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\,\mathsf{G}_{\mathfrak{E}}]$ are extremal positive transformation.

which leads to

$$\boldsymbol{A}l(\omega_\pi) = \begin{bmatrix} -A_{11} + A_{13} \\ -A_{21} + A_{23} \\ 0 \end{bmatrix} \in \mathfrak{S}_+. \tag{4.18}$$

Remembering the $\mathfrak{S}_+$ definition given in Eq.(4.1) the last condition is equivalent to

$$(-A_{11} + A_{13})^2 + (-A_{21} + A_{23})^2 \leqslant 0 \tag{4.19}$$

which is satisfied if and only if

$$A_{11} = A_{13}, \qquad A_{21} = A_{23}. \tag{4.20}$$

Consequently the $[a_0]$-contractions in Eq. (4.14) take the general form

$$\boldsymbol{A} = \begin{bmatrix} A & C & A \\ B & D & B \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \tag{4.21}$$

whose positivity is ensured by

$$\boldsymbol{A}l(\omega_\theta) = \begin{bmatrix} A\cos\theta + C\sin\theta + A \\ B\cos\theta + D\sin\theta + B \\ \frac{1}{2}(1+\cos\theta) \end{bmatrix} \in \mathfrak{S}_+ \qquad \forall\theta \in [0, 2\pi]. \tag{4.22}$$

By ordinary trigonometric relations one has

$$\begin{aligned} \begin{bmatrix} A\cos\theta + C\sin\theta + A \\ B\cos\theta + D\sin\theta + B \\ \frac{1}{2}(1+\cos\theta) \end{bmatrix} &= (1+\cos\theta) \begin{bmatrix} A + C\frac{\sin\theta}{1+\cos\theta} \\ B + D\frac{\sin\theta}{1+\cos\theta} \\ \frac{1}{2} \end{bmatrix} \\ &= (1+\cos\theta) \begin{bmatrix} A + C\tan\theta/2 \\ B + D\tan\theta/2 \\ \frac{1}{2} \end{bmatrix} \in \mathfrak{S}_+ \qquad \forall\theta \in [0, 2\pi]. \end{aligned} \tag{4.23}$$

namely

$$(A + C\tan\theta/2)^2 + (B + D\tan\theta/2)^2 \leqslant \frac{1}{4} \qquad \forall\theta \in [0, 2\pi]. \tag{4.24}$$

A further simplification came from the following limit

$$\lim_{\theta\to\pi} \tan\theta/2 = \infty \qquad \Rightarrow \qquad C = 0,\ D = 0. \tag{4.25}$$

Finally the contractions in the equivalence class $[a_0]$ are all the matrixes

$$\boldsymbol{A} = \begin{bmatrix} A & 0 & A \\ B & 0 & B \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \qquad \forall A, B \quad \text{such that} \quad A^2 + B^2 \leqslant \frac{1}{4}. \tag{4.26}$$

Among these contractions the extremal ones are the matrixes which achieve the equality $A^2 + B^2 = \frac{1}{4}$. Whence

$$\boldsymbol{A} \in \mathrm{Extr}[a_0] \;\Leftrightarrow\; \boldsymbol{A} = \frac{1}{2} \begin{bmatrix} \pm\cos\theta & 0 & \pm\cos\theta \\ \sin\theta & 0 & \sin\theta \\ 1 & 0 & 1 \end{bmatrix}, \qquad \theta \in [0, 2\pi] \tag{4.27}$$

Extremal deterministic contractions. Now we need Extr[e] which are the extremal deterministic contractions. In particular this class include the set of automorphism of $\mathfrak{S}$. This set of states is defined in Eq. (4.4) and is represented by the red disc in Fig. 4.1. The set of automorphism for a disc is the set of its rotations and reflections. The rotation $\mathscr{R}_\phi$ around the cone axis are represented in block form by the following matrixes

$$\boldsymbol{R}_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \phi \in [0, 2\pi], \tag{4.28}$$

while the disc reflections $\mathscr{S}_\phi$ through the axis at ϕ are represented as

$$\boldsymbol{S}_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \phi \in [0, \pi]. \tag{4.29}$$

A generic contraction $\mathscr{D}$ in [e] has to satisfy Eq. (4.15) and then

$$\boldsymbol{D} = \begin{bmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix} \in \mathfrak{S}, \qquad \forall\theta \in [0, 2\pi] \tag{4.30}$$

namely

$$\begin{aligned} &(A^2 + D^2)\cos^2\theta + (B^2 + E^2)\sin^2\theta + C^2 + F^2 + 2(AB + DF)\cos\theta\sin\theta + \\ &2(AC + DF)\cos\theta + 2(BC + EF)\sin\theta \leqslant 1 \qquad \forall\theta \in [0, 2\pi]. \end{aligned} \tag{4.31}$$

In this expression the extremal contractions achieve the equal and from the particular chooses $\theta = 0$, $\theta = \pi$, $\theta = \pi/2$ and $\theta = 3\pi/2$ we get

$$\begin{aligned} A^2 + D^2 + C^2 + F^2 + 2(AC + DF) &= 1 \\ A^2 + D^2 + C^2 + F^2 - 2(AC + DF) &= 1 \\ B^2 + E^2 + C^2 + F^2 + 2(BC + EF) &= 1 \\ B^2 + E^2 + C^2 + F^2 - 2(BC + EF) &= 1 \end{aligned} \tag{4.32}$$

Matching the first two equalities one has $AC + DF = 0$ and, in the same way, the other conditions imply $BC + EF = 0$. Eqs. (4.32) reduce to the couple

$$\begin{aligned} A^2 + D^2 + C^2 + F^2 &= 1 \\ B^2 + E^2 + C^2 + F^2 &= 1 \end{aligned} \tag{4.33}$$

and then $(A^2 + D^2) = (B^2 + E^2) = 1 - C^2 - F^2$. This identifications inserted in Eq. (4.31) provide $AB + DE = 0$. In conclusion the constraints for the variables A, B, C, D, E, F are

$$\begin{aligned} AC + DF &= 0 \\ BC + EF &= 0 \\ AB + DE &= 0 \\ (A^2 + D^2) &= 1 - C^2 - F^2 \\ (B^2 + E^2) &= 1 - C^2 - F^2, \end{aligned} \tag{4.34}$$

whose solution are the contractions having as representative the sets of matrixes

elliptical-transformations
ellipsoid

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \forall\theta \in [0, 2\pi] \tag{4.35}$$

and

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \forall\theta \in [0, \pi], \tag{4.36}$$

Comparing the result with Eqs. (4.28) and (4.29) we get

$$\mathrm{Extr}[e] = \mathsf{Aut}(\mathfrak{S}). \tag{4.37}$$

As usual the set $\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\mathsf{G}_{\mathfrak{E}}]$ can be found by acting with the automorphism over $\mathrm{Extr}[\mathsf{G}_{\mathfrak{E}}]$. Naturally the action over $\mathrm{Extr}[e]$ reproduces the group of automorphism while the action over $\mathrm{Extr}[a_0]$ give $\mathrm{Extr}[a_\phi]$ for $\phi \in [0, 2\Pi]$. Whence[3]

$$\boldsymbol{A}^\phi \in \mathrm{Extr}[a_\phi] \;\Leftrightarrow\; \boldsymbol{A}^\phi = \boldsymbol{A}\boldsymbol{R}_\phi \qquad \forall \boldsymbol{A} \in \mathrm{Extr}[a_0] \tag{4.38}$$

and explicitly

$$\boldsymbol{A}^\phi = \frac{1}{2}\begin{bmatrix} \pm\cos\theta\cos\phi & \pm\cos\theta\sin\phi & \pm\cos\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \sin\theta \\ \cos\phi & \sin\phi & 1 \end{bmatrix}, \qquad \theta,\phi \in [0, 2\pi]. \tag{4.39}$$

The local structure of the clock is more complicated than the square one of the Popescu-Rohrlik model. in fact in this case we have

$$\mathrm{Extr}[\mathsf{Aut}(\mathfrak{S})\,\mathsf{G}_{\mathfrak{E}}] \subset \mathrm{Extr}(\mathfrak{L}_+), \tag{4.40}$$

namely the set of automorphism of $\mathfrak{S}$ plus the transformations in Eq. (4.39) are not all the extremals of $\mathfrak{L}_+$. We have been forgetting some extremals.

The whole set $\mathfrak{L}_+$. It's easy to verify that the extremals contractions of $\mathfrak{L}_+$ are the transformations which send $\mathrm{Extr}(\mathfrak{S})$ into an elliptical conic of $\mathsf{Erays}\mathfrak{S}_+$,[4] we will denote such transformations as **elliptical-transformations** (see also the **ellipsoid** paragraph in Ssec. 1.1.4) . There exist three different kinds of elliptical conics:

- *Circles.* In these case the map $\mathscr{A}$ send $\mathrm{Extr}(\mathfrak{S})$ into a circle (which is a particular ellipse) and then $\mathfrak{S}$ into a disc. Naturally these transformations are the set of transformation proportional to the automorphism of $\mathsf{Aut}(\mathfrak{S})$. They have already been taken into account.

[3] To generate $\mathrm{Extr}[a_\phi]$ from $\mathrm{Extr}[a_0]$ we can use both the set of rotations or the set of reflections because the result is the same.

[4] A conic section, or just a conic, is a curve obtained by intersecting a cone (more precisely, a circular conical surface) with a plane.

- *Degenerate conics.* A conic is said to be degenerate when the intersection between the cone and the plane is a line, namely the plane is tangent to the cone. In these case the map $\mathscr{A}$ send Extr($\mathfrak{S}$) into an extremal ray of $\mathfrak{S}$. Also these transformations have already been taken into account, in fact it' s easy to verify by calculation that are exactly the ones in Eq. (4.39) or transformations proportional to them.

- *True ellipses.* In these case Extr($\mathfrak{S}$) is send into a true ellipse and the transformations which provide such mappings are extremal even if the the corresponding effect is not.

Now we have to find the extremal transformations of the last kind in order to classify all the elements in $\mathfrak{L}_+$. We can characterize such transformations by the perspective function (see Chap. 1 for its definition). in fact a general linear contraction maps $\mathfrak{S}$ in a convex $\mathscr{A}\mathfrak{S}$ lying inside $\mathfrak{S}_+$ and its perspective imagine $P(\mathscr{A}\mathfrak{S})$ is a convex tightly included in $\mathfrak{S}$ while for each elliptical-transformation we get

$$P(\mathscr{A}\mathfrak{S}) = \mathfrak{S}. \tag{4.41}$$

We can understand this property in Fig.4.2 where we have considered, for example, the non extremal effect

$$\lambda(a) = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{3}{4} \end{bmatrix}. \tag{4.42}$$

We can find a conic-transformation in its equivalence class having representative

$$\boldsymbol{A} = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} \tag{4.43}$$

and whose action is in the left pictures of Fig.4.2. The convex $\mathscr{A}\mathfrak{S}$ is the full ellipse inside the cone and its perspective function is represented by the cyan disc at the top or in the projection in the plane at $z = -1$. The effect a in Eq. (4.42) can also be achieved as convex combinations of other effect but there is no way to achieve $\mathscr{A}$ by the same convex combination of elements in their equivalence classes. The result of such convex combination will be similar to the one in the right picture of Fig.4.2 where we can observe the perspective map sending the oval inside the cone into the cyan oval tightly included into the disc representing $\mathfrak{S}$. For this reason $\mathscr{A}$ is an extremal transformation and by the application of the automorphism in $\mathsf{Aut}(\mathfrak{S})$ on the left we obtain all the extremal transformations in $[a]$; the application on the right provide all the transformations in $[a \circ \mathsf{Aut}(\mathfrak{S})]$. We can generalize this results for the other non extremals effects whose equivalence classes contains elliptical-transformations. These leads to the following matrixes

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 0 & (1-\lambda) \\ 0 & \sqrt{2\lambda - 1} & 0 \\ (1-\lambda) & 0 & \lambda \end{bmatrix}, \qquad \lambda \in \left(\frac{1}{2}, 1\right). \tag{4.44}$$

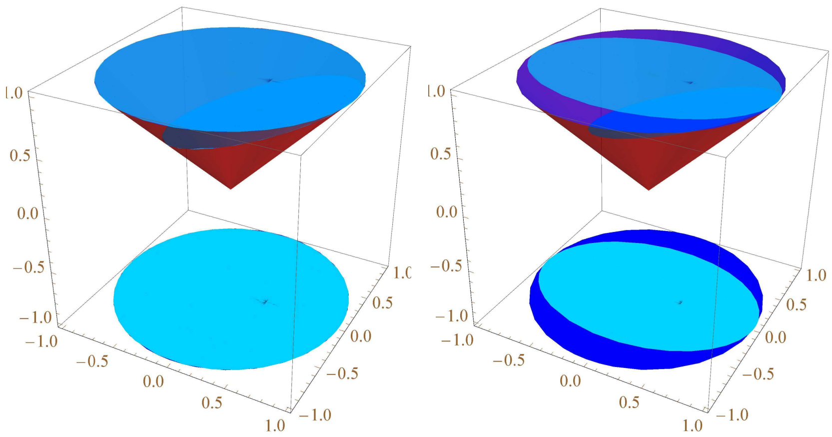


Figure 4.2: **Left figure:** The action of an elliptical-transformation $\mathscr{A}$ in the equivalence class of a non extremal effect a (see Eq. (4.42)). The cyan full ellipse inside the cone represent $\mathscr{A}\mathfrak{S}$ while the cyan disc at the top represent $P(\mathscr{A}\mathfrak{S})$ namely the perspective imagine of $\mathscr{A}\mathfrak{S}$. Notice that such disc is $\mathfrak{S}$. The disc in the plane $z = -1$ is the projection in the last plane of the perspective imagine. **Right figure:** Here are represented the same objects as in the figure on the left but relative to a map $\mathscr{A}$ achieved as a particular convex combination of transformations staying respectively in equivalence classes of effects having a as the same convex combination. In this case the perspective imagine is tightly included in $\mathfrak{S}$.

Naturally the whole set of extremals needs the automorphism application and then by applying rotation to $\mathscr{A}$ on the left we achieve all the elliptical-transformations in the equivalence class of the last row of $\boldsymbol{A}$

$$\lambda(a) = \begin{bmatrix} 1-\lambda \\ 0 \\ \lambda \end{bmatrix}, \qquad \lambda \in \left(\frac{1}{2}, 1\right) \tag{4.45}$$

while the application of rotations and reflections on the right provide the elliptical transformations for the sequent non extremal effect

$$\begin{aligned} a \circ \mathscr{R}_\phi = \boldsymbol{R}_\phi \lambda(a) &= \begin{bmatrix} (1-\lambda)\cos\phi \\ -(1-\lambda)\sin\phi \\ \lambda \end{bmatrix}, \qquad \phi \in [0, 2\pi],\ \lambda \in \left(\frac{1}{2}, 1\right) \\ a \circ \mathscr{S}_\phi = \boldsymbol{S}_\phi \lambda(a) &= \begin{bmatrix} (1-\lambda)\cos 2\phi \\ (1-\lambda)\sin 2\phi \\ \lambda \end{bmatrix}, \qquad \phi \in [0, \pi],\ \lambda \in \left(\frac{1}{2}, 1\right) \end{aligned} \tag{4.46}$$

complete probabilistic theory

Here are reported explicitly the transformations corresponding to $\mathscr{S}_\phi \circ \mathscr{A} \circ \mathscr{S}_\theta{}^\tau$

$$\boldsymbol{S}_\phi \boldsymbol{A} \boldsymbol{S}_\theta{}^\tau =$$

$$\begin{bmatrix} \lambda\cos 2\theta\cos 2\phi + \sqrt{2\lambda-1}\sin 2\theta\sin 2\phi & \lambda\cos 2\theta\sin 2\phi - \sqrt{2\lambda-1}\sin 2\theta\cos 2\phi & (1-\lambda)\cos 2\theta \\ \lambda\sin 2\theta\cos 2\phi - \sqrt{2\lambda-1}\cos 2\theta\sin 2\phi & \lambda\sin 2\theta\sin 2\phi + \sqrt{2\lambda-1}\cos 2\theta\cos 2\phi & (1-\lambda)\sin 2\theta \\ (1-\lambda)\cos 2\phi & (1-\lambda)\sin 2\phi & \lambda \end{bmatrix} \quad (4.47)$$

$$\phi, \theta \in [0, \pi] \ \lambda \in \left(\frac{1}{2}, 1\right).$$

but the others three combinations $\mathscr{R}_\phi \circ \mathscr{A} \circ \mathscr{R}_\theta{}^\tau$, $\mathscr{R}_\phi \circ \mathscr{A} \circ \mathscr{S}_\theta{}^\tau$ and $\mathscr{S}_\phi \circ \mathscr{A} \circ \mathscr{R}_\theta{}^\tau$ are exactly the same for lass than signs.

Observation 4.1 *Naturally we could include all the positive transformations in a single class. In fact the transformations corresponding to a degenerate conic are exactly included in the class of true elliptical transformations where the parameter λ is $\frac{1}{2}$ while the transformations corresponding to circles needs λ to be* 1. *On the other hand, in the classifications of the bipartite states, it will result to be useful keeping these three classes separated.*

4.1.4 The complete two-clocks probabilistic theory with local self-duality

Physical transformations

The probabilistic theory, having the clock as local self-dual system, with the largest set of physical transformations is achieved by taking

$$\mathfrak{T}_+ = \mathfrak{L}_+. \quad (4.48)$$

All the positive maps are declared completely positive maps and then physical transformations of the bipartite system. For this reason we will nominate **complete probabilistic theory** the arising model.

The bipartite cone of states

From the isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$ in Eq. (4.11) induced by the preparationally faithful state we are going to generate the bipartite set of states. For convenience we can divide the pure states in the same tree categories in which extremals transformations have been classified according to the relative conics.

- *Circles.* These states correspond to the transformations in $\mathsf{Aut}(\mathfrak{S})$ and then

$$\begin{aligned} \boldsymbol{\Psi} = \boldsymbol{R}_\phi{}^\tau &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \phi \in [0, 2\pi], \\ \boldsymbol{\Psi} = \boldsymbol{S}_\phi{}^\tau &= \begin{bmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \phi \in [0, \pi]. \end{aligned} \quad (4.49)$$

factorized bipartite pure states
non local bipartite pure states

- *Degenerate conics.* Obviously are the states corresponding to the transformations in Eq. (4.39) which send $\mathfrak{S}$ into an extremal ray of $\mathfrak{S}_+$. Whence, after normalization, we get

$$\boldsymbol{\Psi} = \begin{bmatrix} \pm\cos\theta\cos\phi & \sin\theta\cos\phi & \cos\phi \\ \pm\cos\theta\sin\phi & \sin\theta\sin\phi & \sin\phi \\ \pm\cos\theta & \sin\theta & 1 \end{bmatrix}, \qquad \theta,\phi \in [0,2\pi]. \tag{4.50}$$

- *True ellipses.* For our further purpose these states are the most important and correspond to the elliptical-transformations namely

$$\begin{aligned} \Psi &= \frac{(\mathscr{S}_\theta, \mathscr{S}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a\circ\mathscr{S}_\theta)} & \Psi &= \frac{(\mathscr{S}_\theta, \mathscr{R}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a\circ\mathscr{S}_\theta)} & & \phi\in[0,\pi],\ \theta\in[0,2\pi], \\ \Psi &= \frac{(\mathscr{R}_\theta, \mathscr{R}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a\circ\mathscr{R}_\theta)} & \Psi &= \frac{(\mathscr{R}_\theta, \mathscr{S}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a\circ\mathscr{R}_\theta)} & & \phi\in[0,\pi],\ \theta\in[0,2\pi], \end{aligned} \tag{4.51}$$

with $\mathscr{A}$ the map in Eq. (4.44). For completeness we report explicitly the matrixes representing the states associated to the transformations $\mathscr{S}_\phi \circ \mathscr{A} \circ \mathscr{S}_\theta$ remembering that the others differentiates from these ones only in signs

$$\boldsymbol{\Psi} = \frac{\boldsymbol{\Phi}(\boldsymbol{S}_\phi \boldsymbol{A} \boldsymbol{S}_\theta{}^\tau)^\tau}{\lambda(e)^\tau \boldsymbol{\Phi}\lambda(a\circ\mathscr{S}_\theta)} = \frac{(\boldsymbol{S}_\phi \boldsymbol{A} \boldsymbol{S}_\theta{}^\tau)^\tau}{\lambda} = \\ \begin{bmatrix} \cos 2\theta\cos 2\phi + \frac{\sqrt{2\lambda-1}}{\lambda}\sin 2\theta \sin 2\phi & \sin 2\theta\cos 2\phi - \frac{\sqrt{2\lambda-1}}{\lambda}\cos 2\theta\sin 2\phi & \frac{(1-\lambda)}{\lambda}\cos 2\phi \\ \cos 2\theta\sin 2\phi - \frac{\sqrt{2\lambda-1}}{\lambda}\sin 2\theta\cos 2\phi & \sin 2\theta\sin 2\phi + \frac{\sqrt{2\lambda-1}}{\lambda}\cos 2\theta\cos 2\phi & \frac{(1-\lambda)}{\lambda}\sin 2\phi \\ \frac{(1-\lambda)}{\lambda}\cos 2\theta & \frac{(1-\lambda)}{\lambda}\sin 2\theta & 1 \end{bmatrix}, \\ \phi,\theta\in[0,\pi]\ \lambda\in\left[\frac{1}{2},1\right] \tag{4.52}$$

Notice that the states in Eq. (4.50) are the **factorized bipartite pure states** given by

$$l(\omega_\theta)\otimes l(\omega_\phi) \qquad \forall\omega_\theta,\omega_\phi\in \mathrm{Extr}(\mathfrak{S}), \tag{4.53}$$

while the states in Eqs. (4.49) and (4.51) are the **non local bipartite pure states** of the "complete" model.

The bipartite set of effects.

The developed two clocks probabilistic theory is self dual at the single system level. Then the bilinear form Φ introduced in Ssec. 4.1.2 was also an isomorphism between the local cone of states and effects and we were able to choose it as the bipartite faithful state of the theory. Remember that the Jordan form of the bilinear form Φ coincide with this last one $|\Phi| = \Phi$. On the other hand the model is not self dual at the bipartite system level. The cone of states $\mathfrak{S}_+^{\odot 2}$ strictly include the cone of effects $\mathfrak{E}_+^{\odot 2}$. Firstly observe that for each local factorized pure state

$$\Psi = \omega\otimes\zeta \qquad \forall\omega,\zeta\in\mathrm{Extr}(\mathfrak{S}), \tag{4.54}$$

the corresponding representative $\boldsymbol{\Psi}$ represents an effect too, quite precisely the effect

$$F = a_\omega \otimes b_\zeta, \qquad \text{where} \qquad \Phi(a_\omega, \cdot) = \omega, \Phi(b_\zeta, \cdot) = \zeta. \tag{4.55}$$

We could assert that the self duality for the local system induces a partial self-duality for the composed system in correspondence to the local component of the bipartite cones. On the other hand such property is not preserved by the remaining component of the cones which is the non local one. For every bipartite non local pure state Ψ it's possible to find another bipartite non local pure state Ψ' such that

$$\Psi(\Psi') < 0, \tag{4.56}$$

which imply that Ψ is not proportional to a bipartite effect. An explicit example is given in the following paragraph about teleportation.

Teleportation

The probabilistic theory introduced in this Subsesection doesn't allow teleportation because the inverse of the preparationally faithfull effect is not a bipartite effect. In fact considering the following state in $\mathfrak{S}_+^{\odot 2}$

$$\Psi = (\mathscr{I}, \mathscr{R}_\pi)\Phi \equiv \boldsymbol{R}_\pi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.57}$$

we get

$$\Phi^{-1}(\Psi) = -1 \tag{4.58}$$

which is negative. More precisely we get

$$\Phi^{-1}(\Psi) \leqslant 0 \qquad \forall \Psi = (\mathscr{I}, \mathscr{R}_\phi)\Phi \;\text{ with }\; \phi \in [5\pi/6, 7\pi/6]. \tag{4.59}$$

Anyway We have developed a probabilistic model which is complete[5]while in general the set of physical transformations $\mathfrak{T}_+$ can be strictly included in the set of positive transformations. Reducing the number of physical transformation also the set $\mathfrak{S}_+^{\odot 2}$ is reduced and, the same time, the set of bipartite effects $\mathfrak{E}_+^{\odot 2}$ grows. A good question is how much the set $\mathfrak{T}_+$, and then $\mathfrak{S}_+^{\odot 2}$, has to be restricted in order to achieve a theory which allows teleportation preserving the purification property. This problem will not be further investigated in this thesis.

Purifiability at the single system's level

We know that a probabilistic theory is said to be purificable at the single system level if and only if for every local state ω in $\mathfrak{S}$ there exist a pure bipartite state Ω having ω as marginal state, namely

$$\forall \omega \in \mathfrak{S} \qquad \exists \Omega \in \mathfrak{S}^{\odot 2} \;\text{ such that }\; \Omega(\cdot, e) = \omega. \tag{4.60}$$

[5]Indeed by construction we have identified the set of positive transformations for the local system and the set of physical (completely positive) transformations $\mathfrak{T}_+$.

This is the case in two-clocks "complete" model. In our representation the marginalization on the second system of a bipartite state is simply the last column of its representative matrix. Then we can verify that the bipartite states corresponding to the elliptical-transformations provide purifications for all local states. Consider, for example, the pure bipartite states in Eq. (4.52) including the cases in which $\lambda = \frac{1}{2}$ and $\lambda = 0$ (see Obs. 4.1). Taking their marginalizations we find the set of local states

$$\Psi(\cdot, e) \equiv \mathbf{\Psi}\lambda(e) = \begin{bmatrix} \frac{(1-\lambda)}{\lambda}\cos\phi \\ \frac{(1-\lambda)}{\lambda}\sin\phi \\ 1 \end{bmatrix} \qquad \phi \in [0, 2\pi],\ \lambda \in \left[\frac{1}{2}, 1\right]. \tag{4.61}$$

This set coincides with $\mathfrak{S}$

$$\left\{ \begin{bmatrix} \frac{(1-\lambda)}{\lambda}\cos\phi \\ \frac{(1-\lambda)}{\lambda}\sin\phi \\ 1 \end{bmatrix} \text{for every } \phi \in [0, 2\pi],\ \lambda \in \left[\frac{1}{2}, 1\right] \right\} = \mathfrak{S} \tag{4.62}$$

ensuring purifiabilility of the model. From the other pure bipartite states new purifications are achieved.

Purifiability is unique under the group of automorphism. It's easy to verify that the purification is unique module the group of automorphisms. We briefly recall the meaning of this property. Given a local state $\omega \in \mathfrak{S}$ and two different purifications of such state—say Ω_1 and Ω_2—, then

$$\exists \mathscr{A} \in \mathsf{Aut}(\mathfrak{S}) \text{ such that } \Omega_2 = (\mathscr{A}, \mathscr{I})\Omega_1. \tag{4.63}$$

We can see how the automorphism acts on the space where the marginalization is performed. In order to check the uniqueness it's sufficient to look at a purification of a state ω in $\mathfrak{S}$ as state in $\text{Extr}(\mathfrak{S}^{\odot 2})$ whose last column coincides with the representative of ω. If $\Omega \in \mathfrak{S}^{\odot 2}$ is a purification of ω then the states $(\mathscr{I}, \mathscr{R}_\phi)\Omega$ and $(\mathscr{I}, \mathscr{S}_\phi)\Omega$ are purifications of the same state, in fact their last column is the same as the $\mathbf{\Omega}$'s one, whence

$$\begin{aligned} (\mathscr{I}, \mathscr{R}_\phi)\Omega(\cdot, e) &= \omega \qquad \forall \phi \in [0, 2\pi], \\ (\mathscr{I}, \mathscr{S}_\phi)\Omega(\cdot, e) &= \omega \qquad \forall \phi \in [0, \pi]. \end{aligned} \tag{4.64}$$

The coefficient $\frac{1-\lambda}{\lambda}$ is the third column of a pure states Ω from elliptical transformations, represents the ray of the circumferences in $\mathfrak{S}$ on which the marginal state $\omega = \Omega(\cdot, e)$ lies. All the other purifications of ω are all the possible pure states having the same third column and then are the states in Eq. (4.64) plus the ones achieved by automorphism acting on the first space leaving the third column unchanged. But for all these last states there exist an automorphism that achieve them acting on the second space ensuring the uniqueness module $\mathsf{Aut}(\mathfrak{S})$. In particular the elements in $\mathsf{Aut}(\mathfrak{S})$ are all the purifications for the state χ at the centre of the disc representing $\mathfrak{S}$. Pointedly these purifications are connected by local automorphism since $\mathsf{Aut}(\mathfrak{S})$ is a group.

The uniqueness of the purification at the bipartite level doesn't hold. The present model satisfy the PURIFY postulate plus the uniqueness of the purification module the action of the set of local automorphisms. On the other hand teleportation is not achievable. According to THEOREM, if there were uniqueness of purification at the bipartite level then teleportation should be achievable. This probabilistic model give us the following important information about purifiability

purifiability for all the elementary systems plus uniqueness module the set of local automorphisms	$\not\Rightarrow$	**purifiability for the composed systems plus uniqueness module the set of local automorphisms.**

The proof of the THEOREM draw heavily on the unique purifications for the bipartite system and that's the reason because teleportation is not achievable in the complete two clocks model here developed.

4.2 An alternative two-clocks complete probabilistic theory

The studied two clocks complete model is self dual at the single system level. A good question is if it's possible to construct a self dual complete theory at the bipartite system level by identification of the cones of states $\mathfrak{S}_+^{\odot 2}$ and effects $\mathfrak{E}_+^{\odot 2}$. In order to achieve such identification the faithful state assumed in Ssec. 4.1.2 must be changed. The set of automorphisms $\mathsf{Aut}(\mathfrak{S})$ depends only of the local system and are always rotations and reflections if we choose the clock. In a self dual theory each bipartite state associates to a positive transformation must be proportional to a bipartite effect. If we include $\mathsf{Aut}(\mathfrak{S})$ in $\mathfrak{T}_+$ the corresponding states are

$$\begin{aligned} \Psi = (\mathscr{I}, \mathscr{R}_\phi)\Phi \Rightarrow \mathbf{\Psi} = \mathbf{\Phi}^{-1}\boldsymbol{R_\phi}^\tau, \\ \Psi = (\mathscr{I}, \mathscr{S}_\phi)\Phi \Rightarrow \mathbf{\Psi} = \mathbf{\Phi}^{-1}\boldsymbol{S_\phi}^\tau, \end{aligned} \tag{4.65}$$

where Φ is the new faithful state. Such matrixes would represent elements of $\mathfrak{E}_+^{\odot 2}$ if and only if $\mathbf{\Phi}$ is as follows

$$\mathbf{\Phi} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \tag{4.66}$$

Then we can choose as faithful state of the theory the bipartite functional

$$\Phi(a, b) = \frac{1}{\sqrt{2}} \boldsymbol{\lambda}(a) \cdot \boldsymbol{\lambda}(b) + \lambda_3(a) \cdot \lambda_3(b), \tag{4.67}$$

whose matrix representative is as in Eq. (4.66). With this particular choice rotations and reflections are both bipartite states and effects. However the self duality at the elliptical-transformations

bipartite system level destroys the local self duality, in fact it's no longer $\Phi = |\Phi|$. If the state in Eq. (4.67) is faithful then it is an isomorphism between the local cones $\mathfrak{S}_+$ and $\mathfrak{E}_+$. The matrix representation of Φ clearly display a restriction of the cone $\mathfrak{S}_+$ with respect to $\mathfrak{E}_+$ according to the isomorphism

$$\Phi(a, \cdot) = \omega \Rightarrow l(\omega) = \mathbf{\Phi}\lambda(a). \tag{4.68}$$

Naturally if the local system is not self dual not even the bipartite one is self-dual because the local component of the bipartite cones of states and effects doesn't coincide. We can anyway further investigate the probabilistic theory arising from this new local system. The local cones of states becomes

$$\mathfrak{S}_+ = \left\{ l(\omega) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 \leqslant \frac{t^2}{\sqrt{2}},\ t \geqslant 0 \right\}, \tag{4.69}$$

whose dual cone of effects,

$$\mathfrak{E}_+ = \left\{ \lambda(a) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 \leqslant \sqrt{2}t^2,\ t \geqslant 0 \right\}, \tag{4.70}$$

strictly include it. Naturally the set of states is yet a disc

$$\mathfrak{S} = \left\{ l(\omega) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 = \frac{1}{\sqrt{2}},\ z = 1 \right\}, \tag{4.71}$$

and the truncation of $\mathfrak{E}_+$ providing the set of effects is

$$\mathfrak{E} = \left\{ \lambda(a) = \begin{bmatrix} x \\ y \\ t \end{bmatrix},\ x^2 + y^2 \leqslant \min\left(\left(\frac{t}{2^{1/4}}\right)^2, \left(1 - \frac{t}{2^{1/4}}\right)^2 \right),\ t \in [0, 1] \right\}. \tag{4.72}$$

The local cones of states and effects are represented in Fig. 4.3.

Physical transformations

As in the previous model we must take all the positive maps as physical maps if we want a complete model

$$\mathfrak{T}_+ = \mathfrak{L}_+. \tag{4.73}$$

The positive transformation are the maps which preserve the cone of states $\mathfrak{S}_+$ in Eq. (4.69). We know that the block form of a positive contraction has an effect as last row and the arguments used in the previous model lead us to the same classifications of extremal contractions. This is obvious because the local system is the same, *i.e.* the clock. Whence $\mathrm{Extr}(\mathfrak{T}_+)$ are the **elliptical-transformations**, thus, with respect to the relative conics, the classification is

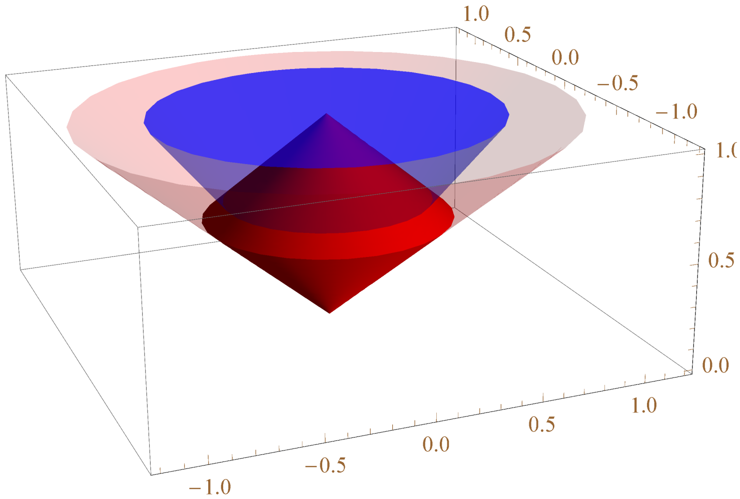


Figure 4.3: The blue disc at the top represent the set of states $\mathfrak{S}$. The blue transparent cone represent the cone $\mathfrak{S}_+$ while the red transparent cone is $\mathfrak{E}_+$. We can see how the cone of effects strictly include the cone of states which makes the model not self-dual at the single system level. The red solid inside the cone $\mathfrak{E}_+$ represent the convex set of effects $\mathfrak{E}$ which is the $\mathfrak{E}_+$-truncation given by the condition $a \leqslant e$ where a is a generic effect and e the deterministic one.

- *Circles*. As usual these transformations are the ones proportional to an automorphism in $\mathsf{Aut}(\mathfrak{S})$, namely rotations and reflections.

- *Deganerate conics*. The transformations that send the set of states in an extremal ray of the relative cone. The representative of the contractions in this class are as follows

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} \pm\cos\theta\cos\phi & \pm\cos\theta\sin\phi & \pm\frac{1}{2^{1/4}}\cos\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \frac{1}{2^{1/4}}\sin\theta \\ 2^{1/4}\cos\phi & 2^{1/4}\sin\phi & 1 \end{bmatrix}, \qquad \theta,\phi \in [0,2\pi]. \tag{4.74}$$

- *True ellipses*. These are the transformation which send $\mathrm{Extr}(\mathfrak{S})$ into a conic of $\mathfrak{S}_+$. The contractions of this class are represented by the maps $\mathscr{R}_\phi \circ \mathscr{A} \circ \mathscr{R}_\theta{}^\tau$, $\mathscr{R}_\phi \circ \mathscr{A} \circ \mathscr{S}_\theta{}^\tau$, $\mathscr{S}_\phi \circ \mathscr{A} \circ \mathscr{R}_\theta{}^\tau$ and $\mathscr{S}_\phi \circ \mathscr{A} \circ \mathscr{S}_\theta{}^\tau$ where the representative of $\mathscr{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 0 & \frac{1}{2^{1/4}}(1-\lambda) \\ 0 & \sqrt{2\lambda - 1} & 0 \\ 2^{1/4}(1-\lambda) & 0 & \lambda \end{bmatrix}, \qquad \lambda \in \left(\frac{1}{2}, 1\right). \tag{4.75}$$

As usual we report explicitly only the block representatives of $\mathscr{S}_\phi \circ \mathscr{A} \circ \mathscr{S}_\theta{}^\tau$, because the others are analogue

$$\boldsymbol{S}_\phi \boldsymbol{A} \boldsymbol{S}_\theta{}^\tau = \begin{bmatrix} \lambda\cos 2\theta\cos 2\phi + \sqrt{2\lambda-1}\sin 2\theta\sin 2\phi & -\lambda\cos 2\theta\sin 2\phi + \sqrt{2\lambda-1}\sin 2\theta\cos 2\phi & \frac{1-\lambda}{2^{1/4}}\cos 2\theta \\ \lambda\sin 2\theta\cos 2\phi - \sqrt{2\lambda-1}\cos 2\theta\sin 2\phi & \lambda\sin 2\theta\sin 2\phi + \sqrt{2\lambda-1}\cos 2\theta\cos 2\phi & \frac{1-\lambda}{2^{1/4}}\sin 2\theta \\ 2^{1/4}(1-\lambda)\cos 2\phi & 2^{1/4}(1-\lambda)\sin 2\phi & \lambda \end{bmatrix}$$

$$\phi, \theta \in [0,\pi], \ \lambda \in \left(\frac{1}{2}, 1\right). \tag{4.76}$$

The bipartite cone of states

The set of bipartite states is as usual generated from $\mathfrak{T}$ via the isomorphism $\mathfrak{S}_+^{\odot 2} \simeq \mathfrak{T}$ induced by the new preparationally faithful state Φ in Eq. (4.67). The pure bipartite states can be classified according to the relative extremal transformations and then according to the three different kind of conics as follows:

- *Circles.* Here we find the states corresponding to the automorphism in $\mathsf{Aut}(\mathfrak{S})$, namely

$$\boldsymbol{\Psi} = \boldsymbol{\Phi}\boldsymbol{R}_\phi{}^\tau = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \qquad \phi \in [0, 2\pi],$$
$$\boldsymbol{\Psi} = \boldsymbol{\Phi}\boldsymbol{S}_\phi{}^\tau = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \qquad \phi \in [0, \pi]. \tag{4.77}$$

- *Degenerate conics.* These are the local factorized pure states given by

$$\boldsymbol{\Psi} = l(\omega_\theta) \otimes l(\omega_\phi) = \frac{1}{\sqrt{2}}\begin{bmatrix} \pm\cos\theta\cos\phi & \sin\theta\cos\phi & 2^{1/4}\cos\phi \\ \pm\cos\theta\sin\phi & \sin\theta\sin\phi & 2^{1/4}\sin\phi \\ \pm 2^{1/4}\cos\theta & 2^{1/4}\sin\theta & \sqrt{2} \end{bmatrix} \tag{4.78}$$
$$\forall \omega_\theta, \omega_\phi \in \mathrm{Extr}(\mathfrak{S}). \tag{4.79}$$

- *True ellipses.* Finally there are the non local pure states

$$\Psi = \frac{(\mathscr{S}_\theta, \mathscr{S}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a \circ \mathscr{S}_\theta)} \qquad \Psi = \frac{(\mathscr{S}_\theta, \mathscr{R}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a \circ \mathscr{S}_\theta)} \qquad \phi \in [0,\pi], \ \theta \in [0, 2\pi],$$
$$\Psi = \frac{(\mathscr{R}_\theta, \mathscr{R}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a \circ \mathscr{R}_\theta)} \qquad \Psi = \frac{(\mathscr{R}_\theta, \mathscr{S}_\phi \circ \mathscr{A})\Phi}{\Phi(e, a \circ \mathscr{R}_\theta)} \qquad \phi \in [0,\pi], \ \theta \in [0, 2\pi], \tag{4.80}$$

with $\mathscr{A}$ the map in Eq. (4.75). The matrixes representing the first set of states in the last equation are

$$\boldsymbol{\Psi} = \frac{\boldsymbol{\Phi}(\boldsymbol{S}_\phi \boldsymbol{A} \boldsymbol{S}_\theta{}^\tau)^\tau}{\lambda(e)^\tau \boldsymbol{\Phi} \lambda(a \circ \mathscr{S}_\theta)} = \frac{\boldsymbol{\Phi}(\boldsymbol{S}_\phi \boldsymbol{A} \boldsymbol{S}_\theta{}^\tau)^\tau}{\lambda} =$$
$$\frac{1}{\sqrt{2}} \begin{bmatrix} \cos 2\theta \cos 2\phi + \frac{\sqrt{2\lambda-1}}{\lambda} \sin 2\theta \sin 2\phi & \sin 2\theta \cos 2\phi - \frac{\sqrt{2\lambda-1}}{\lambda} \cos 2\theta \sin 2\phi & \frac{2^{1/4}(1-\lambda)}{\lambda} \cos 2\phi \\ \cos 2\theta \sin 2\phi - \frac{\sqrt{2\lambda-1}}{\lambda} \sin 2\theta \cos 2\phi & \sin 2\theta \sin 2\phi + \frac{\sqrt{2\lambda-1}}{\lambda} \cos 2\theta \cos 2\phi & \frac{2^{1/4}(1-\lambda)}{\lambda} \sin 2\phi \\ \frac{2^{1/4}(1-\lambda)}{\lambda} \cos 2\theta & \frac{2^{1/4}(1-\lambda)}{\lambda} \sin 2\theta & 1 \end{bmatrix},$$
$$\phi, \theta \in [0, \pi] \ \lambda \in \left(\frac{1}{2}, 1\right) \tag{4.81}$$

The bipartite cone of effects. In the present model we have loosed the local self-duality but as already noticed we haven't achieved a bipartite self-dual probabilistic theory. In fact for each local factorized pure state

$$\Psi = \omega \otimes \zeta \qquad \forall \omega, \zeta \in \mathrm{Extr}(\mathfrak{S}), \tag{4.82}$$

the correspondent matrix $\boldsymbol{\Psi}$ certainly represent an element in $\mathfrak{E}_+$ too but it is not on the extremal rays of the bipartite cone of effects. The reason is the non local self-duality from which the extremal bipartite effect corresponding to the state in Eq. (e:4.82) is

$$F = a_\omega \otimes b_\zeta \qquad \text{where } \Phi(a_\omega, \cdot) = \omega, \Phi(a_zeta, \cdot) = \zeta. \tag{4.83}$$

Remember that the faithful state Φ in this model restrict the cone of states with respect to the cone of effects and then the matrix $\boldsymbol{F}$ doesn't coincide with the previous $\boldsymbol{\Psi}$, in particular we get

$$\boldsymbol{F} = \boldsymbol{\Phi} \Psi \boldsymbol{\Phi}^\tau, \tag{4.84}$$

which shows the squeezing of the bipartite cone of effects to produce the cone of states in correspondence of the local factorized extremals. On the other hand the maximally non local states, which obviously are the states corresponding to the set $\mathsf{Aut}(\mathfrak{S})$ are effects too by construction. All the other non local states aren't also proportional to effects.

Teleportation and purifiability

As for the previous model we can purify all the local states. Also the uniqueness of the purification holds module the local automorphisms. Teleportation is not achievable because the inverse of Φ is the matrix

$$\boldsymbol{\Phi}^{-1} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.85}$$

which is not an effect. In fact we know that the maximally non local component of the bipartite system is self-dual and then Φ^{-1} should be a state in order to be an effect too. This is not the case and taking for example the state

$$\boldsymbol{\Psi} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.86}$$

we get $\Phi^{-1}(\Psi) = -1$. The unique purification for the bipartite system cannot be satisfied.

equatorial q-bit
ball
Block sphere
local observability principle

4.3 A non local feature from the local system's structure

In Part III we will show how the strongest degree of correlations achievable in a probabilistic theory, which certainly is a crucial non local feature of the theory, has a local origin. Here we can observe another global property of a probabilistic theory which come from the local structure of the convex set of states. Observe that the situation in the two models developed in this section is the opposite. In the first model we took a local self-dual system but this forced us to choose a faithful state inducing a non self-dual bipartite system. Anyway when a system is not self dual we can speak about degree of non self duality in correspondence to two extremal correspondent rays of the cone of states and effects meaning how much the cones are separated in that direction. According to this, in the first model we have perfect coincidence between extremal rays of the bipartite dual cones in correspondence to the local component of the bipartite system; while increasing the non locality of the state and effect in exam the cones are even more squeezed one with respect to the other. In the second model the situation is exactly the opposite having identification of the extremal rays in correspondence to the maximally "entangled" states (the ones associated to local automorphisms) while the greatest squeezing correspond to the local component of the bipartite system. The situation is summarised in the flow chart 4.4.

AsWe have understood that it's impossible to achieve a probabilistic theory self dual at the bipartite system level having the clock as local system. All attempts to make the bipartite system self-dual lead to a bipartite faithful state which destroy the local self-duality and then make impossible the bipartite self-duality too. Then a global property of the system has local origin in the structure of the local system.

4.4 A hidden quantum model.

In the class of probabilistic theories having the clock as single system we can find an interesting model which is the **equatorial q-bit**. The set of normalized states of the q-bit system is a 3-dimensional **ball** known as **Block sphere**, then if we take a q-bit in its equatorial plane we achieve a 2-dimensional ball as convex set of states which is exactly a clock. This model is very interesting because it provide an example of violation of the **local observability principle**. Corresponding to the canonical basis $l = \{l_i\}$ and $\lambda = \{\lambda_i\}$ with $i = 1, 2, 3$ for $\mathfrak{E}_{\mathbb{R}}$ and $\mathfrak{S}_{\mathbb{R}}$ embedded into $\mathbb{R}^3$ as euclidean space, with

$$(l_i, \lambda_j) = l_i(\lambda_j) = \lambda_j(l_i) = \delta_{ij} \tag{4.87}$$

in terms of which we can write

$$(a, \omega) = \omega(a) = a(\omega) = l(\omega) \cdot \lambda(a) := \sum_{i=1}^{3} l_i(\omega)\lambda_i(a). \tag{4.88}$$

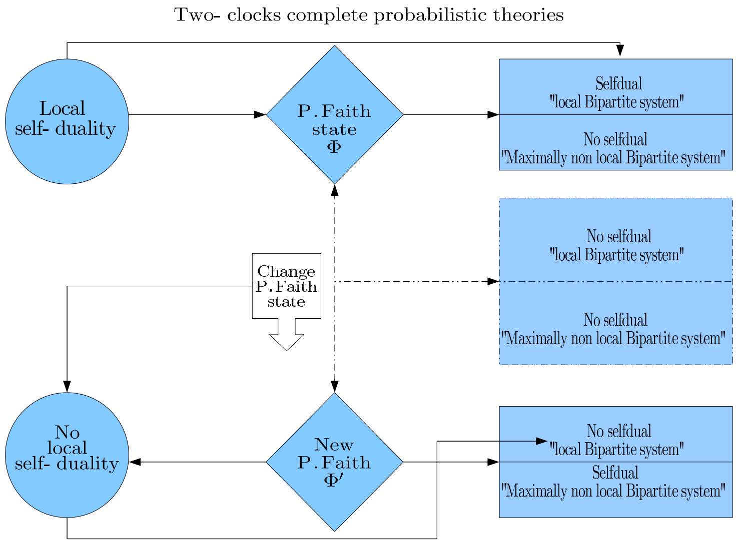


Figure 4.4: In this flow diagram we show the main steps which lead to the impossibility of achieving a complete probabilistic theory having the clock as local system and a self-dual bipartite system at the same time.

Upon defining the operator vector

$$\sigma = \begin{bmatrix} \sigma_z \\ \sigma_x, \\ I \end{bmatrix} \tag{4.89}$$

and introducing the canonical orthonormal basis $\{u_j\}$ for $\mathbb{R}^3$, we define the following map

$$\Upsilon : r \in \mathbb{R}^3 \leftrightarrow \Upsilon(r) \in \mathsf{Her}(\mathbb{R}^2), \quad \begin{cases} \Upsilon(r) = \frac{1}{\sqrt{2}} r \cdot \sigma, \\ \Upsilon^{-1}(A) = \frac{1}{\sqrt{2}} \operatorname{Tr}[A\sigma] \cdot u, \end{cases} \tag{4.90}$$

with pairing[6]

$$\Upsilon(r) \cdot \Upsilon(s) := \operatorname{Tr}[\Upsilon(r)\Upsilon(s)], \quad \operatorname{Tr}[AB] = \Upsilon^{-1}(A) \cdot \Upsilon^{-1}(B). \tag{4.91}$$

In terms of the canonical basis one has

$$\Upsilon(u_i) = \tfrac{1}{\sqrt{2}}\sigma_i, \quad (l_j, \lambda_i) = \tfrac{1}{2}\Upsilon^{-1}(\sigma_i) \cdot \Upsilon^{-1}(\sigma_j). \tag{4.92}$$

Specializing the map to states and effects we have

$$\omega \in \mathfrak{S}, \ \rho = \tfrac{1}{\sqrt{2}}\Upsilon(l(\omega)) \in \mathsf{St}(\mathbb{R}^2), \quad a \in \mathfrak{E}, \ A = \sqrt{2}\Upsilon(\lambda(a)) \in \mathsf{Lin}_+(\mathbb{R}^2), \tag{4.93}$$

[6]One has: $\Upsilon^{-1}(A)\cdot\Upsilon^{-1}(B) = \frac{1}{2}\operatorname{Tr}[A\sigma]\cdot\operatorname{Tr}[B\sigma] = \operatorname{Tr}[(A\otimes B)\frac{1}{2}\sum_i \sigma_i\otimes\sigma_i] = \operatorname{Tr}[AB^\tau] - \operatorname{Tr}[(A\otimes B)\sigma_y\otimes\sigma_y] = \operatorname{Tr}[AB]$.

with Born rule

$$\mathrm{Tr}[A\rho] = \Upsilon(l(\omega)) \cdot \Upsilon(\lambda(a)) \equiv (a, \omega), \tag{4.94}$$

$\mathsf{St}(\mathbb{R}^2)$ denoting the set of symmetric real matrices with unit trace. The extension of the map to tensor product is given by the "commutation rule" $\Upsilon\otimes = \otimes\Upsilon$, namely

$$\Upsilon(r \otimes s) := \Upsilon(r) \otimes \Upsilon(s), \quad \Upsilon^{-1}(A \otimes B) = \Upsilon^{-1}(A) \otimes \Upsilon^{-1}(B). \tag{4.95}$$

In the following we will use the abbreviate notation $\Upsilon(\omega) := \Upsilon(l(\omega))$ for states and $\Upsilon(a) := \Upsilon(\lambda(a))$ for effects. The faithful state has coefficients $\Phi_{ij} = l_i \otimes l_j(\Phi) \equiv \Phi(l_i, l_j) = \delta_{ij}$, whence the corresponding operator is given by

$$\tfrac{1}{2}\Upsilon(\Phi) = \tfrac{1}{2}\sum_{i=1}^{3} \Upsilon(\lambda_i) \otimes \Upsilon(\lambda_i) = \tfrac{1}{4}(I \otimes I + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z), \tag{4.96}$$

which is an Hermitian (non positive) operator with unit trace. Notice that such operator differs from the maximally entangled state by the term $\frac{1}{4}\sigma_y \otimes \sigma_y \notin \mathsf{Lin}(\mathbb{R}^2) \otimes \mathsf{Lin}(\mathbb{R}^2)$, since

$$\tfrac{1}{2}|I\rangle\langle I| = \tfrac{1}{4}(I \otimes I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z). \tag{4.97}$$

The term $\sigma_y \otimes \sigma_y \in \mathsf{Lin}(\mathbb{R}^4)$ corresponds to a null linear form over $\mathbb{R}^3 \otimes \mathbb{R}^3$ given by

$$\forall R \in \mathbb{R}^3 \otimes \mathbb{R}^3,\ \Xi(R) = \mathrm{Tr}[\sigma_y \otimes \sigma_y \Upsilon(R)] = 0. \tag{4.98}$$

Notice that the transposition acts as the identity map over $\Upsilon(\mathbb{R}^3)$, since transposition leaves σ_x, σ_z and I invariant, whence

$$\Upsilon^{-1}[\Upsilon(a)^\tau] = \mathscr{I}(a). \tag{4.99}$$

Using this identity one can also see that the maximally entangled state is another equivalent representation of the faithful state Φ, since $\forall r, s \in \mathbb{R}^3$ one has

$$\tfrac{1}{2}\langle I|\Upsilon(\lambda(a)) \otimes \Upsilon(\lambda(b))|I\rangle = \tfrac{1}{2}\mathrm{Tr}[\Upsilon(\lambda(a))\Upsilon(\lambda(b))^\tau] = \lambda(a) \cdot \lambda(b) = \Phi(a, b). \tag{4.100}$$

since transposition works as the identity over σ_x, σ_z and I.

Let's now represent maps in the hidden quantum model. A generic bipartite state is represented as

$$\Psi = \sum_{ij} \Psi_{ij}\lambda_i \otimes \lambda_j, \quad \Psi_{ij} = \Psi(l_i, l_j), \tag{4.101}$$

and the local action of the transformation $\mathscr{A}$ is given by

$$\begin{aligned}(\mathscr{A}, \mathscr{I})\Psi(l_i, l_j) &= \Psi(l_i \circ \mathscr{A}, l_j) = \sum_k M_{ik}(\mathscr{A})\Psi(l_k, l_j) = \\ &\sum_{nklm} M_{nl}\Psi_{km}(\mathscr{A})\lambda_n(l_i)l_l(\lambda_k)\lambda_m(l_j) = \tfrac{1}{2}\mathrm{Tr}[\Upsilon(\mathscr{A}) \star \Upsilon(\Psi)\sigma_i \otimes \sigma_j]\end{aligned} \tag{4.102}$$

where

$$\Upsilon(\mathscr{A}) := \tfrac{1}{2}\sum_{il} M_{il}(\mathscr{A})\sigma_i \otimes \sigma_l, \tag{4.103}$$

and

$$A \star B = \mathrm{Tr}_2[(A \otimes I)(I \otimes B)]. \tag{4.104}$$

Equivalently, one has

$$(\mathscr{A}, \mathscr{I})\Psi(l_i, l_j) = \tfrac{1}{2}\,\mathrm{Tr}[(\mathcal{A} \otimes \mathfrak{I})\Upsilon(\Psi)\sigma_i \otimes \sigma_j] = \tfrac{1}{2}\,\mathrm{Tr}[\Upsilon(\Psi)\tilde{\mathcal{A}}(\sigma_i) \otimes \sigma_j], \tag{4.105}$$

whence

$$\Upsilon[(\mathscr{A}, \mathscr{I})\Psi] = \Upsilon(\mathscr{A}) \star \Upsilon(\Psi) = (\mathcal{A} \otimes \mathfrak{I})\Upsilon(\Psi) = \Upsilon(\Psi)(\tilde{\mathcal{A}} \otimes \mathfrak{I}). \tag{4.106}$$

Let's now examine bipartite effects. Let's first take the generalized effect corresponding to the inverse matrix of Φ, *i.e.* which would achieve teleportation, and let see if it is a true effect. The matrix multiplication between two (considering Φ^{-1} as a map) is given by

$$\delta_{ij} = \tfrac{1}{2}\,\mathrm{Tr}[\Upsilon(\Phi^{-1}) \star \Upsilon(\Phi)\sigma_i \otimes \sigma_j], \tag{4.107}$$

and

$$(E^{\Phi}, \Phi) = \alpha(\Phi^{-1}, \Phi) = \alpha\Upsilon(\Phi^{-1}) \cdot \Upsilon(\Phi) = 3\alpha, \tag{4.108}$$

whence $\alpha = 1/3$, and one has probability of successful teleportation

$$(E^{\Phi}, e)(\omega, \Phi) = \alpha\,\mathrm{Tr}[(\Upsilon(\Phi^{-1}) \otimes \tfrac{1}{\sqrt{2}}I)(\Upsilon(\omega) \otimes \Upsilon(\Phi))] = \frac{1}{3}. \tag{4.109}$$

The space $\mathfrak{T}_{\mathbb{R}} \equiv \mathsf{Lin}(\mathfrak{S}_{\mathbb{R}}) = \mathsf{Lin}(\mathbb{R}^3)$ of linear maps over $\mathbb{R}^3$ can be obtained as follows

$$\mathfrak{T}_{\mathbb{R}} = \mathsf{Span}\{\mathscr{A}_{11}, \mathscr{A}_{22}, \mathscr{A}_{33}, \Re\mathscr{A}_{12}, \Re\mathscr{A}_{13}, \Re\mathscr{A}_{23}, \Im\mathscr{A}_{14}, \Im\mathscr{A}_{24}, \Im\mathscr{A}_{34}\} \tag{4.110}$$

where

$$\mathscr{A}_{ij}\omega := \Upsilon^{-1}[\sigma_i\Upsilon(\omega)\sigma_j], \tag{4.111}$$

and $\sigma_4 := \sigma_y$ Using Table 4.1 we obtain all the matrices $M(\mathscr{A}_{ij})$ in Eq. (4.110), which are reported in Table 4.2.

The automorphisms of the convex set of states are given by the rotations $\mathscr{T}_\phi$, $\phi \in [0, 2\pi)$ along with the reflections $\mathscr{R}_\phi$, $\phi \in [0, \pi)$ through the axis at ϕ, corresponding to $\mathscr{R}_\phi = \mathscr{T}_\phi\mathscr{R}_0\mathscr{T}_\phi^{-1}$. In addition to the automorphisms, we have the non deterministic transformations

$$M(\mathscr{T}_\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M(\mathscr{R}_\phi) = \begin{bmatrix} \cos 2\phi & -\sin 2\phi & 0 \\ -\sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.112}$$

	$\cdot\sigma_z$	$\cdot\sigma_x$	$\cdot I$	$\cdot\sigma_y$
$\sigma_z\cdot$	$\begin{bmatrix}\sigma_z\\-\sigma_x\\I\end{bmatrix}$	$\begin{bmatrix}\sigma_x\\\sigma_z\\i\sigma_y\end{bmatrix}$	$\begin{bmatrix}I\\i\sigma_y\\\sigma_z\end{bmatrix}$	$\begin{bmatrix}\sigma_y\\iI\\-i\sigma_x\end{bmatrix}$
$\sigma_x\cdot$	$\begin{bmatrix}\sigma_x\\\sigma_z\\-i\sigma_y\end{bmatrix}$	$\begin{bmatrix}-\sigma_z\\\sigma_x\\I\end{bmatrix}$	$\begin{bmatrix}-i\sigma_y\\I\\\sigma_x\end{bmatrix}$	$\begin{bmatrix}-iI\\\sigma_y\\i\sigma_z\end{bmatrix}$
$I\cdot$	$\begin{bmatrix}I\\-i\sigma_y\\\sigma_z\end{bmatrix}$	$\begin{bmatrix}i\sigma_y\\I\\\sigma_x\end{bmatrix}$	$\begin{bmatrix}\sigma_z\\\sigma_x\\I\end{bmatrix}$	$\begin{bmatrix}-i\sigma_x\\i\sigma_z\\\sigma_y\end{bmatrix}$
$\sigma_y\cdot$	$\begin{bmatrix}\sigma_y\\-iI\\i\sigma_x\end{bmatrix}$	$\begin{bmatrix}iI\\\sigma_y\\-i\sigma_z\end{bmatrix}$	$\begin{bmatrix}i\sigma_x\\-i\sigma_z\\\sigma_y\end{bmatrix}$	$\begin{bmatrix}-\sigma_z\\-\sigma_x\\I\end{bmatrix}$

	$\cdot\sigma_z$	$\cdot\sigma_x$	$\cdot I$	$\cdot\sigma_y$
$\sigma_z\cdot$	$\begin{bmatrix}x\\-y\\z\end{bmatrix}$	$\begin{bmatrix}y\\x\\0\end{bmatrix}$	$\begin{bmatrix}z\\0\\x\end{bmatrix}$	$\begin{bmatrix}0\\z\\-y\end{bmatrix}$
$\sigma_x\cdot$	$\begin{bmatrix}y\\x\\0\end{bmatrix}$	$\begin{bmatrix}-x\\y\\z\end{bmatrix}$	$\begin{bmatrix}0\\z\\y\end{bmatrix}$	$\begin{bmatrix}-z\\0\\x\end{bmatrix}$
$I\cdot$	$\begin{bmatrix}z\\0\\x\end{bmatrix}$	$\begin{bmatrix}0\\z\\y\end{bmatrix}$	$\begin{bmatrix}x\\y\\z\end{bmatrix}$	$\begin{bmatrix}y\\-x\\0\end{bmatrix}$
$\sigma_y\cdot$	$\begin{bmatrix}0\\-z\\y\end{bmatrix}$	$\begin{bmatrix}z\\0\\-x\end{bmatrix}$	$\begin{bmatrix}-y\\x\\0\end{bmatrix}$	$\begin{bmatrix}-x\\-y\\z\end{bmatrix}$

Table 4.1: **Left table:** Quantum operations $\sigma_\alpha \cdot \sigma_\beta$ over the vector of operators $[\sigma_z, \sigma_x, I]^\tau$ for $\alpha, \beta = z, x, 0, y$. **Right table:** corresponding transformation of the point on the cone. The off-diagonal terms within the three-by-three submatrix correspond to the action $\sigma_\alpha \cdot \sigma_\beta$+h.c., whereas the three off-diagonal terms on the border correspond to the action $\sigma_\alpha \cdot \sigma_\beta$ − h.c.. Notice how the CP maps $\sigma_x \cdot \sigma_x + \sigma_z \cdot \sigma_z - I \cdot I$ and $\sigma_y \cdot \sigma_y$ are indistinguishable.

	$\cdot\sigma_z$	$\cdot\sigma_x$	$\cdot I$	$\cdot\sigma_y$
$\sigma_z\cdot$	$\begin{matrix}1&0&0\\0&-1&0\\0&0&1\end{matrix}$	$\begin{matrix}0&1&0\\1&0&0\\0&0&0\end{matrix}$	$\begin{matrix}0&0&1\\0&0&0\\1&0&0\end{matrix}$	$\begin{matrix}0&0&0\\0&0&1\\0&-1&0\end{matrix}$
$\sigma_x\cdot$		$\begin{matrix}-1&0&0\\0&1&0\\0&0&1\end{matrix}$	$\begin{matrix}0&0&0\\0&0&1\\0&1&0\end{matrix}$	$\begin{matrix}0&0&-1\\0&0&0\\1&0&0\end{matrix}$
$I\cdot$			$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}$	$\begin{matrix}0&1&0\\-1&0&0\\0&0&0\end{matrix}$
$\sigma_y\cdot$				$\begin{matrix}-1&0&0\\0&-1&0\\0&0&1\end{matrix}$

Table 4.2: Matrix representation of the quantum operations $\sigma_\alpha \cdot \sigma_\beta$ over the vector of operators $[\sigma_z, \sigma_x, I]^\tau$. The off-diagonal terms within the three-by-three submatrix correspond to the action $\sigma_\alpha \cdot \sigma_\beta$ + h.c., whereas the three off-diagonal terms on the border correspond to the action $\sigma_\alpha \cdot \sigma_\beta$ − h.c.. Again, notice how the CP maps $\sigma_x \cdot \sigma_x + \sigma_z \cdot \sigma_z - I \cdot I$ and $\sigma_y \cdot \sigma_y$ are indistinguishable.

n-dimensional ball
spin-factor

Chapter 5

Spin-factors probabilistic model

5.0.1 The system, its states and effects

This model is a generalization of the two-clocks model in which the convex set of states is an **n-dimensional ball**. The local self-dual system is the **spin-factor**, whose cone of states is given by

$$\mathfrak{S}_{+} = \left\{ l(\omega) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix};\ x_1^2 + \ldots + x_n^2 \leqslant x_{n+1}^2 \right\}, \tag{5.1}$$

and, as usual, the set of states is its section at $x_{n+1} = 1$

$$\mathfrak{S} = \left\{ l(\omega) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{bmatrix};\ x_1^2 + \ldots + x_n^2 \leqslant 1 \right\}, \tag{5.2}$$

where x_{n+1} is the coordinate pointing along the cone's axis. Whence the set $\mathfrak{S}$ is simply a unitary n-dimensional sphere and produce a self dual theory in the same way as in the clocks theory. In our representation the self duality is evident, in fact the cone of effects is again as follows

$$\mathfrak{E}_{+} = \left\{ \lambda(a) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix};\ x_1^2 + \ldots + x_n^2 \leqslant x_{n+1}^2 \right\}. \tag{5.3}$$

and its truncation, from the order relation $0 \leqslant a \leqslant e$, give the set of effects

$$\mathfrak{E}_+ = \left\{ l(\omega) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix}; \; x_1^2 + \ldots + x_n^2 \leqslant \min(x_{n+1}^2, (1 - x_{n+1})^2), \; x_{n+1} \in [0, 1] \right\}, \tag{5.4}$$

In the following it will be usefull the following parametrization for the unitary n-dimensional

$$\begin{aligned} x_1 &= \cos\phi_1 \\ x_2 &= \sin\phi_1 \cos\phi_2 \\ &\vdots \\ x_{n-1} &= \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2} \cos\phi_{n-1} \\ x_n &= \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{n-2} \sin\phi_{n-1} \end{aligned} \tag{5.5}$$

5.0.2 The bipartite system and the faithful state

As already observed for the previous probabilistic model, which is a particular case of this one, there isn't a pre-existing theory which estabilsh the correlation for the bipartite system, then we have no information about the faithfull state. We only have to choose a state providing the isomorphism between the con of states and effects. Then there will be a lot of theories having the a spin-factor as local system and a different bipartite system which means that the theories are not the same. The bipartite functional

$$\Phi(a, b) = \lambda(a) \cdot \lambda(b) \tag{5.6}$$

realizes the cone isomorphism $\mathfrak{S}_+ \simeq \mathfrak{E}_+$ via the map

$$\omega_a := \Phi(a, \cdot) = a \tag{5.7}$$

and then can be taken as pure preparationally faithfull state for our local self-dual model. In terms of the canonical basis one has $\Phi = I_{n+1}$, namely

$$\Phi = \sum_{i=1}^{n+1} \lambda_i \otimes \lambda_i. \tag{5.8}$$

As usual accordin to the isomorphism $\mathfrak{S}_+^{\odot 2} \simeq \mathfrak{T}_+$ given by

$$\Psi = (\mathscr{A}, \mathscr{I})\Phi \Rightarrow \mathbf{\Psi} = \mathbf{A}^\tau, \tag{5.9}$$

the whole set of bipartite states $\mathfrak{S}_+^{\odot 2}$ can be generated from set $\mathfrak{T}_+$ of completely positive maps. Notice that each bipartite state has the same matrix of the corresponding transformation with the chosen faithful state. We are left with the problem of classify the contractions for the spin-factor system.

5.0.3 Positive transformations

generators

The classification of the contractions for this probabilistic model is a trivial generalization of the two-clocks one. Being $\mathfrak{S}$ a n-sphere we get

$$\mathsf{Aut}(\mathfrak{S}) = \mathbf{O}(n) \tag{5.10}$$

namely the set of $\mathfrak{S}$-automorphisms is the orthogonal group $\mathbf{O}(n)$. The minimal **generators** of $\mathfrak{E}$ under the $\mathbf{O}(n)$ action are only three effects as in the clocks case. in fact an effect lying on the sphere at $x_{n+1} = 1/2$ plus the deterministic and the null effects constitute a good set of generators. For convenience consider the set $\{a_0, e, o\}$ where a_0 is represented by the vector

$$\lambda(a_0) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{bmatrix}. \tag{5.11}$$

A contractions $\mathscr{A}$ has to satisfy the usual condition

$$\boldsymbol{A} l(\omega) \in \mathfrak{S}_+ \qquad \forall \omega \in \mathfrak{S}. \tag{5.12}$$

Taking the general state in $\mathfrak{S}$ as

$$l(\omega) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{bmatrix}, \tag{5.13}$$

with $x_1, \ldots, x_n$ parametrized as in Eq. (5.5), and following the same same steps as in Eqs. (4.14)-(4.26), we get the general contraction in the equivalence class $[a_0]$ to have a block representation

$$\boldsymbol{A} = \begin{bmatrix} A_1 & 0 & \ldots & 0 & A_1 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ A_n & 0 & \ldots & 0 & A_n \\ 1/2 & 0 & \ldots & 0 & 1/2 \end{bmatrix} \tag{5.14}$$

where

$$\sum_{i=1}^{n} A_i^2 \leqslant 1/4. \tag{5.15}$$

In particular the extremal contractions in $[a_0]$ achieve the equality in the last equation. The extremal contraction in $[e]$ are the automorphism in $\mathbf{O}(n)$. In this way we forget some extremal contractions, precisely the ones associated to the non extremal effects. A possible complete classification came from the $(n-1)$-dimensional elliptical conics of Extr($\mathfrak{S}_+$) as a generalization of the clocks positive transformations[1]. The positive transformations for a n-dimensional spin-factor are the maps which send $\mathfrak{S}$ in a convex set whose perspective map is yet $\mathfrak{S}$ and then such transformations must send Extr($\mathfrak{S}$) in an elliptical conic of $\mathfrak{S}_+$. Whence three different classes of maps can be found[2] according to the kind of conic in which Extr($\mathfrak{S}$) is sent by the transformation.

- $(n-1)$-*dimensional balls*. This class include the transformation $\mathscr{A}$ which send the convex set of states $\mathfrak{S}$ in the same convex set $\mathfrak{S}$ or in a a n-dimensional ball which rescaled by a constant is still $\mathfrak{S}$. Clearly these maps are all the maps $\mathscr{A}$ proportional to the automorphisms in $\mathbf{O}(n)$

$$\mathscr{A} = \lambda\mathscr{D} \qquad \forall\lambda > 0, \forall\mathscr{D} \in \mathsf{Aut}(\mathfrak{S}) = \mathbf{O}(n). \tag{5.16}$$

- *Degenerate conics* As usual these maps send $\mathfrak{S}$ in an extremal ray of $\mathfrak{S}_+$ which is a degenerate conic of the same cone. Such transformations are the ones in Eq. (5.14). In this equation are included the set of maps in the equivalence class $[a_0]$ (see Eq. (5.11)), in order to achieve the transformations for all the other extremal effects on the sphere at $x_{n+1} = 1/2$ is sufficient to act by the group $\mathbf{O}(n)$ on the $[a_0]$ ones being a_0 a generator for $\mathfrak{E}$.

- *true ellipsoids.* These class contains the map sending $\mathfrak{S}$ into an n-dimensional ellipsoid having an ellipsoid conic of $\mathfrak{S}_+$ as extremal set.

[1]The n-dimensional elliptical conics of Extr($\mathfrak{S}_+$) are achievable by intersecting Extr($\mathfrak{S}_+$) with n-dimensional hyperplanes. Naturally in this way one cold find also the others conics (hyperbolic and parabolic ones) but we are interested in the elliptical ones

[2]We know that a single class is sufficient from a mathematical point of view but the elements in the three classes have different physical properties. See Obs. 4.1.

CHSH
simplex
classical bit
trit

Chapter 6

Classical mechanics model

A classical theory doesn't allows non local states, namely non local probability rules, then all the classical theories are local. The boundary between local and non local theories is tracked by the **CHSH** inequality[1]. The possibility of achieving a non local theory is tightly connected to the local system as proved in Sec. 7.2 and in particular the CHSH inequality is always satisfied if the local convex set of states is a **simplex** as in classical mechanics[2]. In this chapter we want to show how the classical mechanics can be classified as a particular probabilistic theory. Differently from the previous models the classical ones can be easily investigated on a generic dimension. In dimension 2 the local simplex of states is a segment and the local system is the well known **classical bit**. The first non trivial case is the local system in dimension three, namely the **trit**. We will develop the model having the trit as local system. The particular representation of the cones of states and effects in the Euclidean space isn' t a significant feature of the probabilistic theory, then will show two equivalent representation of the classical self-dual theory in three dimension at the single system level.

6.1 The probability simplex representation of the two-trits model

6.1.1 Local convex sets of states and effects.

The convex set of states $\mathfrak{S}$ is the 2-dimensional simplex, namely a triangle. A possible representation of $\mathfrak{S}$ is the 2-dimensional probability simplex[3] whose three vertexes ω_1,ω_2 and ω_3 are represented by the vectors

$$l(\omega_1) = \lambda_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad l(\omega_2) = \lambda_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad l(\omega_3) = \lambda_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}. \qquad (6.1)$$

[1] See Sec. 7.2
[2] See Ssec. 1.1.4 for the definition of simplex
[3] See Ssec. 1.1.4 for the definition of probability simplex.

The set of states, which is the convex hull of the vertexes[4] in Eq. (6.1), is given by

$$\mathfrak{S} = \left\{ l(\omega) \in \mathbb{R}^3_+ \mid l_1(\omega) + l_2(\omega) + l_3(\omega) = 1 \right\}, \quad (6.2)$$

and the cone based on it is

$$\mathfrak{S}_+ = \mathbb{R}^3_+ = \left\{ l(\omega) \in \mathbb{R}^3 \mid l(\omega) \succeq 0 \right\}, \quad (6.3)$$

where the symbol $\succeq$ denotes componentwise inequality.[5] Of course, the cone dual cone of effects $\mathfrak{E}_+$ is the same cone $\mathbb{R}^3_+$

$$\mathfrak{E}_+ = \mathbb{R}^3_+ = \left\{ \lambda(a) \in \mathbb{R}^3 \mid \lambda(a) \succeq 0 \right\}. \quad (6.4)$$

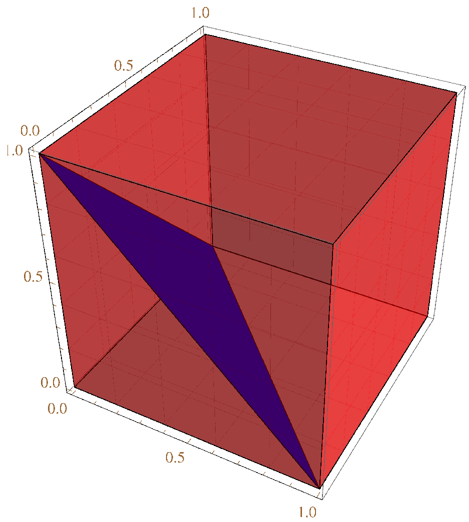


Figure 6.1: In blue the the simplex rapresenting the convex set of states $\mathfrak{S}$ for the 3-dimensional classical mechanics model. In yellow the unit cube representing the convex set of effects $\mathfrak{E}$ for the same model.

Thus the theory is self-dual at the single-system level and the convex set of effects is the truncation of $\mathfrak{E}_+$ given by

$$\mathfrak{E} = \left\{ \lambda(a) \in \mathbb{R}^3 \mid 0 \preceq \lambda(a) \preceq 1 \right\}. \quad (6.5)$$

From a geometrical point of view the set $\mathfrak{E}$ is simply the unit cube in the positive orthant (see the red cube in Fig. 6.1). Assuming the dual pairing rule for states and effects given by

$$\omega(a) = l(\omega) \cdot \lambda(a) = \sum_{i=1,2,3} l_i(\omega)\lambda_i(a) \quad (6.6)$$

[4]Notice that, differently from the Popescu-Rohrlich and the two-clocks models, in this case $\lambda_1, \lambda_2, \lambda_3$ are elements of $\mathfrak{S}$ and than they are true states.

[5]Componentwise or vector inequality in $\mathbb{R}^n$:$w \succeq v$ means $w_i \geqslant v_i$ for $i = 1, \dots, n$.

probability-vectors

the deterministic effect must be represented by the vector

$$\lambda(e) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{6.7}$$

Indeed we get

$$\omega(e) = l(\omega) \cdot \lambda(e) = l_1(\omega) + l_2(\omega) + l_3(\omega) = 1 \qquad \forall \omega \in \mathfrak{S}. \tag{6.8}$$

The vertexes of $\mathfrak{E}$ are the effects a_1, a_2, a_3, a_0 and e represented by the vectors

$$\lambda(a_1) = l_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda(a_2) = l_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \lambda(a_3) = l_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \lambda(a_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \lambda(e) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{6.9}$$

Notice that the deterministic effect points in the cone direction as in the others models. We can also observe that the maximally chaotic state χ is as usual the centre of the convex of states.

$$l(\chi) = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda(e). \tag{6.10}$$

Observation 6.1 *In this classical model, vectors representing local states in $\mathfrak{S}$ are* **probability-vectors***, indeed the sum of their components is always one as trivially follows from the condition $l(\omega) \cdot \lambda(e)$=1. The convex combinations of the vertex states $\lambda_1, \lambda_2, \lambda_3$ are obviously single-system mixed states and we notice that a mixture of state is a well known classical features in a probabilistic theory.*

6.1.2 The bipartite system

The bipartite cone of states and effects

The bipartite system has real dimension $3^2 = 9$. According to Theorem 7.1 the convex set of bipartite states $\mathfrak{S}^{\odot 2}$ is the 8-dimensional polytope having the lowest number of vertexes in order to be a simplex. Than there are nine vertexes corresponding to the nine factorized states

$$\Psi_{ij} = \omega_i \otimes \omega_j \qquad \text{for} \qquad i, j = 1, 2, 3, \tag{6.11}$$

represented by the matrixes

$$\mathbf{\Psi}_{ij} = \lambda_i \otimes \lambda_j \qquad \text{for} \qquad i, j = 1, 2, 3. \tag{6.12}$$

The bipartite cone of states is the cone based on the 8-dimensional simplex given by the convex hull of the vertexes above. In this model the local self-duality imply the self-duality at the bipartite level because there are only bipartite states and effects given by the factorization of the local ones. In Sec. **??** we will investigate this aspect of a classical theory.

PFAITH

Observation 6.2 *Notice that the representative of the extremal bipartite states are the nine* 3×3 *matrixes having the* ij*-entry equal to one and all the other entries equal to zero. This shows that the set* $\mathfrak{S}^{\odot 2}$ *is the* 8*-dimensional probability simplex. The structure is preserved from the local to the bipartite system.*

The faithful state

In a classical theory there not exists a pure bipartite non local state and in particular there not exist a pure preparationally faithful state as stated in Corollary 7.2. Then Postulate **PFAITH** doesn't hold. A faithful state must provide the isomorphism $\mathfrak{E}_+ \simeq \mathfrak{S}_+$ and because of the self-duality it must send the cone of states in the same cone. The only pure bipartite states are the ones in Eq. (6.11). This states, if looked as bilinear functional over effects acts as follows

$$\Psi_{ij}(\cdot, a) = \omega_j(a)\omega_i = \lambda_j(a)\omega_i \qquad \forall a \in \mathfrak{E}, \tag{6.13}$$

they send all the effects in the same extremal ray of $\mathfrak{S}$ and then they don't achieve the required isomorphism. All the possible faithful states must be proportional to a permutation of the $\mathfrak{S}$ vertexes, in particular we can assume the identical permutation

$$\Phi = \frac{1}{3} \sum_{i=1,2,3} \omega_i \otimes \omega_i = \sum_{i=1,2,3} \lambda_i \otimes \lambda_i. \tag{6.14}$$

The factor $1/3$ ensure the normalization. We can explicitly see that Φ is the convex combination of other pure states, namely it isn't extremal. The matrix representing Φ is

$$\boldsymbol{\Phi} = \frac{\boldsymbol{I}}{3} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.15}$$

The cones isomorphism $\mathfrak{S}_+ \approx \mathfrak{E}_+$ realizes via the map

$$\omega_a = \frac{\Phi(a, \cdot)}{\Phi(a, e)} = a. \tag{6.16}$$

Transformations

The cone of physical transformations of the model $\mathfrak{T}_+$ is isomorphic to the cone $\mathfrak{S}_+^{\odot 2}$ via usual the relation

$$\forall \Psi \in \mathfrak{S}_+^{\odot 2} \quad \exists \mathscr{A} \in \mathfrak{T}_+ \text{ such that } \psi = (\mathscr{I}, \mathscr{A})\Phi, \tag{6.17}$$

leading to the relation between representatives

$$\boldsymbol{A}^\tau = \boldsymbol{\Phi}^{-1}\boldsymbol{\Psi}. \tag{6.18}$$

The extremal transformation are associated to the extremal states in Eq. (6.11). The matrixes representing such transformations are

$$\boldsymbol{A}_{ij}{}^\tau = 3\boldsymbol{I}\boldsymbol{\Psi}_{ij} = 3\boldsymbol{\Psi}_{ij} \propto \lambda_i \otimes \lambda_j. \tag{6.19}$$

According to Obs. 6.2 the extremal transformation in $\mathfrak{T}_+$ are the 3×3 matrixes having only one entry different from zero. From these ones all the completely positive maps of the model are achieved by convex combination and multiplication by scalars.

Automorphism. The set of automorphism of $\mathfrak{S}$ is the permutation group $\mathbf{S}_3$

$$\mathsf{Aut}(\mathfrak{S}) = \mathbf{S}_3. \tag{6.20}$$

The 3! permutations correspond to the three independent rotation-symmetries plus the three independent reflection-symmetries of the triangle of states. The six automorphisms are summaruzed in Tab. 6.1.2 The six bipartite states related by the cones

automorphism	action
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	identity transformation
$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$	$\frac{2\pi}{3}$ clockwise rotation around the cone axes
$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$\frac{4\pi}{3}$ clockwise rotation around the cone axes
$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	reflection exchanging ω_1 with ω_2
$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	reflection exchanging ω_1 with ω_3
$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	reflection exchanging ω_2 with ω_3

Table 6.1: In the first column the six automorphism of $\mathfrak{S}$ corresponding to the six elements of the permutation group S_3. In the second column the action of the automorphisms is specified.

isomorphism in Eq. (6.18 to the six automorphism are represented by rhe same matrixes in Tab. 6.1.2. These states all isomorphism between $\mathfrak{E}_+$ and $\mathfrak{S}_+$ and then are all potential preparationally faithful states of the theory. As already noticed none of them is pure.

Positive transformations. It's easy to show that the cone of completely positive maps for the two-trits model coincides with the cone of positive maps

$$\mathfrak{T}_+ \simeq \mathfrak{L}_+. \tag{6.21}$$

A map $\mathscr{A}$ in $\mathfrak{L}_+$ preserve the cone $\mathfrak{S}_+$, namely

$$\mathscr{A}\omega_i \in \mathfrak{S}_+ \qquad \forall\ i = 1, 2, 3. \tag{6.22}$$

If $\boldsymbol{A}$ is the representative of $\mathscr{A}$ then the condition in Eq. (6.22) reads $\boldsymbol{A}l(\omega_i) \in \mathfrak{S}_+$. Then all the columns of the matrix $\boldsymbol{A}$ must lay in $\mathfrak{S}_+$. Recalling that $\mathfrak{S}_+ = \mathbb{R}^3_+$ we are left with the following positive transformations

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \qquad A_{ij} \geqslant 0. \tag{6.23}$$

The cone of these transformation is the same cone generated by the nine extremal completely positive transformations found in the two-trits model and then $\mathfrak{T}_+ = \mathfrak{L}_+$. In particular a among the positive maps, the contraction satisfy the bound

$$\sum_i A_{ij} \leqslant 1 \qquad \forall j = 1, 2, 3, \tag{6.24}$$

or, which is the same, their columns are probability-vectors[6].

Teleportation

In the classical model it's easy to verify thath Φ^{-1} is in $\mathfrak{E}^{\odot 2}$. Indeed $\Phi^{-1} \propto \Phi$ and the theory is self-dual at the bipartite system level. The probabilistic teleportation is then achievable and we can find the constant α peculiar of the probabilistic model in exam establishing the probability of teleport states. The bipartite effect proportional to Φ^{-1} achieving teleportation is

$$F = \frac{\Phi^{-1}}{3} = \sum_{i=1,2,3} l_i \otimes l_i, \tag{6.25}$$

fixing the constant of the theory α to $1/3$. The idea of teleportation is not very interesting in classical theory because there aren't non local states to be teleported. We have only local states perfectly discriminable. The factor $\frac{1}{3}$ arise from the existence of three different perfectly dicriminable local pure states.

6.2 Alternative representation of the two-trits model

In this section we give an alternative representation of the Two-trits self-dual model comparable with the representations of the others models studied in the present Part

6.2.1 Local convex sets of states and effects.

The vertexes ω_1, ω_2 and ω_3 of the simplex of states $\mathfrak{S}$ are

$$l(\omega_1) = \begin{bmatrix} 0 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \qquad l(\omega_2) = \begin{bmatrix} -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}. \qquad l(\omega_3) = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \tag{6.26}$$

[6] if in Eq.(6.24) the equal is achieved the set of transformation related to $\lambda = 1$ is $\mathfrak{S}$ preserving.

whose convex hull is the triangle

$$\mathfrak{S} = \mathsf{Co}(l(\omega_1), l(\omega_2), l(\omega_3)) \tag{6.27}$$

The cone of states is given by

$$\mathfrak{S}_+ = \mathsf{Co}_+(l(\omega_1), l(\omega_2), l(\omega_3)). \tag{6.28}$$

The three vertexes of $\mathfrak{S}$ in Eq. (6.26) are orthogonal vectors in $\mathbb{R}^3$ from which the self-duality of the local system

$$\mathfrak{E}_+ = \mathfrak{S}_+. \tag{6.29}$$

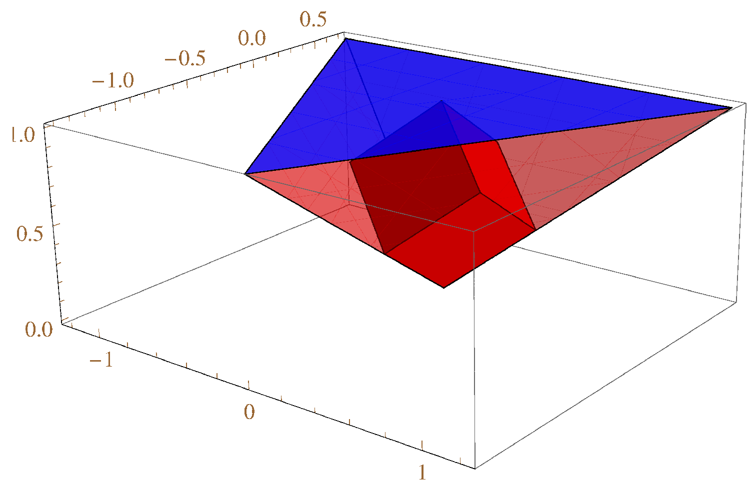


Figure 6.2: In blue the the simplex rapresenting the convex set of states $\mathfrak{S}$ for the two-trits classical model. In transparent red the cones of states and effects showing explicitly the self-duality of the system. The red cube inside the cones is the truncation of the cone $\mathfrak{E}_+$ representing the convex set of effects $\mathfrak{E}$ for the model.

The convex set of effects $\mathfrak{E}$ is the truncation of $\mathfrak{E}_+$ represented in Fig. 6.2 by the red cube. The pairing relation between states and effects is as usual

$$\omega(a) = l(\omega) \cdot \lambda(a), \tag{6.30}$$

and the deterministic effect is the vector

$$\lambda(e) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6.31}$$

Explicitly the eight vertexes of $\mathfrak{E}$ are the effects $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and e represented by the eight vectors

$$\lambda(a_1) = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{3} \\ \frac{1}{3} \end{bmatrix} \qquad \lambda(a_2) = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{6} \\ \frac{1}{3} \end{bmatrix} \qquad \lambda(a_3) = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{6} \\ \frac{1}{3} \end{bmatrix} \tag{6.32}$$

$$\lambda(a_4) = \begin{bmatrix} -0 \\ \frac{\sqrt{2}}{3} \\ \frac{2}{3} \end{bmatrix} \qquad \lambda(a_5) = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2}{3} \end{bmatrix} \qquad \lambda(a_6) = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2}{3} \end{bmatrix} \tag{6.33}$$

$$\lambda(a_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \lambda(e) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6.34}$$

Notice that the truncation is more complicate than in the previous probabilistic models. The origin and the deterministic effect behave as usual but there isn't simply a triangular section at middle height. There are two triangular sections at different height, the first one at $z = \frac{1}{3}$ is the convex hull of the vertexes in Eq. (6.32) while the other is the convex hull of the points in Eq. (6.33).[7] Accordingly to the usual Block representation l_3 has been identified with the deterministic effect, thereby the deterministic component of an object is fixed along the z axis which is also the axis of the cones. The maximally chaotic state is again the centre of the simplex of states

$$l(\chi) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{6.35}$$

6.2.2 the bipartite system

The situation is exactly the splitting image of the probabilistic simplex representation. Obviously there aren't theoretical differences because the local system is exactly the same and is self-dual in both cases. Moreover the local system is a simplex and accordingly to Sec. 7.3 the bipartite system is forced to be a simplex too.

the bipartite cones of states and effects

The vertexes of $\mathfrak{S}^{\odot 2}$ are the nine states

$$\Psi_{ij} = \omega_i \otimes \omega_j \qquad \forall i, j = 1, 2, 3, \tag{6.36}$$

generating the bipartite simplex. The cone $\mathfrak{S}_+^{\odot 2}$ having this simplex as base is also the cone $\mathfrak{E}_+ \odot 2$ of bipartite effect, namely the theory is self dual at the bipartite system level.

[7] If the convex set of states is a regular polytope with an even number of wedges—say $2n$—then the truncation of $\mathfrak{E}$ has $2n + 2$ vertexes, $2n$ of them has the regular polytope as $\mathfrak{S}$ at middle height of the cone, while the remaining two are the origin and the deterministic effect. If the number of wedges is odd—say $2n + 1$— then the vertexes of $\mathfrak{E}$ are $2(2n + 1) + 2$ because there are two dentical $\mathfrak{S}$-like polytopes between $z = 0$ and $z = 1$.

The faithful state and the transformations

As previously we assume as faithful state the bilinear form over effects given by

$$\Phi = \frac{1}{3} \sum_{i=1,2,3} \omega_i \otimes \omega_i, \qquad \text{with } \boldsymbol{\Phi} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.37}$$

which obviously is not an extremal state.

bipartite state	automorphism	action
$\Psi = \Phi = \frac{1}{3}(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	identity transformation
$\Psi = \frac{1}{3}(\omega_2 \otimes \omega_1 + \omega_3 \otimes \omega_2 + \omega_1 \otimes \omega_3)$	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{2\pi}{3}$ clockwise rotation around the cone axes
$\Psi = \frac{1}{3}(\omega_3 \otimes \omega_1 + \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_3)$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{4\pi}{3}$ clockwise rotation around the cone axes
$\Psi = \frac{1}{3}(\omega_2 \otimes \omega_1 + \omega_1 \otimes \omega_2 + \omega_3 \otimes \omega_3)$	$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection exchanging ω_1 with ω_2
$\Psi = \frac{1}{3}(\omega_3 \otimes \omega_1 + \omega_1 \otimes \omega_3 + \omega_2 \otimes \omega_2)$	$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection exchanging ω_1 with ω_3
$\Psi = \frac{1}{3}(\omega_3 \otimes \omega_2 + \omega_2 \otimes \omega_3 + \omega_1 \otimes \omega_1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	reflection exchanging ω_2 with ω_3

Table 6.2: In the first column the six bipartite states achieving the isomorphism $\mathfrak{E}_+ \simeq \mathfrak{S}_+$ and then potential candidates for the faithful state of the theory. In the second column the relative transformations if the identical permutation is chosen as faithful state. These maps are the six automorphism of $\mathfrak{S}$ and then are the six permutations in $\mathbf{S}_3$. In the last column the action of the automorphisms is specified.

The extremal physical transformations, according to the isomorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$ induced by Φ are the rays associated to the nine vertexes of $\mathfrak{S}^{\odot 2}$, namely the maps represented by the matrixes

$$\boldsymbol{A}_{ij}{}^{\tau} \propto \Phi^{-1}\boldsymbol{\Psi}_{ij} = l(\omega_i) \otimes l(\omega_j) \qquad \forall i, j = 1, 2, 3. \tag{6.38}$$

All the physical transformation are in the convex hull of the ones above and in particular all the automorphisms of $\mathfrak{S}$ are not extremal maps. Recalling that $\mathsf{Aut}(\mathfrak{S}) = \mathbf{S}_3$ we can summarize the representative of the six permutation in Tab. 6.2.2. In the same table are also reported the states corresponding to such transformations whose representatives are the same of the automorphisms' ones rescaled by a factor $\frac{1}{3}$. Naturally these six states cold have been chosen as faithful state of the theory because all of them achieve the isomorphisms $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$ and $\mathfrak{E}_+ \simeq \mathfrak{S}_+$. Notice that all the automorphisms' representatives has the deterministic effect as last row according to the Block representation. Moreover the last rows of the extremal transformations in Eq. (6.38) are proportional to the extremal effects in Eqs. (6.32) and 6.33).

Teleportation

Clearly teleportation is achievable and the theory's parameter α is still $\frac{1}{3}$. In fact the functional Φ^{-1} is in the bipartite cone of effects but a scale factor $\frac{1}{3}$ is required to make it a true effect in $\mathfrak{E}^{\odot 2}$.

Part III

Complements

Chapter 7

Global features from the local system structure

In this thesis we have noticed a lot of times how the local structure of a probabilistic theories contains information about the global structure of the same theory. For example in Part II we showed that it's impossible to get a probabilistic complete theory having the clock as local system and at the same time a self-dual bipartite system. This is not the only case in which the local system forces a feature of the composed one and in this chapter we are going to explore this possibility.

7.1 The faithful state

In this work we use very often the property of a faithful state to be an isomorphism between the local cones of state and effects too. This is a very interesting property because from the local system we can find out a set of functional including all possible faithful state.

Consider a system S and the bipartite system $\mathsf{S} \odot \mathsf{S}$. From Chap. 2 we know that a system is defined as collection of tests and starting from this simple assumption arises the structure of the system, which is the set of transformation $\mathfrak{T}(\mathsf{S})$, the set of effects $\mathfrak{E}(\mathsf{S})$ and finally the set of states $\mathfrak{S}(\mathsf{S})$. From these set we get the respective cones $\mathfrak{T}_+(\mathsf{S})$,$\mathfrak{E}_+(\mathsf{S})$ and $\mathfrak{S}_+(\mathsf{S})$. A preparationally faithful state for the bipartite system $\mathsf{S} \odot \mathsf{S}$ with respect to the local system S must achieve the isomorphism

$$\mathfrak{E}_+ \simeq \mathfrak{S}_+. \tag{7.1}$$

Denoting by $\mathsf{Isom}(\mathfrak{E}, \mathfrak{S})$ the set of cones isomorphisms between $\mathfrak{E}$ and $\mathfrak{S}$ it will include all admissible faithful states for any probabilistic theory having S as local system

$$\Phi \in \mathsf{Isom}(\mathfrak{E}, \mathfrak{S}) \qquad \forall\Phi \text{ preparationally faithful w.r.t. } \mathsf{S}. \tag{7.2}$$

We can summarize this section in the following flux chart 7.1.

non locality
no-signaling

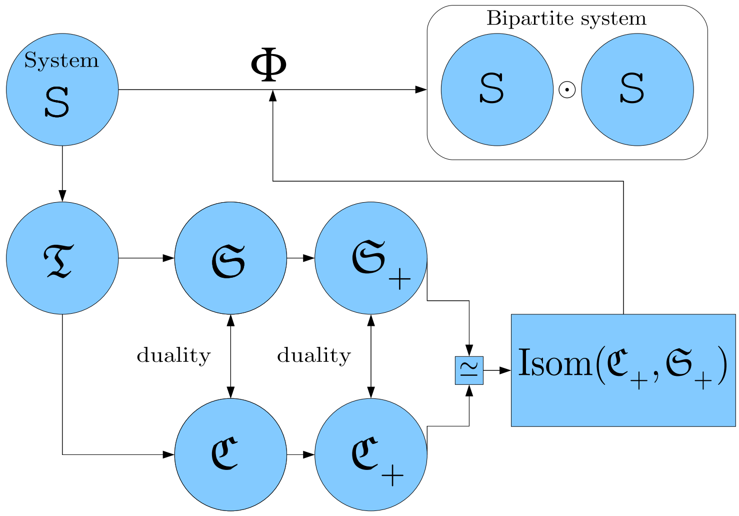


Figure 7.1: In this flow diagram we show the logical steps leading to a set including all the admissible faithful state for all the possible probabilistic theories having a particular local system S. Such set is the set of isomorphisms between the local cones $\mathfrak{E}_+$ and $\mathfrak{S}_+$. Then the faithful state, which is tightly connected to non local features of a probabilistic theory, belong to a set $\mathsf{Isom}(\mathfrak{E}, \mathfrak{S})$ having local origin.

7.2 Non locality from locality

7.2.1 Boundaries between local, non local and superluminal theories

It is well known that the **non locality** can not be assumed as a principle in order to achieve quantum mechanics among the set of probabilistic theories developed in PartII. In fact there are other theories which are non local and are no quantum mechanics. Then one could imagine that quantum mechanics is the theory which achieve the most strong correlation compatible with the **no-signaling** principle[1]. But we know that it is possible to construct a theoretical system which does not allow superluminal signaling, yet which is more non local than quantum mechanics. This theoretical model are the Popescu-Rohrlich boxes in [RP95]. We have showed in Part II that this system is a

[1]Consider a system S and assume the **postulate NSF** in Chap.2 which states that the marginal probability $\sum_{\mathscr{B}_j\in\mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A})$ of any event $\mathscr{A}$ is independent on test $\mathbb{B}$, and is equal to the probability with no test $\mathbb{B}$. In other words future tests cannot modify the present marginal probability of any event. We know from Chap.2 that the **postulate NSF** is implied by a purely dynamical notion of system independence. From this follows that for N systems in the joint state Ω the marginal state $\Omega|_i$ of the i-th system is independent of any deterministic transformations, *i.e.* any test, that is performed on systems different from the i-th,

$$\Omega|_n(a) = \Omega(e, \ldots, e, a, e, \ldots, e) \qquad \forall a \in \mathfrak{E}, \tag{7.3}$$

which is equivalent to the no-signaling condition.

Bell inequality
Cirel'son bound
CHSH
correlations
local theory
CHSH

particular probabilistic model. In order to quantify the non locality of a theory some bounds had been found which fix the boundaries between, for example, a classical theory, quantum theory and a superluminal theory:

- Bell in [Bel64] proved that some predictions of quantum theory cannot be reproduced by any theory of local physical variables according to the famous **Bell inequality** whose upper bound 2 tracks the boundaries between local hidden variable models and non local theories.
- A generalization of the Bell's inequality is the **Cirel'son bound** in [Cir94] which gives the upper bound $2\sqrt{2}$ for quantum mechanic.
- Finally the maximal possible value for the Clauser-Horne-Simony-Holt (**CHSH**) expression in [CHSH69] compatible with the no-signaling principle is found to be 4 compared to $2\sqrt{2}$ for quantum theory or 2 for any local theory. It is well known that the Popescu-Rohrlich model achieves this upper bound and we will show it in our probabilistic framework in this chapter.

What we are going to show in this chapter is that given a probabilistic theory, the maximal non locality achievable is written in the local structure of the theory. In a nutshell is the structure of the local convex of states that determines the maximal degree of non locality. This is very suggestive because a global property depends only on the local system.

7.2.2 Correlations and the CHSH inequality

As in [RP94] consider a system with two parts far from one another. The first part is held by Alice which can perform two measurements A and A' both taking binary values $a = 0, 1$ and $a' = 0, 1$. Bob held the second part and can perform two measurements too, B and B'. Also Bob measurements take value $b = 0, 1$ and $b' = 0, 1$. We denote by

$$\mathbb{P}(a, b) \tag{7.4}$$

the joint probability of obtaining $A = a$ and $B = b$ when A and B are measured. The correlation of A and B is defined as

$$C(A, B) = \mathbb{P}_{AB}(0, 0) + \mathbb{P}_{AB}(1, 1) - \mathbb{P}_{AB}(0, 1) - \mathbb{P}_{AB}(1, 0). \tag{7.5}$$

Consider then the combination of **correlations**

$$\mathbf{J} = C(A, B) + C(A, B') + C(A', B) - C(A', B'), \tag{7.6}$$

whose value lies between −2 and 2 in any theory of local hidden variable (or shortly for any **local theory**) according to the **CHSH** inequality

$$-2 \leqslant \mathbf{J} \leqslant 2. \tag{7.7}$$

In quantum mechanic for certain choices of measurements and for particular probability rule (given by particular entangled states), this inequality can be violated. Beside 2

CHSH
quantum entanglement

there are other two important numbers on the **CHSH** relation as introduced in the previous section. If the four correlations in **J** were independent, the absolute value of the sum would be 4. For quantum correlations (**quantum entanglement**) this value is not achievable and the sum of correlations is bounded in absolute value by $2\sqrt{2}$ (Cirel'son bound). 4 is also the maximal value compatible with the no-signaling and is achieve by the Popescu-Rohrlik boxes. We will reproduce this result for our generalized model (see Chap.3) in Sec.

7.2.3 Maximal correlations for a probabilistic theory

We can describe the situation exposed in the last Subsection in the framework of the probabilistic theories introduced in Chap.2. Assume the **NSF postulate**[2]. Consider the bipartite system $\mathsf{S}_A \odot \mathsf{S}_B$ where S_A and S_B are two identical systems ($\mathsf{S}_A = \mathsf{S}_B = \mathsf{S}$) respectively held by Alice and Bob. Then Alice and Bob share a bipartite system allowing some kind of correlations between them. The question is how strong this correlations could be? Which is the upper bound? From what this upper bound depends? Obviously the local system's structure would influence the characteristics of the bipartite one and then the non locality achievable in a probabilistic model. On the other hand, at first sight, seem strange that one could determine the maximal non locality of a probabilistic theory observing only the local system. In this context the measurements of Alice and Bob becomes transformations of particular tests. Whence Alice can perform the sequent two tests on her system

$$\mathbb{A} = \left\{\mathscr{A}_0, \mathscr{A}_1\right\}, \qquad \mathbb{A}' = \left\{\mathscr{A}'_0, \mathscr{A}'_1\right\}. \tag{7.8}$$

Each test has only two possible outcomes represented by the transformations $\mathscr{A}_0, \mathscr{A}_1$ and $\mathscr{A}'_0, \mathscr{A}'_1$[3]. At the same time Bob can perform the following dicotomic tests

$$\mathbb{B} = \left\{\mathscr{B}_0, \mathscr{B}_1\right\}, \qquad \mathbb{B}' = \left\{\mathscr{B}'_0, \mathscr{B}'_1\right\}. \tag{7.9}$$

We have

$$\mathscr{A}_0 \in [a_0] \qquad \mathscr{A}_1 \in [a_1] \qquad \text{with} \quad a_0 + a_1 = e, \tag{7.10}$$
$$\mathscr{A}'_0 \in [a'_0] \qquad \mathscr{A}'_1 \in [a'_1] \qquad \text{with} \quad a'_0 + a'_1 = e, \tag{7.11}$$
$$\mathscr{B}_0 \in [b_0] \qquad \mathscr{B}_1 \in [b_1] \qquad \text{with} \quad b_0 + b_1 = e, \tag{7.12}$$
$$\mathscr{B}'_0 \in [b'_0] \qquad \mathscr{B}'_1 \in [b'_1] \qquad \text{with} \quad b'_0 + b'_1 = e. \tag{7.13}$$

where $a_0, a_1, a'_0, a'_1, b_0, b_1, b'_0, b'_1$ are local effects in $\mathfrak{E}(\mathsf{S})$. The correlation in Eq. (7.5) is now as follow

$$C(\mathbb{A}, \mathbb{B}) = \Phi(a_0, b_0) + \Phi(a_1, b_1) - \Phi(a_0, b_1) - \Phi(a_1, b_0), \tag{7.14}$$

[2]In the general definition of a probabilistic theory this postulate is assumed. Remember that in Chap.2 we pointed out how the no-signaling condition is implied by the **local observability principle** which is a consequence of the postulate **PFAITH**

[3]Naturally the transformations with index 0 correspond to the outcome $a = 0$ in the last Subsection and the same holds for the index 1 corresponding to $a = 1$.

where Φ is a bipartite state in $\mathfrak{S}^{\odot 2}(\mathsf{S})$. Is this state which provide the joint probability rule in Eq. (7.4). The sum of correlations $\mathbf{J}$ is achieved by inserting the last expression in Eq. (7.5), whence we have CHSH

$$\begin{aligned}\mathbf{J} =&\Phi(a_0, b_0) + \Phi(a_1, b_1) - \Phi(a_0, b_1) - \Phi(a_1, b_0)+\\ &\Phi(a_0, b'_0) + \Phi(a_1, b'_1) - \Phi(a_0, b'_1) - \Phi(a_1, b'_0)+\\ &\Phi(a'_0, b_0) + \Phi(a'_1, b_1) - \Phi(a'_0, b_1) - \Phi(a'_1, b_0)-\\ &\Phi(a'_0, b'_0) - \Phi(a'_1, b'_1) + \Phi(a'_0, b'_1) + \Phi(a'_1, b'_0).\end{aligned} \tag{7.15}$$

Replacing a_1, a'_1, b_1, and $b1'_1$ by (see Eq. (7.10))

$$a_1 = e - a_0 \qquad a'_1 = e - a'_0, \tag{7.16}$$

$$b_1 = e - b_0 \qquad b'_1 = e - b'_0, \tag{7.17}$$

our expression for $\mathbf{J}$ becomes

$$\mathbf{J} = 4\Big[\Phi(a_0 + a'_0, b_0) + \Phi(a_0 - a'_0, b'_0) - \Phi(a_0, e) - \Phi(e, b_0)\Big] + 2. \tag{7.18}$$

This is a very significant expression because it contains both the important numbers 4 and 2. In a classical theory the term in the square bracket would result to be lesser or equal to 0 in order to satisfy the **CHSH** inequality, while in the Popescu-Rohrlich case the square bracket must be equal to $\frac{1}{2}$. Quantum mechanic is an intermediate situation. All this results will be checked in the following.

The joint probability rule. We know that the joint probability rule is given by a bipartite state. In order to achieve the maximal correlations this state must be, in quantum language, "maximally entangled". In a general probabilistic theory the "maximally entangled" states are the states preparationally faithful with respect to the local system $\mathsf{S} = \mathsf{S}_A = \mathsf{S}_B$. Accordingly a praparationally faithful state Φ achieves the isomorphism $\mathfrak{S}_+(\mathsf{S}) \simeq \mathfrak{E}_+(\mathsf{S})$ between the cones of states and effects

$$\Phi(\cdot, c) = \omega_c \tag{7.19}$$

. Then our possible choices are limited by

$$\Phi \in \mathsf{Isom}(\mathfrak{S}_+, \mathfrak{E}_+). \tag{7.20}$$

The maximal correlations. Observing the expression for $\mathbf{J}$ in Eq. (7.18) we find the following dependance

$$\mathbf{J} = \mathbf{J}_\Phi(a_0, a'_0, b_0, b'_0), \tag{7.21}$$

where

$$\Phi \in \mathfrak{S}^{\odot 2}(\mathsf{S}), \qquad a_0, a'_0, b_0, b'_0 \in \mathfrak{E}(\mathsf{S}). \tag{7.22}$$

If we want to achieve the maximal correlations we must find the upper and the lower bound for $\mathbf{J}$. Consider for example the upper one, namely

$$\sup_{\substack{\Phi \in \mathfrak{S}^{\odot 2}(\mathsf{S}) \\ a_0, a'_0, b_0, b'_0 \in \mathfrak{E}(\mathsf{S})}} \mathbf{J}_\Phi(a_0, a'_0, b_0, b'_0). \tag{7.23}$$

The maximal value of the last expression is certainly achieved by a preperationally faithful state and then

$$\Phi \in \mathfrak{S}^{\odot 2}(\mathsf{S}) \cap \mathsf{Isom}(\mathfrak{S}_+, \mathfrak{E}_+), \tag{7.24}$$

where the set $\mathfrak{S}^{\odot 2}(\mathsf{S}) \cap \mathsf{Isom}(\mathfrak{S}_+, \mathfrak{E}_+)$ is the set of isomorphisms $\mathfrak{S}_+(\mathsf{S}) \simeq \mathfrak{E}_+(\mathsf{S})$ which are also bipartite states. Finally we are left with the problem of determine the following supreme

$$\sup_{\substack{\Phi \in \mathfrak{S}^{\odot 2}(\mathsf{S}) \cap \mathsf{Isom}(\mathfrak{S}_+, \mathfrak{E}_+) \\ a_0, a_0', b_0, b_0' \in \mathfrak{E}(\mathsf{S})}} \mathbf{J}_\Phi(a_0, a_0', b_0, b_0'). \tag{7.25}$$

The upper bound comes from the local structure. It's easy to convince ourselves that the supreme in the last equation depends only on the local structure of the probabilistic theory. $\mathbf{J}$ depends on the effects a_0, a_0', b_0 and b_0' which lie in the local convex set $\mathfrak{E}$, achieved as truncation of the cone $\mathfrak{E}_+$ dual of the cone of states $\mathfrak{S}_+$ which is local too. Moreover we have the dependance on a bipartite faithful state which is also an isomorphism between $\mathfrak{S}_+$ and $\mathfrak{E}$ and then an object achievable looking at the local system. In conclusion there exist an upper bound for the non locality achievable in a probabilistic theory which is written in the geometrical configuration of the local system of the same theory. Some known example are given in the following Section.

7.2.4 Maximal correlations for some probabilistic models

In this section we want to find the maximal correlations for the classical model, the two-clocks model and the Popescu-Rohrlich generalized models. We can start from the last one.

Popescu-Rohrlich model.

The local system is defined in Sec. 3.2. We have to maximise $\mathbf{J}$ over all the possible faithful states Φ and the effects a_0, a_0', b_0, b_0'. In practice a faithful state induces the cone isomorphism $\mathfrak{E}_+ \simeq \mathfrak{S}_+$ through the well known map $\Phi(\cdot, a) = \omega_a$, then looking at the $\mathbf{J}$ expression we can rewrite it as

$$\mathbf{J}_\Phi(a_0, a_0', b_0, b_0') = 4\Big[\omega_{b_0}(a_0 + a_0') + \omega_{b_0'}(a_0 - a_0') - \chi(a_0) - \chi(b_0)\Big] + 2. \tag{7.26}$$

Thus choosing the effects b_0 and b_0' in $\mathfrak{E}$ is equivalent to choose the states ω_{b_0} and $\omega_{b_0'}$ among the truncation of $\mathfrak{S}_+$ image of $\mathfrak{E}$ under the map induced by Φ. So the maxim value of $\mathbf{J}$ is certainly achieved by a faithful state Φ among the ones mapping $\mathfrak{E}$ in the largest truncation of $\mathfrak{S}_+$. These are exactly the eight states $\Phi_{\alpha\beta\gamma}$ corresponding to the probability rules in Eq. (3.10), namely the non local pure bipartite states of the probabilistic theory which correspond via the automorphism $\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}$ to the local automorphisms of $\mathfrak{S}$. In Chap. 3 we showed that Φ_{000} was a good pure symmetric faithful state for the model but all the eight states above would reproduce the same theory. For concreteness consider $\mathbf{J}_{\Phi_{000}}$ with

$$\mathbf{\Phi}_{000} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{7.27}$$

In the first picture in Fig. 7.2 is shown the action of the faithful state Φ_{000} on a section of the set $\mathfrak{E}$; the square section at height z is sent into the internal square corresponding to the section of the cone $\mathfrak{S}_+$ at the same height. Precisely Φ_{000} performs a $\pi/4$ clockwise rotation composed to a reflection with respect to the plane xz. In practice the objects $\Phi(a,b)$ are scalar products between vectors in $\mathfrak{E}$ and vectors in the $\mathfrak{E}$ image under Φ. According to Fig. 3.2 the great scalar product would be achieved using vectors at height $z = \frac{1}{2}$ where the sections of the cones of states and effects are maxim. Therefore we can fix to $\frac{1}{2}$ the last component of the vectors representing the effects a_0, a_0', b_0, b_0'.

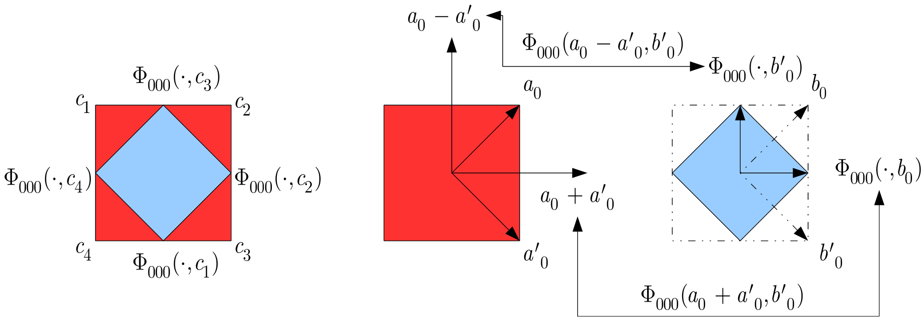


Figure 7.2: In the first picture is represented the action of the faithful state Φ_{000} over a constant height section of $\mathfrak{E}$. In the others pictures are represented the effects a_0, a_0', b_0, b_0' maximising $\mathbf{J}_{\Phi_{000}}$.

This result in the following simplified expression of $\mathbf{J}_{\Phi_{000}}$

$$\mathbf{J}_{\Phi_{000}}(a_0, a_0', b_0, b_0') = 4\left[\Phi_{000}(a_0 + a_0', b_0) + \Phi_{000}(a_0 - a_0', b_0')\right] \tag{7.28}$$

where for convenience we have kept the same notation even if the only components x an y of the states and effects must be considered (the third component is fixed and it's contribute in the scalar product has already been included in the last expression). It's easy to convince yourself that the effects maximising the last expression of $\mathbf{J}_{\Phi_{000}}$ are the following ones

$$\lambda(a_0) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \lambda(a_0') = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \lambda(b_0) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \lambda(b_0') = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}. \tag{7.29}$$

Remembering that the last component of these vectors have already been considered in Eq. (7.28), the effects involved in it are two dimensional vectors in the plane at height $z = \frac{1}{2}$. The situation is well represented in Fig. 7.2. In the third pictures we see that the vectors representing $\Phi_{000}(\cdot, b_0)$ and $\Phi_{000}(\cdot, b_0')$ are among the longest one achievable in the admissible blue set. At the same time, in the second picture, we can notice that the vectors representing the sum and the differences between a_0 and a_0' are the vectors pointing respectively in the direction of $\Phi_{000}(\cdot, b_0)$ and $\Phi_{000}(\cdot, b_0')$ achieving the greatest scalar product with them. A simple vector calculus shows that the maxim value of **J** is

$$4\left[\Phi_{000}(a_0 + a_0', b_0) + \Phi_{000}(a_0 - a_0', b_0')\right] = 4\left[\frac{1}{2} + \frac{1}{2}\right] = 4, \tag{7.30}$$

which reproduces the well know result of the Popescu-Rohrlich model, namely the maximal violation of the CHSH inequality compatible with the no-signaling principle.

The two-clocks model.

Consider the first model developed in Chap. 4, namely the local self-dual two-clocks model whose local system is defined in Ssec. 4.1.1. The situation is exactly the same as in the Popescu-Rohrlich model. The bipartite faithful pure states mapping the truncated dual cone $\mathfrak{E}$ into the largest truncation of $\mathfrak{S}_+$ are the states corresponding to the automorphisms of the local convex set of states $\mathfrak{S}$. Newly the effects maximising the **J**-value must be looked for in the section of $\mathfrak{E}$ at middle height ($z = \frac{1}{2}$). For convenience we choose as faithful state the one represented by the identity ensuring $\Phi(\cdot, a) = a$. The effects a_0, a_0', b_0, b_0' must be chosen applying the same arguments analysed in Popescu-Rohrlich. It's sufficient to choose effects a_0 and a_0' whose representing vectors are orthogonal in the plane at $z = \frac{1}{2}$ and the vectors representing effects b_0 and b_0' as the longest ones pointing respectively in the direction of $a_0 + a_0'$ and $a_0 - a_0'$. A possible choice is as follows

$$\lambda(a_0) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \qquad \lambda(a_0') = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \lambda(b_0) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}, \qquad \lambda(b_0') = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}. \tag{7.31}$$

achieving the maximum

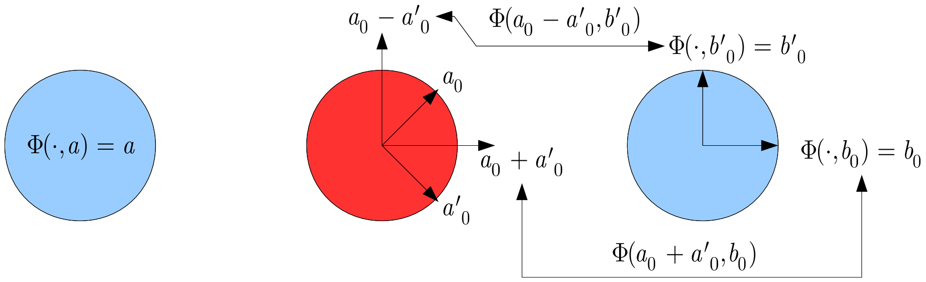
$$\mathbf{J}_\Phi = 4\left[\Phi(a_0 + a_0', b_0) + \Phi(a_0 - a_0', b_0')\right] = 4\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 2\sqrt{2}, \tag{7.32}$$

The situation is summarised in Fig. 7.3

Observation 7.1 *If instead of the local self-dual theory we should have chosen the alternative non local self-dual theory in Chap. 4 the result should be the same upper* **J***-value. In fact if the local cones are squeezed respect each other the faithful state accordingly changes keeping the induce scalar products between states and effects. Thereby the maximum of* **J** *would be still the same.*

Classical mechanics

In the following section we explore some properties of the syplectic probabilistic theories, namely probabilistic theories having simplexes as convex sets of states. Naturally classical mechanics (see the trit model in Chap. 6) belongs to this set of theories and the maximal correlations achievable will be found later as a general property of simplex theories.



trit
bit
simplex
simplex theory

Figure 7.3: In the first picture is represented the action of the faithful state Φ (corresponding to the identity transformation) over a constant height section of the set $\mathfrak{E}$. In the other pictures the effects a_0, a'_0, b_0, b'_0 maximising $\mathbf{J}_\Phi$ are geometrically represented.

7.3 Simplex theories

7.3.1 Local and global siplectic structure

In Sec. 6 we explored a probabilistic classical model. In particular in that Section has been investigated the local system denoted by **trit** which is a three dimensional generalization of the well known **bit**. The local convex set of states in that case was a triangle which is the 2-dimensional **simplex**[4]. We now show that a probabilistic theory having a simplex as local convex set of states, namely a **simplex theory**, is always a classical theory. In fact the bipartite set of states must be a simplex too and non local bipartite states are not admissible. This is an other interesting global feature arising from a local one *i.e.* the local convex set of states.

Theorem 7.1 :induced simplex structure at the bipartite system. *A probabilistic theory has a simplex as local convex set of states if and only if the bipartite set of states is a simplex too.*

Proof. Consider a system S having dimension $n+1$ and with a simplex as convex set of states $\mathfrak{S}$. Remembering that $\dim \mathsf{S} = \dim \mathfrak{S}_+ = \dim \mathfrak{S}_\mathbb{R}$ we get

$$\dim \mathfrak{S} = n \tag{7.33}$$

and then $\mathfrak{S}$ is an n-dimensional simplex. We can denote by $\omega_1, \omega_2, \ldots, \omega_{n+1}$ the vertexes of the simplex $\mathfrak{S}$

$$\{\omega_1, \omega_2, \ldots, \omega_{n+1}\} = \mathrm{Extr}(\mathfrak{S}). \tag{7.34}$$

Now introduce the set of functional in $\mathfrak{E}_\mathbb{R}$

$$\{a_1, a_2, \ldots, a_n + 1\}, \tag{7.35}$$

[4] see Ssec. 1.1.4 for the definition of simplex.

permutation group

such that

$$a_i(\omega_j) = \delta_{ij}{}^5 \quad (7.36)$$

Naturally these functional are the vertexes of $\mathfrak{E}$ which is an n-dimensional simplex too. The set of positive transformation $\mathfrak{L}_+$ has dimension $\dim(\mathfrak{L}_+) = \dim \mathfrak{L}_{\mathbb{R}} = (n+1)^2$ and the extremal rays of this cone are the transformations

$$\lambda\omega_i \otimes a_j \qquad \forall i, j = 1, \ldots, n+1, \ \forall \lambda > 0. \quad (7.37)$$

where λ is a multiplicative constant spanning the whole rays generated by the $(n+1)^2$ transformations $\omega_i \otimes a_j$. Such maps send the convex set $\mathfrak{S}$ into an extremal ray of $\mathfrak{S}_+$[6]. Now consider a probabilistic theory having S as local system. The probabilistic theory admits a preparationally faithful state Φ providing the isomorphism between the cone of states and effect. The bipartite states correasponding to the transformations in Eq. (7.37) are the local factorized bipartite states

$$\omega_i \otimes \omega_j \qquad \forall i, j = 1, \ldots, n+1. \quad (7.38)$$

Since the isomorphism

$$\mathfrak{T}_+ \simeq \mathfrak{S}_+^{\odot 2}, \quad (7.39)$$

holds and the only extremals of $\mathfrak{T}_+$ are the transformations in Eq. (7.37) then the only extremal states of $\mathfrak{S}^{\odot 2}$ are the $(n+1)^2$ states in Eq. (7.38). In conclusion the bipartite set of states is a $((n+1)^2 - 1)$-dimensional convex set having $(n+1)^2$ vertexes which is a simplex. The opposite implication is trivially true. If the bipartite convex set of states is a simplex then set of local states is a simplex too. Consider for example a $(n^2 - 1)$-dimensional bipartite simplex, then $\mathfrak{S}^{\odot 2}$ has only n^2 pure states. Naturally the local system's set of states couldn't admit more than n vertexes, from which it's a simplex. ■

Corollary 7.1 *If a probabilistic theory has a simplex as local convex set of states then the local automorphisms cannot be extremal transformations.*

Proof. The set of automorphisms for a n-dimensional simplex is the **permutation group** of order n

$$\mathsf{Aut}(\mathfrak{S}) = \mathbf{S}_n, \quad (7.40)$$

which contain the $n!$ different permutation of the set $\mathrm{Extr}(\mathfrak{S})$. From the previous theorem we know that the only extremal transformations are the ones in Eq. (7.38). In fact the automorphism of $\mathfrak{S}$ are not extremal. A general element of $\mathbf{S}_n$ can be identified by a set of indexes

$$J = \{j_1, \ldots, j_{n+1}\}, \quad (7.41)$$

[5]Notice that if we take the **probability simplex** defined in Ssec. 1.1.4 as representation of the n-dimensional simplex in exam the vertexes $\{\omega_1, \omega_2, \ldots, \omega_n + 1\}$ coincide with the orthonormal base $\{\lambda_i\}$ for $\mathfrak{S}_{\mathbb{R}}$ while the functional $\{a_1, a_2, \ldots, a_n + 1\}$ are the orthonormal base $\{l_i\}$ for $\mathfrak{E}_{\mathbb{R}}$. This is the case in the first representation of the trit probabilistic model in Sec. 6

[6]In the probabilistic simplex case these transformations are represented by the $(n+1) \times (m+1)$ matrixes having only a one and all the other entries equal to zero.

representing a permutation of the set $\{1, \dots, n+1\}$. The automorphism associated to such permutation is the map

$$\sum_{i=1,\dots,n+1} \omega_{j_i} \otimes a_i, \tag{7.42}$$

which is manifestly a convex combination of the extremals $\omega_{j_i} \otimes a_i$.[7] ■

Corollary 7.2 *All the preparationally faithful state of a probabilistic theory having a simplex as convex set of states are not pure.*

Proof. For each preparationally faithful state of the theory the bipartite pure states corresponding to the extremal transformation of the set $\mathfrak{T}_+$ are the local factorized ones which are not isomorphisms between the cones $\mathfrak{E}_+$ and $\mathfrak{S}_+$. According to the simplex structure of $\mathfrak{S}^{\odot 2}$ the local factorized states are the only pure bipartite states and then there not exist a pure preparationally faithful state[8] ■

7.3.2 The atomicity of the identical transformation

Theorem 7.2 :refinement of the identity. *In a probabilistic theory the identical transformation admits a refinement if and only the theory is simplex.*

Proof. Consider a probabilistic theory having local system S, with $\dim(\mathsf{S}) = n$. Now suppose that the identity transformation $\mathscr{I}$ admits a refinement

$$\mathscr{I} = \sum_i \mathscr{A}_i, \tag{7.43}$$

on the set of atomic transformations $\{\mathscr{A}_i\}$. Naturally is $\mathscr{I}\omega = \omega$ for every $\omega \in \mathfrak{S}$. In particular, for each $\omega \in \mathrm{Extr}(\mathfrak{S})$, the identity $\sum_i \mathscr{A}_i\omega = \omega$ entails the relations

$$\mathscr{A}_i\omega = p_i^\omega \omega \qquad \text{for } 0 \leqslant p_i^\omega \leqslant 1. \tag{7.44}$$

At the same time, for each atomic transformation $\mathscr{T} \in \mathsf{Erays}(\mathfrak{T}_+)$ we get

$$\mathscr{T}\mathscr{A}_i = q_i^{\mathscr{T}} \mathscr{T} \qquad \text{for } 0 \leqslant q_i^{\mathscr{T}} \leqslant 1. \tag{7.45}$$

In particular we can take the elements of $\{\mathscr{A}_i\}$ as atomic transformations in the last equation. Thus from Eqs. (7.44) and (7.45) follows

$$\mathscr{A}_i\mathscr{A}_i\omega = p_i^\omega \mathscr{A}_i\omega = q_i^{\mathscr{A}_i}\mathscr{A}_i\omega \qquad \forall\omega \in \mathrm{Extr}(\mathfrak{S}). \tag{7.46}$$

If $\mathscr{A}_i\omega \neq 0$ then $p_i^\omega = q_i^{\mathscr{A}_i}$ for each ω, namely p_i^ω are independent on ω and $\mathscr{A}_i$ is proportional to $\mathscr{I}$. This is impossible because $\mathscr{A}_i$ is supposed to be atomic, then, for

[7] In the case of the probability simplex the set of automorphism are the $(n+1) \times (n+1)$ permutation matrixes which are manifestly combinations of the extremal ones (see footnote 7.3.1).

[8] This is particularly manifest in the probabilistic simplex case. In fact in that case the set of isomorphism between the local cones of states and effects is exactly the set of local automorphism $\mathbf{S}_n$ (or maps proportional to these ones) but we know that this transformations are not extremal.

each $\mathscr{A}_i$ there exist at least an extremal state ω such that $\mathscr{A}_i\omega = 0$. Consider then two atomic transformation in $\{\mathscr{A}_i\}$—say $\mathscr{A}_1$ and $\mathscr{A}_2$ with $\mathscr{A}_1 \neq \mathscr{A}_2$. From

$$\mathscr{A}_1\mathscr{A}_1 = q_1^{\mathscr{A}_1}\mathscr{A}_1, \qquad \mathscr{A}_1\mathscr{A}_2 = q_2^{\mathscr{A}_1}\mathscr{A}_1, \tag{7.47}$$

we get $\mathscr{A}_1 \propto \mathscr{A}_2$, namely $\mathscr{A}_1$ and $\mathscr{A}_2$ are the same atomic transformation unless $q_2^{\mathscr{A}_1} = 0$. Thus we are left with a set of atomic transformations

$$\{\mathscr{A}_i\}, \qquad \text{with} \quad \mathscr{A}_i\mathscr{A}_j = \mathscr{A}_j\mathscr{A}_i = 0 \quad \forall i \neq j. \tag{7.48}$$

We can also show that

$$\forall\mathscr{A}_i \in \{\mathscr{A}_i\}, \qquad \exists!\omega \in \mathrm{Extr}(\mathfrak{S}) \text{ such that } \mathscr{A}_i\omega \neq 0. \tag{7.49}$$

The existence is because if $\mathscr{A}_i\omega = 0$ for each $\omega \in \mathrm{Extr}(\mathfrak{S})$ then $\mathscr{A}_i$ is the null transformation. The uniqueness can be proved supposing by absurd that a state ω both satisfies $\mathscr{A}_1\omega \neq 0$ and $\mathscr{A}_2\omega \neq 0$ for $\mathscr{A}_1, \mathscr{A}_2 \in \{\mathscr{A}_i\}$. Then we would have $\mathscr{A}_1\mathscr{A}_2\omega = p_2^{\omega}\mathscr{A}_1\omega = p_2^{\omega}p_1^{\omega}\omega = p_1p_2\omega \neq 0$ in contradiction with Eq. (7.48). Moreover, according to the relation $\sum_i \mathscr{A}_i\omega = \omega$, the only state achieving $\mathscr{A}_i\omega \neq 0$ must satisfy $\mathscr{A}_i\omega = \omega$.In conclusion is $\mathscr{I} = \sum_i \mathscr{A}_1$ where the atomic transformations $\mathscr{A}_i$ are in one to one correspondence with the extremal states $\omega_i \in \mathrm{Extr}(\mathfrak{S})$ according to the condition $\mathscr{A}_i\omega_j = \delta_{ij}\omega_j$, namely $\mathfrak{S}$ is a simplex. The other implication, corresponding to the *if* in the statement of the theorem, is trivial. ■

Corollary 7.3 *For a non simplex probabilistic theory all the local automorphisms are atomic.*

Proof. This is an immediate consequence of the atomicity of the identity. Every automorphism $\mathscr{A}$ is inverted by a transformation[9], namely there exist a $\mathscr{A}^{-1} \in \mathfrak{T}_+$ such that $\mathscr{A}\mathscr{A}^{-1} = \mathscr{T}$, from which the atomicity of $\mathscr{A}$. ■

A criterion to ensure the atomicity of the identity transformation is the following one

Proposition 7.1 *For each probabilistic theory having an atomic deterministic transformation the identical transformation is atomic.*

Proof. Considering a refinement $\mathscr{I} = \sum_i \mathscr{A}_i$ of the identity, according to Eq. (7.45) for a deterministic atomic transformation $\mathscr{T}$ we get

$$\mathscr{T}\mathscr{A}_i\omega(e) = p_i^{\omega}\mathscr{T}\omega(e) = q_i^{\mathscr{T}}\mathscr{T}\omega(e)\ \forall\omega \in \mathrm{Extr}(\mathfrak{S}) \quad \Rightarrow \quad p_i^{\omega} = q_i^{\mathscr{T}}\ \forall\omega \in \mathrm{Extr}(\mathfrak{S}), \tag{7.50}$$

that is the coefficients p_i^{ω} doesn't defend on ω. This is true for each atomic transformation $\mathscr{A}_i$ in the refinement which are all proportional to the identity. Therefore $\mathscr{I}$ is atomic. ■

[9]See the Wigner Theorem in Corollary 2.1.

In a simplex theory naturally doesn't exist a pure deterministic transformation according to Theorem 7.2. As a matter of fact we have constructed the atomic transformations for a simplex theory in Theorem 7.1 and none of them was deterministic. CHSH

We can also state the following result about purification for a simplex theory.

Proposition 7.2 *All the simplex theories doesn't allow purification*

Proof. The proof is an immediate consequence of Lemma 2.6-1 in Chap. 2. There was showed that Postulate PURIFY imply the atomicity of the identity which is not admissible if the convex sets of states are simplexes. ■

7.3.3 CHSH inequality and simplex theories

Here we shows that the simplex theories, namely the classical ones, are the only probabilistic theories satisfying the **CHSH** inequality. This feature is expressed in the following theorem.

Theorem 7.3 *A probabilistic theory satisfy the CHSH bound if and only if it is simplex.*

Proof. We start from the implication

$$\text{simplex theory} \Rightarrow \text{CHSH is satisfied.}$$

Given a simplex probabilistic theory we have to check that the bounds

$$-2 \leqslant \mathbf{J}_\Phi(a_0, a_0', b_0, b_0') \leqslant 2. \tag{7.51}$$

are satisfied for each set of effects a_0, a_0', b_0, b_0', and for each probability rule induced by all the admissible bipartite states Φ. From the explicit expression of $\mathbf{J}_\Phi(a_0, a_0', b_0, b_0')$ in Eq. (7.21) we know that the two bounds read as follows

$$-1 \leqslant \Phi(a_0 + a_0', b_0) + \Phi(a_0 - a_0', b_0') - \Phi(a_0, e) - \Phi(e, b_0) \leqslant 0. \tag{7.52}$$

From Theorem 7.1 we know that the extremal states of $\mathfrak{S}^{\odot 2}$ are the n^2 factorised states $\omega_i \otimes \omega_j$ where n is the dimension of the local system and the states ω_i, $i = 1, \dots, n$ are the vertexes of $\mathfrak{S}$. Since each bipartite state is a convex combinations of the n^2 vertexes above, the bounds further simplify to

$$-1 \leqslant \omega_i(a_0 + a_0')\omega_j(b_0) + \omega_i(a_0 - a_0')\omega_j(b_0') - \omega_i(a_0) - \omega_j(b_0) \leqslant 0 \quad \forall i, j = 1, \dots, n. \tag{7.53}$$

Consider first the upper bound. If $\omega_i(a_0) - \omega_i(a_0') > 0$ then we get

$$\begin{gathered} \omega_i(a_0 + a_0')\omega_j(b_0) + \omega_i(a_0 - a_0')\omega_j(b_0') - \omega_i(a_0) - \omega_j(b_0) \leqslant \\ \omega_i(a_0)\omega_j(b_0) + \omega_i(a_0')\omega_j(b_0) + \omega_i(a_0) - \omega_i(a_0') - \omega_i(a_0) - \omega_j(b_0) = \\ \omega_j(b_0)(\omega_i(a_0) - 1) + \omega_i(a_0')(\omega_j(b_0) - 1) \leqslant 0. \end{gathered} \tag{7.54}$$

While if $\omega_i(a_0) - \omega_i(a'_0) < 0$ we get

$$\begin{aligned}\omega_i(a_0{+}a'_0)\omega_j(b_0) + \omega_i(a_0 - a'_0)\omega_j(b'_0) - \omega_i(a_0) - \omega_j(b_0) \leqslant \\ \omega_i(a_0)\omega_j(b_0) + \omega_i(a'_0)\omega_j(b_0) - \omega_i(a_0) - \omega_j(b_0) = \\ \omega_i(a_0)(\omega_j(b_0) - 1) + \omega_j(b'_0)(\omega_i(a'_0) - 1) \leqslant 0.\end{aligned} \tag{7.55}$$

Now consider the lower bound. If $\omega_i(a_0) - \omega_j(a'_0) < 0$ then

$$\begin{aligned}\omega_i(a_0 + a'_0)\omega_j(b_0) + \omega_i(a_0 - a'_0)\omega_j(b'_0) - \omega_i(a_0) - \omega_j(b_0) \geqslant \\ \omega_i(a_0)\omega_j(b_0){+}\omega_i(a'_0)\omega_j(b_0) + \omega_i(a_0) - \omega_i(a'_0) - \omega_i(a_0) - \omega_j(b_0) = \\ \omega_j(b_0)(\omega_i(a_0) + \omega_i(a'_0) - 1) - \omega_i(a'_0) \geqslant \\ \omega_j(b_0)(\omega_i(a'_0) - 1) - \omega_i(a'_0) \geqslant -1.\end{aligned} \tag{7.56}$$

Naturally if $\omega_i(a_0) - \omega_j(a'_0) > 0$ the same bound is satisfied

$$\begin{aligned}\omega_i(a_0{+}a'_0)\omega_j(b_0) + \omega_i(a_0 - a'_0)\omega_j(b'_0) - \omega_i(a_0) - \omega_j(b_0) \geqslant \\ \omega_j(b_0)(\omega_i(a'_0) + \omega_i(a_0) - 1) - \omega_i(a_0) \geqslant \\ \omega_j(b_0)(\omega_i(a_0) - 1) - \omega_i(a_0) \geqslant -1.\end{aligned} \tag{7.57}$$

Now we check the other implication

$$\text{CHSH satisfied} \;\Rightarrow\; \text{simplex theory.}$$

It's sufficient to show that if the theory is not simplex then the CHSH inequality is violable. As usual the starting point is the $\mathbf{J}$ factor expressed in terms of a bipartite state Φ and four arbitrary effects a_0, a'_0, b_0, b'_0. If for some choice we get

$$\Phi(a_0 + a'_0, b_0) + \Phi(a_0 - a'_0, b'_0) - \Phi(a_0, e) - \Phi(e, b_0) > 0, \tag{7.58}$$

then the CHSH can be violated, occurring $\mathbf{J}_\Phi(a_0, a'_0, b_0, b'_0) > 0$. If the theory is not simplex we can find two extremal effects a_0, a'_0 and two states ω^{a_0} and $\omega^{a'_0}$ such that

$$\begin{aligned}\omega^{a_0}(a_0) = 1, \qquad \omega^{a_0}(a'_0) > 0, \\ \omega^{a'_0}(a'_0) = 1, \qquad \omega^{a'_0}(a_0) > 0.\end{aligned} \tag{7.59}$$

Moreover, even for a non simplex theory, there exist a state $\omega \in \mathfrak{S}$ achieving

$$\omega(a_0 + a'_0) > (c_1\omega^{a_0} + c_2\omega^{a'_0})(a_0 + a'_0) = 1 + c_1\omega^{a_0}(a'_0) + c_2\omega^{a'_0}(a_0) \tag{7.60}$$

for every convex combination $c_1 + c_2 = 1$. From the isomorphism $\mathfrak{E}_+ \simeq \mathfrak{S}_+$ induced by a faithful state Φ we get $\omega = \Phi(\cdot, b_0)/\Phi(e, b_0)$ for some $b_0 \in \mathfrak{E}$ and then the first term in Eq. (7.58) is majored as follows

$$\Phi(a_0 + a'_0, b_0) = \Phi(e, b_0)\omega(a_0 + a'_0) > \Phi(e, b_0)(1 + c_1\omega^{a_0}(a'_0) + c_2\omega^{a'_0}(a_0)). \tag{7.61}$$

Consider now the second term in Eq. (7.58) in the form

$$\Phi(a_0 - a'_0, b'_0) = \Phi(a_0, b'_0) - \Phi(a'_0, b'_0) = \Phi(a_0, e)\omega_{a_0}(b'_0) - \Phi(a'_0, b'_0). \tag{7.62}$$

Let b'_0 be the extremal effect achieving $\omega_{a_0}(b'_0) = 1$, therefore

$$\Phi(a_0 - a'_0, b'_0) = \Phi(a_0, e) - \Phi(a'_0, b'_0). \tag{7.63}$$

From Eqs. (7.60) and (7.63) we get

$$\begin{gathered}\Phi(a_0 + a'_0, b_0) + \Phi(a_0 - a'_0, b'_0) - \Phi(a_0, e) - \Phi(e, b_0) > \\ \Phi(e, b_0)(c_1\omega^{a_0}(a'_0) + c_2\omega^{a'_0}(a_0)) - \Phi(a'_0, b'_0).\end{gathered} \tag{7.64}$$

Recalling the definition of b'_0 we can also write

$$\begin{gathered}\Phi(a_0 + a'_0, b_0) + \Phi(a_0 - a'_0, b'_0) - \Phi(a_0, e) - \Phi(e, b_0) > \\ c_1\Phi(a'_0, b'_0) + c_2\Phi(a_0, b_0'') - \Phi(a'_0, b_0').\end{gathered} \tag{7.65}$$

Now if $\Phi(a_0, b_0'') \geqslant \Phi(a'_0, b'_0)$ choosing $c_1 = c_2 = 1/2$ we find a CHSH violation. On the other hand, if $\Phi(a_0, b_0'') \leqslant \Phi(a'_0, b'_0)$ we can repeat the constriction inverting the roles of a_0 and a'_0 newly violating the CHSH inequality. ■

This theorem is of great importance because it states that classical mechanics, (namely the simplex probabilistic theories), is the only local probabilistic theory.

subsystem
state-preparations

Chapter 8

Subsystems

In this last chapter we give a possible definition of **subsystem**. In Chap. 2 was introduced an operational definition of system based on the notion of test. Here we want to define a subsystem according to that idea and ensuring a sort of compatibility between the system and its subsystem structures.

8.1 Definition

Consider a system S,

$$\mathsf{S} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots, \mathbb{S}\} \tag{8.1}$$

where the test $\mathbb{S}$ is belongs to the **state-preparations** tests.[1] We can define a subsystem of S as a set of tests deducible from the previous ones according to some conditions. Thus a set of tests S'

$$\mathsf{S}' = \{\mathbb{A}', \mathbb{B}', \mathbb{C}', \ldots, \mathbb{S}'\} \tag{8.2}$$

is said to be a subsystem of S, and we will denote it as

$$\mathsf{S}' \subset \mathsf{S}, \tag{8.3}$$

if the following conditions are satisfied:

1. S' is a system having $\mathfrak{S}(\mathsf{S}')$ as convex set of states. The convex set of effects is achieved as usual from $\mathfrak{E}(\mathsf{S}')$ by duality and set $\mathfrak{T}_+(\mathsf{S}')$ contains all the transformations of the system.
2. $\dim(S') < \dim(S)$.
3. There exists a linear map $\mathscr{V}$ from $\mathfrak{S}(\mathsf{S}')$ to $\mathfrak{S}(\mathsf{S})$ such that

$$\forall \omega' \in \mathfrak{S}(\mathsf{S}') \quad \exists!\, \omega \in \mathfrak{S}(\mathsf{S}) \text{ such that } \mathscr{V}\omega' = \omega. \tag{8.4}$$

The map $\mathscr{V}$ is injective with left-inverse $\mathscr{V}^{-1}$ acting from $\mathfrak{S}(\mathsf{S})$ to $\mathfrak{S}(\mathsf{S}')$.

[1] See Ssec. 2.1.1

4. The dual pairing relation is preserved. For each $\omega' \in \mathfrak{S}(\mathsf{S}')$ holds

$$\forall a' \in \mathfrak{E}(\mathsf{S}') \quad \exists!\, a \in \mathfrak{E} \;\text{ such that }\; \omega'(a') = \omega(a) \;\text{ and }\; a = a' \circ \mathscr{V}. \tag{8.5}$$

Then $\mathscr{V}$ is also an injective map between $\mathfrak{E}(\mathsf{S}')$ and $\mathfrak{E}(\mathsf{S})$.

5. The cone $\mathfrak{T}_+(\mathsf{S}')$ of physical transformations for the subsystem is achievable from the cone $\mathfrak{T}(\mathsf{S})$ as follows

$$\mathfrak{T}_+(\mathsf{S}') = \{\mathscr{A}' \,|\, \mathscr{A}' = \mathscr{V}^{-1}\mathscr{A}\mathscr{V},\ \mathscr{A} \in \mathfrak{T}(\mathsf{S})\}. \tag{8.6}$$

Naturally two transformations $\mathscr{A} \neq \mathscr{B}$ in $\mathfrak{T}(\mathsf{S})$ can be identified in $\mathfrak{T}_+(\mathsf{S}')$. Thus the set in Eq. (8.6) must be interpreted as set of distinguishable transformations.

We can give a more restrictive definition of subsystem of a system S requiring a condition on the bipartite systems. As we will see in the next section this condition is indirectly connected to the faithful states of the probabilistic theories built on the system and subsystem. We say that S' is a subsystem of S and denote it as

$$\mathsf{S}' \Subset \mathsf{S}, \tag{8.7}$$

if the further condition is satisfied

6. The bipartite set of states $\mathfrak{S}(\mathsf{S}'^{\odot 2})$ can be achieved from the bipartite states in $\mathfrak{S}(\mathsf{S}^{\odot 2})$, thus for each $\Psi' \in \mathfrak{S}(\mathsf{S}'^{\odot 2})$, there exist a $\Psi \in \mathfrak{S}(\mathsf{S}'^{\odot 2})$ such that

$$\Psi'(a', b') = \Psi(a' \circ \mathscr{V}, a' \circ \mathscr{V}) \qquad \forall a', b' \in \mathfrak{E}(\mathsf{S}'). \tag{8.8}$$

The same condition holds for the bipartite set of effects $\mathfrak{E}(\mathsf{S}'^{\odot 2})$ which can be achieved from $\mathfrak{S}(\mathsf{S}^{\odot 2})$, namely for each $E' \in \mathfrak{E}(\mathsf{S}'^{\odot 2})$, there exist an $E \in \mathfrak{E}(\mathsf{S}'^{\odot 2})$ such that

$$E'(\omega', \zeta') = E(\mathscr{V}\omega, \mathscr{V}\zeta) \qquad \forall \omega, \zeta \in \mathfrak{S}(\mathsf{S}'). \tag{8.9}$$

Observation 8.1 *The definition of a subsystem must reflect the definition of system. We know from Chap. 2 that a system is defines as a set of tests and then a subsystem must in some way to be a subset of test. The best definition would start from this point of view but seem to be convenient to define a subsystem from the structure of its set of states, effects and transformations. Naturally the subset of tests feature will be a consequence of our construction.*

8.2 Relations between systems and subsystems

In this section the definition of subsystem is explored. The goal is to indirectly check the effectiveness of this definition through the consequence of it. Substantially reasonable relations between a system and its subsystem must be attained.

First notice that

$$\begin{aligned} \mathscr{V}^{-1}\mathscr{V} &= \mathscr{I}_{\mathsf{S}'} \\ \mathscr{V}\mathscr{V}^{-1}\big|_{\mathsf{Img}(\mathscr{V})} &= \mathscr{I}_{\mathsf{S}} \end{aligned} \tag{8.10}$$

where $\mathscr{I}_{\mathsf{S}'}$ and $\mathscr{I}_{\mathsf{S}'}$ denote respectively the identical transformation in $\mathfrak{T}_+(\mathsf{S}')$ and $\mathfrak{T}_+(\mathsf{S})$. It's easy to deduce the next relations

$$\begin{aligned} \mathscr{V}\omega'(a) = \omega'(a \circ \mathscr{V}^{-1}) \qquad & \forall \omega' \in \mathfrak{S}(\mathsf{S}'),\ \forall a \in \mathfrak{E}(\mathsf{S}') \circ \mathscr{V}, \\ \mathscr{V}^{-1}\omega(a') = \omega(a' \circ \mathscr{V}) \qquad & \forall \omega \in \mathscr{V}\mathfrak{S}(\mathsf{S}'),\ \forall a' \in \mathfrak{E}(\mathsf{S}'), \end{aligned} \tag{8.11}$$

in fact from Eqs. (8.5) and (8.10) we get

$$\begin{aligned} \omega(a) &= \mathscr{V}\mathscr{V}^{-1}\omega(a) = \mathscr{V}\omega'(a) = \omega'(a') = \omega(a \circ \mathscr{V}^{-1}), \\ \omega'(a') &= \mathscr{V}^{-1}\mathscr{V}\omega'(a') = \mathscr{V}^{-1}\omega(a') = \omega(a) = \omega(a' \circ \mathscr{V}). \end{aligned} \tag{8.12}$$

Proposition 8.1 *The map $\mathscr{V}$ injecting a subsystem in the respective system is isometric.*

Proof. Let d the natural distances on the convex set $\mathfrak{S}(\mathsf{S})$ and $\mathfrak{E}(\mathsf{S})$ defined in Sec. 2.1.7 and d' the analogues for the system S'. Then $\mathscr{V}$ is isometric if

$$d'(\omega', \zeta') = d(\mathscr{V}\omega', \mathscr{V}\zeta') \qquad \forall \omega', \zeta' \in \mathfrak{S}(\mathsf{S}'). \tag{8.13}$$

We can try out this equality for every $\omega', \zeta' \in \mathfrak{S}(\mathsf{S}')$ as follows

$$\begin{aligned} d(\omega, \zeta) &= \sup_{a \in \mathfrak{E}} (\omega - \zeta)(a) = \sup_{a \in \mathfrak{E}} \mathscr{V}(\omega' - \zeta')(a) = \sup_{a \in \mathfrak{E}} (\omega' - \zeta')(a \circ \mathscr{V}^{-1}) \\ &= \sup_{a \in \mathfrak{E} \circ \mathscr{V}^{-1}} (\omega' - \zeta')(a) = \sup_{a' \in \mathfrak{E}'} (\omega' - \zeta')(a') = d'(\omega', \zeta'), \end{aligned} \tag{8.14}$$

where we have used the relation $\mathfrak{E} \circ \mathscr{V}^{-1} = \mathfrak{E}(\mathsf{S}')$.[2] Naturally by duality we also get

$$d'(a', b') = d(a' \circ \mathscr{V}, b' \circ \mathscr{V}) \qquad \forall a', b' \in \mathfrak{E}(\mathsf{S}'). \blacksquare \tag{8.15}$$

Observation 8.2 *Notice that we could define a subsystem imposing more restrictive conditions. For example we would have required*

$$\mathscr{V} : \mathfrak{S}_+(\mathsf{S}') \to \mathfrak{S}_+(\mathsf{S}), \quad \mathscr{V} : \mathfrak{E}_+(\mathsf{S}') \to \mathfrak{E}_+(\mathsf{S}) \tag{8.16}$$

to be cone homomorphisms. In such case $\mathscr{V}$ is automatically an isometric map because it send extremal rays into extremal rays

$$\mathscr{V} : \mathsf{Erays}(\mathfrak{S}_+(\mathsf{S}')) \to \mathsf{Erays}(\mathfrak{S}_+(\mathsf{S})), \quad \mathscr{V} : \mathsf{Erays}(\mathfrak{E}_+(\mathsf{S}')) \to \mathsf{Erays}(\mathfrak{E}_+(\mathsf{S})). \tag{8.17}$$

On the other hand with this further condition systems such as the Popescu-Rohlich one would not admit subsystems (see Sec. 8.3). In particular without the homomorphism condition it's possible that a pure state of the subsystem is a non pure state when embedded in the whole system.

[2] Notice that $\mathscr{V}^{-1}$ is not injective.

8.2.1 Induced faithful state and effect

Preparationally faithful state

From conditions (5)-(6) in the subsystem definition it's easy to prove the statement connecting the faithful states of the probabilistic theory based on the system and on the subsystem.[3]

Proposition 8.2 *If* $\mathsf{S}' \Subset \mathsf{S}$ *and if* Φ *is a symmetric preparationally faithful state with respect to the system* S *then* $\Phi' = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi$ *is a symmetric preparationally faithful state with respect to* S'.

Proof. From condition (6), for each bipartite state $\Psi' \in \mathfrak{S}(\mathsf{S}'^{\odot 2})$ there exist at least a $\Psi \in \mathfrak{S}(\mathsf{S}^{\odot 2})$ such that $\Psi' = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Psi$. Because of the faithfulness of Φ exists a transformation $\mathscr{A} \in \mathfrak{T}(\mathsf{S})$ such that $(\mathscr{A}, \mathscr{I})\Phi = \Psi$, hence

$$\begin{aligned}(\mathscr{A}, \mathscr{I})\Phi = \Psi &\Rightarrow (\mathscr{V}^{-1}, \mathscr{V}^{-1})(\mathscr{A}, \mathscr{I})\Phi = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Psi \\ &\Rightarrow (\mathscr{V}^{-1}\mathscr{A}\mathscr{V}, \mathscr{I})(\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Psi \\ &\Rightarrow (\mathscr{V}^{-1}\mathscr{A}\mathscr{V}, \mathscr{I})\Phi' = \Psi'.\end{aligned} \tag{8.18}$$

From condition (5) we know that $\mathscr{A}' = \mathscr{V}^{-1}\mathscr{A}\mathscr{V} \in \mathfrak{T}(\mathsf{S}')$ and then Eq. (8.18) show that exists a transformation $\mathscr{A}' \in \mathfrak{T}(\mathsf{S}')$ preparing the state Ψ' acting locally on the bipartite state $\Phi' = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi$. This is true for each $\Psi' \in \mathfrak{S}(\mathsf{S}'^{\odot 2})$ and then $\Phi' = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi$ is preparationally faithful for the system $\mathfrak{S}'$. Naturally if ϕ is pure and symmetric Φ' is too. ■

Observation 8.3 *At first sight it seem not obvious that the diagram in Fig. 8.3 would be commutative; however with our hypothesis it is the case when* Φ *and Phi′ are connected as in Proposition 8.2. In fact we get*

$$\omega'_{a'} = \Phi'(\cdot, a') = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi(\cdot, a') = \mathscr{V}^{-1}\Phi(\cdot, a' \circ \mathscr{V}) = \mathscr{V}^{-1}\omega_{a'\circ\mathscr{V}}. \tag{8.19}$$

Teleportation

From Postulate FAITH we know that there exist a bipartite effect $F \in \mathfrak{E}(\mathsf{S}^{\odot 2})$ achieving probabilistically the inverse of the cone isomorphism induced by the faithful state Φ and then the probabilistic teleportation. Here we show when the probabilistic teleportation for a theory having local system S imply the possibility of teleportation for all the probabilistic theories having as local system a S-subsystem. First notice that

Proposition 8.3 *If* E *is a bipartite effect in* $\mathfrak{E}(\mathsf{S}^{\odot 2})$ *then* $E' = E(\mathscr{V}^{-1}, \mathscr{V}^{-1})$ *is a bipartite effect in* $\mathfrak{E}(\mathsf{S}'^{\odot 2})$.

[3] As usual we are only taking into account identical systems, namely $\mathsf{S} \odot \mathsf{S}$ for the system and $\mathsf{S}' \odot \mathsf{S}'$ for the subsystem probabilistic theories.

$$
\begin{array}{ccc}
a' & \xrightarrow{\mathscr{V}} & a' \circ \mathscr{V} = a \\
\Big\downarrow \Phi' & & \Big\downarrow \Phi \\
\omega'_{a'} = \Phi'(\cdot, a') & \xrightarrow{\mathscr{V}} & \mathscr{V}\omega'_{a'} = \Phi(\cdot, a) = \omega_a
\end{array}
$$

Figure 8.1: Subsystems commutativity diagram.

Proof. According to condition (6), for each bipartite state $\Psi' \in \mathfrak{S}(\mathsf{S}'^{\odot 2})$ exists a $\Psi \in \mathfrak{S}(\mathsf{S}^{\odot 2})$ such that $\Psi' = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Psi$, thus

$$E(\mathscr{V}^{-1}, \mathscr{V}^{-1})(\Psi') = E(\mathscr{V}^{-1}, \mathscr{V}^{-1})((\mathscr{V}^{-1}, \mathscr{V}^{-1})\Psi) = E(\Psi) \in [0, 1] \qquad \forall \Psi \in \mathfrak{S}(\mathsf{S}'^{\odot 2}), \tag{8.20}$$

namely $E' = E(\mathscr{V}^{-1}, \mathscr{V}^{-1})$ is an effect in $\mathfrak{E}(\mathsf{S}'^{\odot 2})$. ■

We are now ready to show the main property about teleporatation in the following Proposition:

Proposition 8.4 *Consider a probabilistic theory having* S *as local system and let* $\mathsf{S}' \Subset \mathsf{S}$ *with* $\mathscr{V}$ *the isometric map injecting* S' *into* S*. Let* Φ *and* Φ' *be two preparationally faithful states connected by* $\mathscr{V}$ *as in Proposition 8.2. If* $F \in \mathfrak{E}(\mathsf{S}'^{\odot 2})$ *invert probabilistically* Φ *achieving teleportation then* $F' = F(\mathscr{V}^{-1}, \mathscr{V}^{-1})$ *invert* Φ'.

Proof. First notice that according to Proposition 8.3 $F' = F(\mathscr{V}^{-1}, \mathscr{V}^{-1})$ is a bipartite effect in $\mathfrak{E}(\mathsf{S}'^{\odot 2})$. By hypothesis there exist an $\alpha > 0$ such that

$$F_{12}\Phi_{23}(\cdot, a) = \alpha a_1 \qquad \forall a \in \mathfrak{E}(\mathsf{S}). \tag{8.21}$$

Cosider then the equalities chain

$$
\begin{aligned}
&F'_{12}\Phi'_{23}(\cdot, a') \\
=&F_{12}(\mathscr{V}^{-1}, \mathscr{V}^{-1})(\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi_{23}(\cdot, a') = F_{12}(\mathscr{V}^{-1}, \mathscr{V}^{-1})(\cdot, \mathscr{V}^{-1}\Phi_{23}(\cdot, a' \circ \mathscr{V})) \\
=&F_{12}\Phi_{12}(\cdot, a)\mathscr{V}^{-1} = \alpha a_1 \circ \mathscr{V}^{-1} = \alpha a'_1 \qquad \forall a' \in \mathfrak{E}(\mathsf{S}').
\end{aligned} \tag{8.22}
$$

showing the assertion. ■

Observation 8.4 *Notice that the teleportation probability* α *for the system* S *is only a lower limit for the teleportation probability in a* S*-subsystem* S'*; indeed in general it is possible to find out a bipartite effect in* $\mathfrak{E}(\mathsf{S}'^{\odot 2})$ *which achieves teleportation with a probability greater than* α*. Moreover teleportation could be possible in a subsystem even if it's not achievable for the whole system.*

bit

8.3 Some subsystems examples

isometric map representation. Consider $\mathsf{S}' \Subset \mathsf{S}$ with $\mathscr{V}$ the isometric map injecting S' in S. Given a minimal informationally complete observable, denoted as usual by $\{l'_j\}$, for the d'-dimensional system S', it's always possible to find an informationally complete observable $\{l_j\}$ for the d-dimensional system S such that

$$\{l_j\} = \{l'_1 \circ \mathscr{V}, l'_2 \circ \mathscr{V}, \ldots, l'_{d'+1} \circ \mathscr{V}, l_{d'+2}, \ldots, l_{d+1}\}. \tag{8.23}$$

This info-complete observable induce a representation for the two systems S and S'. The matrixes representing the preparationally faithfull states are as usual introduced by their action over the info-complete observables:

$$\Phi(l_i, l_j) = \Phi_{ij}, \qquad \Phi'(l'_i, l'_j) = \Phi'_{ij}, \tag{8.24}$$

while the representative of $\mathscr{V}$ is find to be the $(d+1) \times (d'+1)$ matrix V satisfying the relation

$$l'_j \circ \mathscr{V} = \sum_{k=1,\ldots,d+1} V_{kj} l_k. \tag{8.25}$$

In this representation, the relation $\Phi(\mathscr{V}^{-1}, \mathscr{V}^{-1})$ between faithful states becomes

$$(\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi(l'_i, l'_j) = \Phi(l'_i \circ \mathscr{V}, l'_j \circ \mathscr{V}) = \sum_{k,q=1,\ldots,d+1} V_{ki} \Phi_{ij} V_{jq} = \Phi'_{ij}, \tag{8.26}$$

namely

$$\boldsymbol{\Phi}' = \boldsymbol{V}^\tau \boldsymbol{\Phi} \boldsymbol{V}. \tag{8.27}$$

The Popescu-Rohrlich model subsystems

The only subsystem of local Popescu-Rohrlich system is the classical **bit**. There are only two really different way to embed a bit local system in the Popescu-Rohrlich one and in both cases is $\mathsf{S}' \subset \mathsf{S}$ and not $\mathsf{S}' \Subset \mathsf{S}$. In Fig. 8.2 we pictorially show the two possibilities. In the pictures at the top the local system is self-dual while in the bottom pictures it is not. From observing the figures emerges that in both cases no both pure state and pure effects in the subsystem correspond to pure states and effect in the Popescu-Rohrlich system. Then coherently with Observation 8.2 no subsystems of the Popescu-Rohrlich one would exist if we require the map $\mathscr{V}$ to be a cones homomorphism. Now we briefly explore the two different subsystems.

Self-dual bit. In the first case the bit local system, denoted by S' reserving the symbol S for the Popescu-Rohrlich local system, is self-dual, $\mathfrak{S}_+(\mathsf{S}') \simeq \mathfrak{E}_+(\mathsf{S}')$, as showed in the top imagine in Fig. 8.2. The set of states $\mathfrak{S}(\mathsf{S}')$ is injected in a diagonal of the square of states $\mathfrak{S}(\mathsf{S})$. The extremals of $\mathfrak{S}(\mathsf{S}')$ are the states ω'_1 and ω'_2 represented by the vectors[4]

$$l'(\omega'_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad l'(\omega'_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{8.28}$$

[4]Remember that $\{l_i\}$ is the canonical base for $\mathfrak{E}_\mathbb{R}$ and $\{\lambda_i\}$ is the canonical base for $\mathfrak{S}_\mathbb{R}$. This allows the representation of the systems in an Euclidean space. See the Block representation in Sec. 2.5 and Part II for further details.

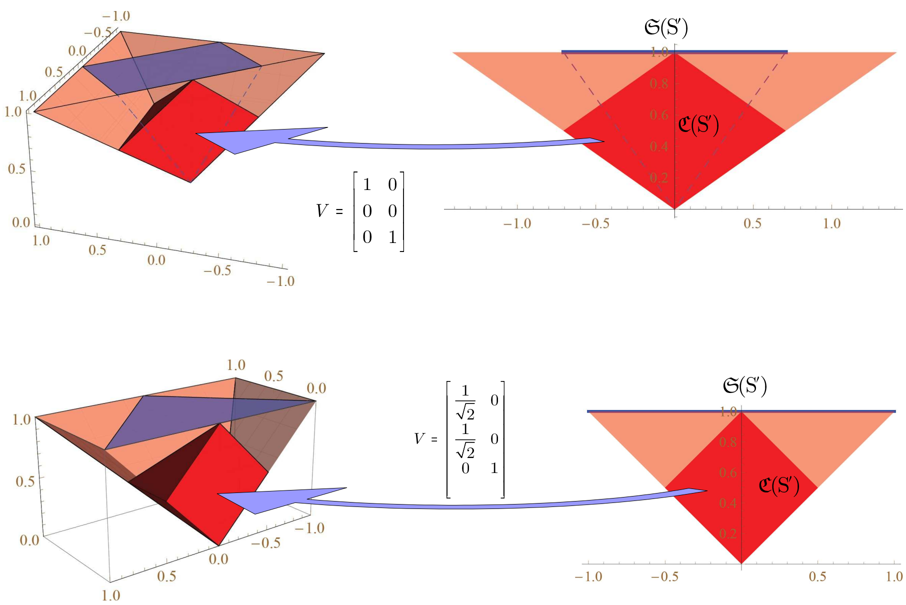


Figure 8.2: The possible subsystems of the Popescu-Rohrlich local one.**Upper figure:** on the right is represented the self-dual bit system while on the left is showed the section of the Popescu-Rohrlich local system corresponding to the image of the bit system under the action of the isometric map $\mathscr{V}$. The isometric matrix $\boldsymbol{V}$ represent the map $\mathscr{V}$. **Lower figure:** here there is the same representation corresponding to a no self-dual bit system.

and the isometric map $\mathscr{V}$ must inject them in $\mathfrak{S}_{\mathbb{R}}(\mathsf{S})$ as follows

$$\mathscr{V}\omega_1' = \omega_{00}, \quad l(\omega_{00}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathscr{V}\omega_2' = \omega_{01}, \quad l(\omega_{01}) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{8.29}$$

Therefore the matrix representing $\mathscr{V}$ is find to be

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{8.30}$$

and the left-inverse map $\mathscr{V}^{-1}$ is represented by

$$\boldsymbol{V}^{\tau} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.31}$$

Naturally $\boldsymbol{V} : \mathbb{R}^2 \Rightarrow \mathbb{R}^3$ while $\boldsymbol{V}^\tau : \mathbb{R}^3 \Rightarrow \mathbb{R}^2$, where $\mathbb{R}^2 \simeq \mathfrak{S}_\mathbb{R}(\mathsf{S}') \simeq \mathfrak{E}_\mathbb{R}(\mathsf{S}')$ and $\mathbb{R}^3 \simeq \mathfrak{S}_\mathbb{R}(\mathsf{S}) \simeq \mathfrak{E}_\mathbb{R}(\mathsf{S})$.[5] Moreover, according to the isometric nature of $\mathscr{V}$, we get

$$\boldsymbol{V}^\tau \boldsymbol{V} = \boldsymbol{I}_{\mathbb{R}^2}, \qquad \boldsymbol{V}\boldsymbol{V}^\tau = P_{\mathsf{Img}(\boldsymbol{V})}, \tag{8.32}$$

where $P_{\mathsf{Img}(\boldsymbol{V})}$ is the projector over the real linear span of the $\mathscr{V}$ image. As visualised in Fig. 8.2 the Popescu-Rohrlich system is projected by $P_{\mathsf{Img}(\boldsymbol{V})}$ into the section corresponding to the injection of the subsystem S'. We cannot say that $\mathsf{S}' \Subset \mathsf{S}$ because the further condition (6) is not satisfied. It's easy to show this observing that the relations between faithful states, which is a consequence of (6), cannot be satisfied. In Chap. 3 we showed that the eight non local pure vertexes $\Phi_{\alpha,\beta,\gamma}$ of $\mathfrak{S}(\mathsf{S}^{\odot 2})$ are all faithful states of the Popescu-Rohrlich model. Nevertheless the bipartite states in $\mathfrak{S}(\mathsf{S}'^{\odot 2})$ connected to them by the map $\mathscr{V}$ are not faithful state for the bit system. For example the bipartite subsystem's state Φ' couple with the faithful state Φ_{000} is

$$\boldsymbol{\Phi}' = \boldsymbol{V}^\tau \boldsymbol{\Phi}_{000} \boldsymbol{V} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \tag{8.33}$$

which certainly doesn't achieve the cone isomorphism $\mathfrak{S}_+(\mathsf{S}'^{\odot 2}) \simeq \mathfrak{E}_+(\mathsf{S}'^{\odot 2})$.

No self-dual bit system. As clearly showed in Fig. 8.2 it is no longer $\mathfrak{E} + (\mathsf{S}') = \mathfrak{S}_+(\mathsf{S}')$ and the extremal states of $\mathfrak{S}(\mathsf{S}')$

$$l'(\omega_1') = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}, \qquad l'(\omega_2') = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}, \tag{8.34}$$

are injected in $\mathfrak{S}_\mathbb{R}(\mathsf{S})$ as follows

$$\mathscr{V}\omega_1' = \frac{1}{2}(\omega_{00}+\omega_{11}), \quad l(\mathscr{V}\omega_1') = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \qquad \mathscr{V}\omega_2' = \frac{1}{2}(\omega_{01}+\omega_{10}), \quad l(\mathscr{V}\omega_2') = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}. \tag{8.35}$$

For completeness we also specify the extremal effects of $\mathfrak{E}(\mathsf{S}')$

$$\lambda'(a_1') = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \qquad \lambda'(a_2') = \frac{1}{2}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}, \tag{8.36}$$

whose image in $\mathfrak{E}_\mathbb{R}(\mathsf{S})$ are $a_0^{(1)}$ and $a_0^{(1)}$, among the well known extremal effectsof the Popescu-Rohrlich local system

$$a_1' \circ \mathscr{V} = a_0^{(1)}, \quad \lambda(a_0^{(1)}) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad a_2' \circ \mathscr{V} = a_1^{(1)}, \quad \lambda(a_1^{(1)}) = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \tag{8.37}$$

The isometric matrix representing $\mathscr{V}$ is

$$\boldsymbol{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}, \tag{8.38}$$

[5] Recall that in our representation this isomorphisms became identifications.

and as in the self-dual case the statement of Proposition 8.2 is not satisfied excluding $\mathsf{S}' \Subset \mathsf{S}$.

The clock subsystems

The only subsystem of the clock system is again the classical bit. From fig. 4.3 is plain that the subsystem can be injected in the System in infinite equivalent way. Taking the bit system self-dual as in the first example op the previous model, the extremal states are newly ω_1' and ω_2' represented as in Eq. (8.28). We can take as isometric map $\mathscr{V}$ one of the following matrixes

$$\boldsymbol{V} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}, \tag{8.39}$$

and differently from the Popescu-Rohrlich subsystems here $\Phi' = (\mathscr{V}^{-1}, \mathscr{V}^{-1})\Phi$ is a faithful state for the two-bits probabilistic theory if Φ is faithful for the two-clocks model. In fact taking Φ such that $\boldsymbol{\Phi} = \boldsymbol{I}_{\mathbb{R}^3}$ as in Chap. 4 we get

$$\boldsymbol{\Phi}' = \boldsymbol{V}^{\tau}\boldsymbol{\Phi}\boldsymbol{V} = \boldsymbol{V}^{\tau}\boldsymbol{V} = \boldsymbol{I}_{\mathbb{R}^2}, \tag{8.40}$$

which obviously is a faithful state for the two-bits model (by analogy with the faithful state for the two-trits model in Eq. (6.37)).

The trit subsystems

Naturally the bit is the only subsystem of the trit. Observing Fig. 6.1, or Fig 6.2, we can see that the bit set of states correspond to a side of the triangle representing $\mathfrak{S}(\mathsf{S})$. Then there are three equivalent injection of the subsystem in the system. It's easy to show that $\mathsf{S}' \Subset \mathsf{S}$.

Acknowledgment

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